HOMOMORPHISMS OF IDEALS IN GROUP ALGEBRAS[1]

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1. Let G_1 and G_2 be locally compact abelian groups, let J be a closed ideal in the group algebra $L^1(G_1)$, and let T be a homomorphism of the ideal J into the algebra $M(G_2)$ of bounded and regular Borel measures on G_2. The purpose of this paper is to show that when $\| T \| \leq 1$, then T must come from an affine transformation of the dual group Γ_2 of G_2 into the dual group Γ_1 of G_1. When $J = L^1(G_1)$, this is known and is due to Helson [3]. Helson assumes also that T is an isomorphism onto $L^1(G_2)$, but with a certain modification his argument works without this additional assumption. That T comes from an affine transformation when $J = L^1(G_1)$ and $\| T \| \leq 1$ is also a corollary of the deep result of Cohen [1], [4, ch. 4] that every homomorphism of $L^1(G_1)$ into $M(G_2)$ comes from a piecewise affine transformation of Γ_2 into Γ_1. Although our extension of Helson's theorem is very modest and the proof we offer is not difficult, it does not seem to be possible to obtain this extension from either the results or the arguments of Helson and Cohen.

Let Δ_1 be the open set of χ in Γ_1 such that $\hat{f}(\chi) \neq 0$ for some f in J, where $\hat{f}$ is the Fourier transform of f. Then Δ_1 can be identified with the maximal ideal space of J. Each χ in Δ_1 defines a nontrivial complex homomorphism of J whose value at f in J is $\hat{f}(\chi)$, and every such homomorphism of J is obtained in this way. Moreover, $\hat{J}$ separates points on Δ_1 as J contains every f in $L^1(G_1)$ such that the support of $\hat{f}$ is contained in Δ_1 [4, p. 161], and the topology of Δ_1 as a subspace of Γ_1 is the same as the topology induced on Δ_1 by the functions in $\hat{J}$. Let Δ_2 be the open set of χ in Γ_2 such that $(Tf)^\wedge(\chi) \neq 0$ for some f in J, where $(Tf)^\wedge$ is the Fourier-Stieltjes transform of the measure Tf. Each χ in Δ_2 defines a nontrivial complex homomorphism of J whose value at f in J is $(Tf)^\wedge(\chi)$, and therefore there is $\varphi(\chi)$ in Δ_1 such that $(Tf)^\wedge(\chi) = \hat{f}(\varphi(\chi))$. The map φ from Δ_2 into Δ_1 defined in this way is continuous and we have for f in J,

$$(Tf)^\wedge = 0 \qquad \text{on } \Gamma_2 \backslash \Delta_2$$
$$(Tf)^\wedge = \hat{f}(\varphi) \qquad \text{on } \Delta_2 .$$

Let Σ_2 be the coset in Γ_2 generated by Δ_2. A map π from Σ_2 into Γ_1 is said to be affine if

$$\pi(\alpha\beta\gamma^{-1}) = \pi(\alpha)\pi(\beta)\pi(\gamma)^{-1}$$

for all α, β, γ in Σ_2. Because the norm of a multiplicative linear functional on J is 1, we always have $\| T \| \geq 1$ (unless $T = 0$, and we will always assume $T \neq 0$). We will show:

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THEOREM 1. *Suppose* $\| T \| = 1$. *Then there is an affine transformation* π *from* Σ_2 *into* Γ_1 *such that* φ *is the restriction of* π *to* Δ_2 *and such that* $\pi(\Sigma_2 \backslash \Delta_2)$ *is contained in* $\Gamma_1 \backslash \Delta_1$.

The affine transformation π induces a homomorphism S of $M(G_1)$ into $M(G_2)$ such that

$$(S\mu)^{\wedge} = 0 \qquad \text{on } \Gamma_2 \backslash \Sigma_2$$

$$(S\mu)^{\wedge} = \hat{\mu}(\pi) \qquad \text{on } \Sigma_2$$

and $\| S \| = 1$ [4, p. 79]. Therefore we have as a corollary of Theorem 1:

THEOREM 2. *Every homomorphism with norm* 1 *of an ideal in* $L^1(G_1)$ *into* $M(G_2)$ *can be extended to a homomorphism with norm* 1 *of* $M(G_1)$ *into* $M(G_2)$.

Although a homomorphism of $L^1(G_1)$ must come from a piecewise affine transformation of the dual groups, this is not true when J is smaller than $L^1(G_1)$, and some additional condition, such as $\| T \| = 1$, is necessary to know that T comes from an affine or even a piecewise affine transformation of the dual groups. To give an example, let G_1 and G_2 be compact, let Δ_1 be a Sidon set in Γ_1, and let J be the ideal of all f in $L^1(G_1)$ with $\hat{f} = 0$ on $\Gamma_1 \backslash \Delta_1$. Then J is contained in $L^2(G_1)$ [4, p. 128], and all maps from sets in Γ_2 into Δ_1 that are one-one induce homomorphisms of J into $L^1(G_2)$.

2. We will show that Theorem 1 is true when G_1 and G_2 are compact, and then use an idea of Cohen's [1] to remove this restriction. The argument that we will give for compact groups is similar in some ways to one given in [2]. Assume then that G_1 and G_2 are compact, J is a closed ideal in $L^1(G_1)$, and T is a homomorphism of J into $M(G_2)$ with $\| T \| = 1$. Haar measure on G_k will be denoted by σ_k $(k = 1, 2)$. Haar measures on compact groups will always be such that the measure of the group is 1. We wish to point out that assuming $\sigma_1 G_1 = 1$ is not an additional assumption about T.

LEMMA 1. *Let* $\chi_1, \cdots, \chi_n$ *belong to* Γ_1 *and* $\psi_1, \cdots, \psi_n$ *to* Γ_2. *Suppose that for all complex numbers* $c_1, \cdots, c_n$

$$\int \left| \sum_{k=1}^{n} c_k \psi_k \right| d\sigma_2 \le \int \left| \sum_{k=1}^{n} c_k \chi_k \right| d\sigma_1 . \tag{1}$$

Then both sides of (1) *are the same for all choices of* $c_1, \cdots, c_n$.

Proof. Replace c_k in (1) by $c_k e^{ix_k}$. Reversing the order of integration in

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \int \left| \sum_{k=1}^{n} c_k e^{ix_k} \chi_k \right| d\sigma_1 \, dx_1 \cdots dx_n$$

shows that this integral is

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \left| \sum_{k=1}^{n} c_k e^{ix_k} \right| dx_1 \cdots dx_n .$$

Therefore both sides of (1) as continuous functions of $(x_1, \cdots, x_n)$ have the same mean value over the n-torus, and thus are the same.

There is a closed subgroup H in G_2 such that the range of T is contained in $L^1(H)$. Because G_1 is compact, Δ_1 is contained in J and J is the closed linear span of Δ_1. Let χ in Δ_1 be such that $T\chi \neq 0$. As χ is an idempotent in J, $T\chi$ is an idempotent in $M(G_2)$. Moreover

$$\| T\chi \| \leq \int | \chi | \, d\sigma_1 = 1,$$

and thus $T\chi = \psi\mu$ where ψ belongs to Γ_2 and μ is Haar measure on a closed subgroup H of G_2 [4, p. 62]. We will show that H does not depend on χ.

To this end let χ_k in Δ_1 be such that $T\chi_k \neq 0$ $(k = 1, 2)$. Then $T\chi_k = \psi_k \mu_k$ where ψ_k belongs to Γ_2 and μ_k is Haar measure on a closed subgroup H_k of G_2. Let $H = H_1 \cap H_2$.

Suppose H has infinite index in either H_1 or H_2. Then μ_1 and μ_2 are mutually singular and

$$\| \psi_1 \mu_1 + \psi_2 \mu_2 \| = 2.$$

On the other hand

$$\| \psi_1 \mu_1 + \psi_2 \mu_2 \| \leq \int | \chi_1 + \chi_2 | \, d\sigma_1$$

and so

$$\int | \chi_1 + \chi_2 | \, d\sigma_1 = 2.$$

This in turn implies $| \chi_1 + \chi_2 | = 2$ and therefore $\chi_1 = \chi_2$. But now $H_1 = H_2$, and this contradicts the infinite index assumption.

We have shown that H must have finite index in both H_1 and H_2. Let $m + 1$ be the index of H in H_1 and $n + 1$ the index of H in H_2, and let μ be Haar measure on H. Then, as the trace of μ_1 on H is $\mu/(m + 1)$ and the trace of μ_2 on H is $\mu/(n + 1)$, $\| c_1 \psi_1 \mu_1 + c_2 \psi_2 \mu_2 \|$ is given by

$$(2) \quad \int | c_1\psi_1/(m+1) + c_2\psi_2/(n+1) | \, d\mu + | c_1 | m/(m+1) + | c_2 | n/(n+1),$$

and therefore (2) does not exceed

$$(3) \qquad \int | c_1 \chi_1 + c_2 \chi_2 | \, d\sigma_1 .$$

On the other hand (3) does not exceed

$$(4) \quad \int | c_1 \chi_1/(m+1) + c_2 \chi_2/(n+1) | \, d\sigma_1 + | c_1 | m/(m+1) + | c_2 | n/(n+1),$$

and thus the integral in (2) does not exceed the integral in (4) for all complex numbers c_1 and c_2. Then both integrals are the same (Lemma 1 with G_2 replaced by H), and therefore (3) and (4) are the same. This in turn implies

$$| c_1 \chi_1 + c_2 \chi_2 | = | c_1 \chi_1/(m+1) + c_2 \chi_2/(n+1)| + | c_1 | m/(m+1) + | c_2 | n/(n+1)$$

for all c_1 and c_2, and this is possible only if $m = n = 0$.

We have shown that there is a closed subgroup H of G_2 such that the range of T is contained in $L^1(H)$. Because of the homomorphism of Γ_2 onto the dual group of H obtained by restricting Γ_2 to H, we can and will assume that T takes J into $L^1(G_2)$. Let Π_1 be the set of χ in Δ_1 such that $T\chi \neq 0$. Then $T\Pi_1$ is contained in Γ_2, and because the Fourier transform of a character is the characteristic function of the character, T takes Π_1 in a one-one way onto Δ_2 and $\varphi = T^{-1}$ on Δ_2. To show that φ can be extended to an affine transformation of Σ_2 into Γ_1, we can and will assume that Δ_2 and Δ_1 contain the identities of Γ_2 and Γ_1 and that these identities correspond with respect to φ. Then Σ_2 becomes the subgroup generated by Δ_2, and our problem is to show that φ can be extended to a group homomorphism π of Σ_2 into Γ_1 and that $\pi(\Sigma_2 \backslash \Delta_2)$ is contained in $\Gamma_1 \backslash \Delta_1$.

T carries $P(\Pi_1)$, the linear span of Π_1, onto the linear span of Δ_2, and when f is in $P(\Pi_1)$

$$\int | Tf | \, d\sigma_2 \le \int | f | \, d\sigma_1 . \tag{5}$$

Therefore (Lemma 1) both sides of (5) are the same. Let f be in $P(\Pi_1)$. Then

$$\int | 1 + zTf | \, d\sigma_2 = \int | 1 + zf | \, d\sigma_1 \tag{6}$$

for all complex numbers z since $T(1 + zf) = 1 + zTf$. As f is a bounded function on G_1, when $| z |$ is sufficiently small

$$\begin{aligned} | 1 + zf | &= (1 + zf)^{1/2}(1 + \bar{z}\bar{f})^{1/2} \\ &= \sum_{j,k \ge 0} \binom{\frac{1}{2}}{j} \binom{\frac{1}{2}}{k} z^j \bar{z}^k f^j \bar{f}^k \end{aligned}$$

and the integral on the right side of (6) is given by

$$\sum_{j,k \ge 0} \binom{\frac{1}{2}}{j} \binom{\frac{1}{2}}{k} z^j \bar{z}^k \int f^j \bar{f}^k \, d\sigma_1 . \tag{7}$$

Also, as Tf is a bounded function on G_2, when $| z |$ is sufficiently small the integral on the left side of (6) is given by (7) with f replaced by Tf and σ_1 replaced by σ_2. Because these two series are the same when $| z |$ is small, they must have the same coefficients, and we find

$$\int (Tf)^j (\bar{T}\bar{f})^k \, d\sigma_2 = \int f^j \bar{f}^k \, d\sigma_1 \tag{8}$$

for all nonnegative integers j and k. Let $f_1, \cdots, f_n$ be in $P(\Pi_1)$, and replace

f in (8) by $z_1 f_1 + \cdots + z_n f_n$ and Tf by $z_1\, Tf_1 + \cdots + z_n\, Tf_n$. Both sides of (8) are now polynomials in $z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n$, and we find by identifying coefficients that

$$\int (Tf_1)^{j_1} \cdots (Tf_n)^{j_n} (\bar{T}\bar{f}_1)^{k_1} \cdots (\bar{T}\bar{f}_n)^{k_n}\, d\sigma_2 \tag{9}$$
$$= \int f_1^{j_1} \cdots f_n^{j_n} \bar{f}_1^{k_1} \cdots \bar{f}_n^{k_n}\, d\sigma_1$$

for all nonnegative integers $j_1, \cdots, j_n, k_1, \cdots, k_n$. Now let $\chi_1, \cdots, \chi_n$ be characters in Π_1 and let $k_1, \cdots, k_n$ be any integers. Then because of (9)

$$\int |(T\chi_1)^{k_1} \cdots (T\chi_n)^{k_n} - 1|^2\, d\sigma_2 = \int |\chi_1^{k_1} \cdots \chi_n^{k_n} - 1|^2\, d\sigma_1$$

and therefore

$$(T\chi_1)^{k_1} \cdots (T\chi_n)^{k_n} = 1 \qquad \text{on } G_2$$

if and only if

$$\chi_1^{k_1} \cdots \chi_n^{k_n} = 1 \qquad \text{on } G_1.$$

This shows that *φ can be extended to a group isomorphism π of Σ_2 onto Σ_1*, where Σ_1 is the subgroup of Γ_1 generated by Π_1. There remains to show that $\pi(\Sigma_2\backslash\Delta_2)$ is contained in $\Gamma_1\backslash\Delta_1$, and this amounts to showing that $\Sigma_1 \cap \Delta_1 = \Pi_1$.

Suppose f is in $P(\Pi_1)$ and g is in $P(\Delta_1\backslash\Pi_1)$. Then

$$\int |f|\, d\sigma_1 \leq \int |f + g|\, d\sigma_1$$

since $Tg = 0$ and

$$\int |f|\, d\sigma_1 = \int |Tf|\, d\sigma_2$$
$$= \int |T(f+g)|\, d\sigma_2 \leq \int |f + g|\, d\sigma_1.$$

Suppose in addition that $f \neq 0$ on G_1. Then

$$\int |f|^{-1} f\bar{g}\, d\sigma_1 = 0 \tag{10}$$

as the function

$$\int |f + tg|\, d\sigma_1$$

of the real variable t has a minimum at $t = 0$ and the derivative of this function at $t = 0$ is

$$\tfrac{1}{2} \int |f|^{-1}(f\bar{g} + \bar{f}g)\, d\sigma_1.$$

Dropping the assumption that $f \neq 0$ on G_1 and replacing f in (10) by $1 + zf$ we have when $|z|$ is sufficiently small

$$\int |1 + zf|^{-1}(1 + zf)\bar{g}\, d\sigma_1 = 0.$$

Also when $|z|$ is sufficiently small

$$|1 + zf|^{-1} = \sum_{j,k\geq 0} \binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{k} z^j \bar{z}^k f^j \bar{f}^k$$

and

$$|1 + zf|^{-1}(1 + zf) = \sum_{j,k\geq 0} c_{jk}\, z^j \bar{z}^k f^j \bar{f}^k$$

where none of the c_{jk} are 0. Therefore

$$\sum_{j,k\geq 0} c_{jk}\, z^j \bar{z}^k \int f^j \bar{f}^k \bar{g}\, d\sigma_1 = 0$$

when $|z|$ is small, and we find that

$$\int f^j \bar{f}^k \bar{g}\, d\sigma_1 = 0 \tag{11}$$

for all nonnegative integers j and k. Replacing f in (11) by $z_1 f_1 + \cdots + z_n f_n$ we find that

$$\int f_1^{j_1} \cdots f_n^{j_n} \bar{f}_1^{k_1} \cdots \bar{f}_n^{k_n} \bar{g}\, d\sigma_1 = 0$$

for $f_1, \cdots, f_n$ in $P(\Pi_1)$, g in $P(\Delta_1\backslash\Pi_1)$, and all nonnegative integers $j_1, \cdots, j_n, k_1, \cdots, k_n$. This shows that no character in $\Delta_1\backslash\Pi_1$ can belong to the group generated by Π_1, and this completes the proof of Theorem 1 when G_1 and G_2 are compact.

3. We will now drop the assumption that G_1 and G_2 are compact and imitate Cohen [1], [4, p. 85] to complete the proof of Theorem 1. Let $\bar{G}_k$ be the Bohr compactification of G_k. The dual group of $\bar{G}_1$ is Γ_1 with the discrete topology and so associated with the set Δ_1 is another ideal $\bar{J}$ that consists of all f in $L^1(\bar{G}_1)$ with $\hat{f} = 0$ on $\Gamma_1\backslash\Delta_1$. What is true is that *the map φ from Δ_2 into Δ_1 induces a homomorphism S of $\bar{J}$ into $M(\bar{G}_2)$ with norm* 1 *such that*

$$\begin{aligned} (Sf)^\wedge &= 0 && \text{on } \Gamma_2\backslash\Delta_2 \\ (Sf)^\wedge &= \hat{f}(\varphi) && \text{on } \Delta_2\,. \end{aligned} \tag{12}$$

Let μ be Haar measure on $\bar{G}_1$.

LEMMA 2. *Let Δ be an open set in Γ_1, let*

$$f = \sum_{k=1}^{n} c_k \chi_k \tag{13}$$

where the χ_k are distinct characters in Δ, and let $\varepsilon > 0$. Then there is g in

$L^1(G_1)$ such that $\hat{g}$ has compact support contained in Δ, $\hat{g}(\chi_k) = c_k$ $(k = 1, \cdots, n)$, and

$$\int |g| \, d\sigma_1 \leq (1 + \varepsilon) \int |f| \, d\mu.$$

Proof. For each compact neighborhood N of the identity in Γ_1 choose a nonnegative function g_N in $L^1(G_1)$ such that $\hat{g}_N(1) = 1$ and $\hat{g}_N = 0$ on $\Gamma_1 \backslash N$. The net of positive measures $g_N \, d\sigma_1$ converges in the weak star topology of $M(\bar{G}_1)$ to the Haar measure of $\bar{G}_1$, and therefore there is a compact neighborhood N' such that N contained in N' implies

$$\int |f| \, g_N \, d\sigma_1 \leq (1 + \varepsilon) \int |f| \, d\mu.$$

Choose N contained in N' such that for $k = 1, \cdots, n$ the sets $\chi_k N$ are pairwise disjoint and $\chi_k N$ is contained in Δ. Then the function $g = fg_N$ will have the desired properties as

$$\hat{g}(\chi) = \sum_{k=1}^{n} c_k \, \hat{g}_N(\chi \bar{\chi}_k).$$

Let f be in $P(\Delta_1)$, the linear span of Δ_1, and define a linear functional L on $P(\Gamma_2)$ by

$$L(F) = \sum_{\Delta_2} \hat{F}(\chi) \hat{f}(\varphi(\chi)).$$

With f given by (13), let Δ be the open set in Γ_1 obtained by removing from Δ_1 the finitely many points of the form $\varphi(\chi)$ where χ is in Δ_2, $\hat{F}(\chi) \neq 0$, and $\varphi(\chi) \neq \chi_k$ $(k = 1, \cdots, n)$. Now choose g as in Lemma 2. Then

$$L(F) = \sum_{\Delta_2} \hat{F}(\chi) \hat{g}(\varphi(\chi)).$$

As $\hat{g}$ has compact support contained in Δ, g belongs to J, and therefore

$$L(F) = \sum \hat{F}(\chi)(Tg)^{\wedge}(\chi) = \int F(-x) \, d(Tg)(x)$$

and this shows that

$$|L(F)| \leq \| F \|_\infty (1 + \varepsilon) \int |f| \, d\mu.$$

Thus L is a bounded linear functional on $P(\Gamma_2)$ with norm not exceeding the $L^1(\bar{G}_1)$ norm of f, and therefore there is a measure ν in $M(\bar{G}_2)$ with

$$\| \nu \| \leq \int |f| \, d\mu$$

and

$$L(F) = \int F(-x) \, d\nu(x).$$

The definition of L shows that $\hat{\nu} = 0$ on $\Gamma_2 \backslash \Delta_2$ and $\hat{\nu} = \hat{f}(\varphi)$ on Δ_2, and thus there is a homomorphism S of $\bar{J}$ into $M(\bar{G}_2)$ with norm 1 given by (12). This completes the proof of Theorem 1.

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