GROTHENDIECK GROUPS OF INTEGRAL GROUP RINGS

BY

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1. Introduction

Let A be a ring, and consider the category of A-modules. Unless otherwise stated, A-modules are assumed to be left modules which are finitely generated. Recall that the *Grothendieck group* $K^0(A)$ of this category is the abelian additive group defined by means of generators and relations, as follows: the generators are the symbols $[M]$, where M ranges over all A-modules; the relations are given by

$$[M] = [M'] + [M''],$$

corresponding to all short exact sequences of A-modules

$$0 \to M' \to M \to M'' \to 0.$$

In particular, let G be a finite group, and let R = alg. int. $\{F\}$, the ring of all algebraic integers in the algebraic number field F. Denote by FG the group algebra of G over F, and by RG the integral group ring of G over R. Swan [11] has already demonstrated the importance of the Grothendieck group $K^0(RG)$ for the study of RG-modules, and has recently obtained in [13] some new fundamental results on the structure of the group.

The present authors have given an explicit formula for $K^0(RG)$ under the restriction that F be a splitting field for G (see [9]). This formula involves the ideal theory of the Dedekind ring R, as well as the decomposition numbers of G relative to the set of those prime ideals of R which divide the order of G.

Here we shall generalize this formula to the case where F need not be a splitting field for G. Our results will involve the ideal theory of certain algebraic extension fields of R, as well as analogues of the decomposition matrices.

In our earlier paper, we made use of the following:

THEOREM 1 (Brauer [3], [4]). *If F is a splitting field for G, then the set of maximal size minors of the decomposition matrix of G (relative to any prime ideal of R) has greatest common divisor* 1.

As a by-product of the present approach, an independent proof of Brauer's theorem is obtained.

For the homological algebra used herein, we refer the reader to [5]. As a general reference for the representation theory needed here, we may cite [6].

2. Whitehead groups

This section is devoted to the introduction of notation, and the statement of one of the main results of our previous paper [9]. Let R be any Dedekind

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ring with quotient field F, and let A^* be a finite-dimensional semisimple F-algebra. By an R-order A of A^* is meant a subring A of A^* such that

(i) $1 \in A$,
(ii) A contains an F-basis of A^*, and
(iii) A is finitely generated as R-submodule of A^*.

We may then form the Grothendieck groups $K^0(A^*)$, $K^0(A)$, and $K_t^0(A)$, the last of which is obtained from the category of R-torsion A-modules.

For X an R-torsion-free A-module, we shall denote the A^*-module $F \otimes_R X$ by FX, for brevity, and shall regard X as embedded in FX. If X and Y are a pair of R-torsion-free A-modules for which $FX \cong FY$ as A^*-modules, we may identify FX and FY. Then we define

$$[X /\!\!/ Y] = [X/U] - [Y/U] \in K_t^0(A),$$

where U is any A-submodule of $X \cap Y$ such that $FX = FY = FU$.

Let us recall that the *Whitehead group* $K^1(A^*)$ is the abelian additive group defined by generators and relations as follows: the generators are the symbols $[M, \mu]$, where M ranges over all A-modules, and μ ranges over all automorphisms of M; the relations are, first, those of the form

$$[M, \mu\mu'] = [M, \mu] + [M, \mu']$$

for a pair of automorphisms μ, μ' of M; and second, those of the form

$$[M, \mu] = [M', \mu'] + [M'', \mu'']$$

for every short exact sequence of A^*-modules

$$0 \longrightarrow M' \xrightarrow{\ \varphi\ } M \xrightarrow{\ \psi\ } M'' \longrightarrow 0$$

such that $\mu\varphi = \varphi\mu'$, $\mu''\psi = \psi\mu$.

We quote without proof:

THEOREM 2 (Heller and Reiner [9]). *There is an exact sequence*

$$K^1(A^*) \xrightarrow{\ \delta\ } K_t^0(N) \xrightarrow{\ \eta\ } K^0(A) \xrightarrow{\ \theta\ } K^0(A^*) \longrightarrow 0,$$

with the maps defined as follows:

(i) *Given* $[M, \mu] \in K^1(A^*)$, *let* M_0 *be any* A*-submodule of* M *such that* $FM_0 = M$, *and set*

$$\delta[M, \mu] = [\mu M_0 /\!\!/ M_0] \in K_t^0(A).$$

(ii) *The map* η *is induced by the inclusion of the category of* R*-torsion* A*-modules in the category of all* A*-modules.*

(iii) *For an* A*-module* L, *set* $\theta[L] = [F \otimes_R L]$.

For later use, we shall determine the Whitehead group of a simple algebra A^*. Suppose that A^* is a full matrix algebra over the division ring D, and let W be an irreducible A^*-module. Then we may write $D = \mathrm{Hom}_{A^*}(W, W)$, and view W as a right D-module. As is well known, we have

$$A^* = \mathrm{Hom}_D(W, W).$$

Now each A^*-module M is a direct sum of (say) t copies of W, and each automorphism μ of M is represented by an invertible $t \times t$ matrix $\bar{\mu}$ with entries in D. Let $\tilde{D}$ denote the multiplicative group of non-zero elements of D, and set

$$D^{\#} = \tilde{D}/[\tilde{D}, \tilde{D}],$$

the factor commutator group of $\tilde{D}$. We may then form the Dieudonné determinant $d(\bar{\mu}) \in D^{\#}$. It is easily seen that the relations which serve to define $K^1(A^*)$ are precisely those which characterize the Dieudonné determinant (see [8]). Thus we have

$$K^1(A^*) \cong D^{\#},$$

the isomorphism being given by $[M, \mu] \to d(\bar{\mu})$.

As a special case of the above, we have $K^1(D) \cong D^{\#}$. (In fact, Morita's theorem (see §3) implies that the categories of A-modules and D-modules are isomorphic. Consequently we may conclude that $K^1(A^*) \cong K^1(D)$.)

Suppose now that A is an R-order in the simple algebra A^*, and let W_0 be any A-submodule of W such that $FW_0 = W$. We may write

$$D^{\#} \cong K^1(A^*) \to K_t^0(A),$$

thereby obtaining a map $D^{\#} \to K_t^0(A)$ which we again denote by δ. For $\lambda \in \tilde{D}$, we have

$$\delta(\lambda) = [W_0 \lambda /\!\!/ W_0].$$

3. Maximal orders in central simple algebras

Let A^* be a central simple algebra over the algebraic number field F. Then A^* is isomorphic to a full matrix algebra over a division ring D whose center is F. Let W be an irreducible A^*-module, viewed as right D-module. Then we may write

$$D = \mathrm{Hom}_{A^*}(W, W), \qquad A^* = \mathrm{Hom}_D(W, W).$$

Now let $R = \text{alg. int. } \{F\}$, and let A be a maximal R-order in A^*. Such maximal orders always exist, but need not be unique. From the results of Auslander and Goldman [1], it follows that there exists a maximal R-order $\mathfrak{o}$ in D, and a right projective $\mathfrak{o}$-module M contained in W, such that $W = FM$ and

$$A = \mathrm{Hom}_{\mathfrak{o}}(M, M).$$

We shall use Morita's theorem to set up an isomorphism between the categories of left A-modules and left $\mathfrak{o}$-modules, following an approach due to Bass [2]. The right $\mathfrak{o}$-module M is called a *generator* (of the category of right $\mathfrak{o}$-modules) if given any pair of right $\mathfrak{o}$-modules X and Y, and any non-zero map f in $\text{Hom}_{\mathfrak{o}}(X, Y)$, there exists a map $g \in \text{Hom}_{\mathfrak{o}}(M, X)$ such that fg is not the zero map. It is convenient to rephrase this as follows: The map f induces a map

$$f^* : \text{Hom}_{\mathfrak{o}}(M, X) \to \text{Hom}_{\mathfrak{o}}(M, Y).$$

Then M is a generator if and only if for each $f \in \text{Hom}_{\mathfrak{o}}(X, Y)$, $f \neq 0$ implies $f^* \neq 0$.

We now quote without proof.

THEOREM 3 (Morita [10]; see Bass [2]). *Let M be a right finitely generated projective $\mathfrak{o}$-module which is a generator for the category of right $\mathfrak{o}$-modules. Define $A = \text{Hom}_{\mathfrak{o}}(M, M)$, viewed as a ring of left operators on M, and set $\tilde{M} = \text{Hom}_{\mathfrak{o}}(M, \mathfrak{o})$, a left $\mathfrak{o}$-, right A-module. Then the categories of left A-modules and left $\mathfrak{o}$-modules are isomorphic, and the isomorphism is given as follows: a left $\mathfrak{o}$-module U corresponds to the left A-module $M \otimes_{\mathfrak{o}} U$, and inversely a left A-module V corresponds to the left $\mathfrak{o}$-module $\tilde{M} \otimes V$. Furthermore, $\mathfrak{o} = \text{Hom}_A(M, M)$ as right operator domain on M.*

In order to apply the above, we must verify that in our case M is indeed a generator. Let X, Y be $\mathfrak{o}$-modules, and let $f \in \text{Hom}_{\mathfrak{o}}(X, Y)$, $f \neq 0$. We need only show that $f^* \neq 0$. Let P denote a prime ideal of R, and let a subscript P indicate localization at P. Since $f \neq 0$, then also $f_P \neq 0$ for some P, where $f_P : X_P \to Y_P$. By the results of [1], the R_P-order $\mathfrak{o}_P$ is a hereditary principal ideal ring, so that M_P is a free $\mathfrak{o}_P$-module (see [13]). Consequently M_P is an $\mathfrak{o}_P$-generator, and therefore $(f_P)^* \neq 0$. But $(f_P)^* = (f^*)_P$, and therefore also $f^* \neq 0$, as desired.

Applying Morita's theorem, we have $\mathfrak{o} = \text{Hom}_A(M, M)$, and the category of left A-modules is isomorphic to the category of left $\mathfrak{o}$-modules, under the isomorphisms given above. Therefore we have

$$K^0(A) \cong K^0(\mathfrak{o}).$$

Furthermore, the isomorphism of categories preserves R-torsion, so that also

$$K_t^0(A) \cong K_t^0(\mathfrak{o}), \qquad K^0(A/PA) \cong K^0(\mathfrak{o}/P\mathfrak{o})$$

for each prime ideal P of R. We note further that

$$K_t^0(A) = \sum\nolimits_P^{\oplus} K^0(A/PA) \cong \sum\nolimits_P^{\oplus} K^0(\mathfrak{o}/P\mathfrak{o}) = K_t^0(\mathfrak{o}).$$

Let $I(R)$ denote the abelian multiplicative group of non-zero R-ideals in F. We shall show that $K_t^0(\mathfrak{o}) \cong I(R)$, and in fact shall give two descriptions of this isomorphism. For any R-torsion $\mathfrak{o}$-module X, define

$$\text{ann } X = \{\alpha \in R : \alpha X = 0\} \in I(R).$$

Now let V be any R-torsion $\mathfrak{o}$-module, and let $V_1, \cdots, V_k$ be its $\mathfrak{o}$-composition factors. Define the *order ideal* of V to be

$$\text{ord } V = \prod\nolimits_{i=1}^{k} (\text{ann } V_i) \in I(R).$$

Then ord V is a well-defined ideal in R, and the map $[V] \to \text{ord } V$ defines a homomorphism of $K_t^0(\mathfrak{o})$ into $I(R)$, since the composition factors of an extension module are just those of the submodule together with those of the factor module.

Let us show that the above-defined map is an isomorphism. The additive group $K_t^0(\mathfrak{o})$ has as Z-basis the elements $[\mathfrak{o}/\mathfrak{m}]$, where $\mathfrak{m}$ ranges over all maximal left ideals of $\mathfrak{o}$. This is clear from the fact that every irreducible $\mathfrak{o}$-module is expressible as $\mathfrak{o}/\mathfrak{m}$, for some $\mathfrak{m}$. For fixed $\mathfrak{m}$, let $\mathfrak{p}$ be the unique maximal two-sided ideal of $\mathfrak{o}$ contained in $\mathfrak{m}$. Set $P = \mathfrak{p} \cap R$, a prime ideal in R. Then ord $(\mathfrak{o}/\mathfrak{m}) = P$.

On the other hand, there is a mapping $I(R) \to K_t^0(\mathfrak{o})$, defined as follows. Let P be any prime ideal of R. By [1] there is a unique maximal two-sided ideal $\mathfrak{p}$ of $\mathfrak{o}$ such that $\mathfrak{p} \cap R = P$. The ring $\mathfrak{o}/\mathfrak{p}$ is then a simple ring. If $\mathfrak{m}$ is any maximal left ideal of $\mathfrak{o}$ such that $\mathfrak{p} \subset \mathfrak{m}$, then $\mathfrak{o}/\mathfrak{m}$ is an irreducible $(\mathfrak{o}/\mathfrak{p})$-module, which is determined up to isomorphism by P, since $\mathfrak{o}/\mathfrak{p}$ is simple. Letting $P \to [\mathfrak{o}/\mathfrak{m}]$, we obtain a homomorphism of $I(R)$ onto $K_t^0(\mathfrak{o})$. It follows at once that $K_t^0(\mathfrak{o}) \cong I(R)$, the isomorphisms being given as above.

The referee has kindly pointed out a second description of the above isomorphism, which is more useful for later purposes. For $\mathfrak{a}$ any (non-zero) left ideal in $\mathfrak{o}$, let $N\mathfrak{a}$ be its reduced norm (see [7]). We recall the definition of reduced norm: take any R-composition series of the R-module $\mathfrak{o}/\mathfrak{a}$, and let $N'\mathfrak{a}$ be the product of the annihilators of the composition factors. Let n be the index of the division algebra D, so that $(D:F) = n^2$. The equation

$$(N\mathfrak{a})^n = N'\mathfrak{a}$$

then serves to define an ideal $N\mathfrak{a}$ in R, called the *reduced norm* of $\mathfrak{a}$.

We shall prove that ord $(\mathfrak{o}/\mathfrak{a}) = N\mathfrak{a}$, and for this it suffices to prove that $P = N\mathfrak{m}$, where $\mathfrak{m}$ and P are related as above. The simple ring $\mathfrak{o}/\mathfrak{p}$ is a full matrix algebra $(\bar{k})_r$ over some skewfield $\bar{k}$. Since $\bar{k}$ is a finite extension of R/P, it follows from Wedderburn's theorem that $\bar{k}$ is a field. The ring $\mathfrak{o}/\mathfrak{p}$ is a direct sum of r copies of the irreducible $(\mathfrak{o}/\mathfrak{p})$-module $\mathfrak{o}/\mathfrak{m}$, which implies that $N'\mathfrak{p} = (N'\mathfrak{m})^r$. On the other hand, $\mathfrak{o}/\mathfrak{p}$ is R-isomorphic to a direct sum of f copies of R/P, where $f = (\mathfrak{o}/\mathfrak{p}:R/P)$. Therefore

$$N'\mathfrak{p} = P^f, \qquad N\mathfrak{m} = P^{f/rn},$$

and we need only show that $f = rn$.

This may be accomplished by working over the P-adic completion $\hat{F}$ of the field F. Let $\hat{R}$ be the valuation ring of $\hat{F}$, and $\hat{P}$ its prime ideal. Set $\hat{D} = D \otimes_F \hat{F}$, $\hat{\mathfrak{o}} = \mathfrak{o} \otimes_R \hat{R}$. Then $\hat{D}$ is a simple ring with center $\hat{F}$, but is not necessarily a skewfield. Write $\hat{D} = (\hat{D}_1)_s$, a full matrix algebra over a skewfield $\hat{D}_1$. If we set $n_1^2 = (\hat{D}_1:\hat{F})$, then $n^2 = (\hat{D}:\hat{F}) = s^2 n_1^2$, so $n = sn_1$. As in Theorem 3, we may write

$$\hat{\mathfrak{o}} = (\hat{\mathfrak{o}}_1)_s, \qquad \hat{\mathfrak{p}} = (\hat{\mathfrak{p}}_1)_s,$$

where $\hat{\mathfrak{o}}_1$ is a maximal order in $\hat{D}_1$, and $\hat{\mathfrak{p}}_1$ is a maximal two-sided ideal in $\hat{\mathfrak{o}}_1$. If $f_1 = (\hat{\mathfrak{o}}_1/\hat{\mathfrak{p}}_1 : \hat{R}/\hat{P})$, then

$$f = (\mathfrak{o}/\mathfrak{p} : R/P) = (\hat{\mathfrak{o}}/\hat{\mathfrak{p}} : \hat{R}/\hat{P}) = s^2 f_1 .$$

But also $\mathfrak{o}/\mathfrak{p} \cong (\hat{\mathfrak{o}}_1/\hat{\mathfrak{p}}_1)_s$, and since $\hat{F}$ is a complete P-adic field, it follows from [7] that $\hat{\mathfrak{o}}_1/\hat{\mathfrak{p}}_1$ is a field. Thus $r = s$, and so $f/rn = s^2 f_1/s^2 n_1 = f_1/n_1$. However, since $\hat{F}$ is complete, we have $f_1 = n_1$, which shows that $f = rn$, as claimed.

(Later on we shall need to know the number ν of composition factors of the $(\mathfrak{o}/\mathfrak{p})$-module $\mathfrak{o}/P\mathfrak{o}$. Let us compute this by comparing dimensions over R/P. The dimension of an irreducible $(\mathfrak{o}/\mathfrak{p})$-module is sf_1, while

$$\dim (\mathfrak{o}/P\mathfrak{o}) = s^2 \dim (\hat{\mathfrak{o}}_1/\hat{\mathfrak{p}}_1^{e_1}) = s^2 e_1 f_1 ,$$

where e_1 is the ramification index of P at $\hat{\mathfrak{p}}_1$. Since $\hat{F}$ is a P-adic field, we have $e_1 = f_1 = n_1$, and thus $\nu = s^2 e_1 f_1/sf_1 = se_1 = sn_1 = n$. This shows that $\mathfrak{o}/P\mathfrak{o}$ has n composition factors when viewed as $(\mathfrak{o}/\mathfrak{p})$-module.)

In §2 we had defined a map $\delta : D^{\#} \to K_t^0(A)$. Since

$$K_t^0(A) \cong K_t^0(\mathfrak{o}) \cong I(R),$$

δ gives a map of $D^{\#}$ into $I(R)$; denote by $J(R)$ the image of δ in $I(R)$. For $\lambda \in \tilde{D}$, we have $\delta(\lambda) = [W_0 \lambda /\!\!/ W_0] \in K_t^0(A)$, where W_0 is an A-submodule of A^* such that $FW_0 = W$; indeed, choose W_0 to be the module M in Theorem 3. Since M corresponds to $\mathfrak{o}$ itself in the correspondence given in Theorem 3, we see that

$$\delta(\lambda) = [\mathfrak{o}\lambda /\!\!/ \mathfrak{o}] \in K_t^0(\mathfrak{o}),$$

and hence (in $I(R)$)

$$\delta(\lambda) = \{N(\mathfrak{o}\lambda)\}^{-1}.$$

However, $N(\mathfrak{o}\lambda) = (N\lambda)R$, where $N\lambda$ is the reduced norm of the element λ. This shows that $J(R)$ is the subgroup of $I(R)$ generated by the principal ideals $(N\lambda)R$, where λ ranges over all non-zero elements of D.

As shown in [12], we may describe $J(R)$ explicitly. If P_0 is an infinite prime of R, and F_0 is the P_0-adic completion of F, we call D *ramified* at P_0 if $D \otimes_F F_0$ is a full matrix algebra over the real quaternions. Let U be the divisor of R consisting of all infinite primes P_0 at which D is ramified. Then $J(R)$ is precisely the ray mod U, that is,

$$J(R) = \{xR : x \in F, x > 0 \text{ at each } P_0 \in U\}.$$

We shall briefly discuss the projective class group $P(\mathfrak{o})$, and reduced projective class group $C(\mathfrak{o})$, of the ring $\mathfrak{o}$. The group $P(\mathfrak{o})$ is defined as the Grothendieck group of the category of projective $\mathfrak{o}$-modules, and there is an obvious map $P(\mathfrak{o}) \to K^0(\mathfrak{o})$. However, the ring $\mathfrak{o}$ is hereditary (by [1]), and as pointed out in [13], this easily implies that the above map is an isomorphism: $P(\mathfrak{o}) \cong K^0(\mathfrak{o})$. Since A is also hereditary, we have similarly $P(A) \cong K^0(A)$.

Swan [13] proved that

$$C(A) \cong C(\mathfrak{o}) \cong I(R)/J(R).$$

We may obtain this same result here by use of Theorems 2 and 3. Using the map θ defined in Theorem 2, we have

$$P(\mathfrak{o}) \cong K^0(\mathfrak{o}) \xrightarrow{\theta} K^0(D),$$

which defines a homomorphism (again denoted by θ) of $P(\mathfrak{o})$ into $K^0(D)$. In [13, Prop. 4.1], Swan showed that the kernel of θ (in $P(\mathfrak{o})$) is precisely $C(\mathfrak{o})$.

From Theorems 2 and 3, we obtain a pair of isomorphic exact sequences

$$\begin{array}{ccccccccc} K^1(A) & \xrightarrow{\delta'} & K_t^0(A) & \xrightarrow{\eta'} & K^0(A) & \xrightarrow{\theta'} & K^0(A^*) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ K^1(D) & \xrightarrow[\delta]{} & K_t^0(D) & \xrightarrow[\eta]{} & K^0(\mathfrak{o}) & \xrightarrow[\theta]{} & K^0(D) & \to & 0, \end{array}$$

in which each vertical arrow is an isomorphism. Therefore $\ker\theta' \cong \ker\theta$, that is, $C(A) \cong C(\mathfrak{o})$. Furthermore,

$$C(\mathfrak{o}) \cong \ker\theta = \operatorname{im}\eta \cong K_t^0(\mathfrak{o})/\operatorname{im}\delta \cong I(R)/J(R),$$

which gives the desired result.

As shown in [12] and [13], $C(\mathfrak{o})$ is always a finite group. The group $J(R)$ is an analogue of the group of principal ideals, and $I(R)/J(R)$ is an analogue of the ideal class group of R. Indeed, when $D = F$ then $\mathfrak{o} = R$, and in this case the quotient $I(R)/J(R)$ is precisely the ideal class group of R.

4. Grothendieck groups of group rings

Let G be a finite group, F an algebraic number field, and $R = \text{alg. int. } \{F\}$ Set $A = RG$, $A^* = FG$, and let $\mathfrak{O}$ be any maximal R-order of A^* which contains A. By restriction of operators, each $\mathfrak{O}$-module becomes an A-module, and R-torsion is preserved. Using Theorem 2, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} K^1(A^*) & \xrightarrow{\delta'} & K_t^0(\mathfrak{O}) & \xrightarrow{\eta'} & K^0(\mathfrak{O}) & \xrightarrow{\theta'} & K^0(A^*) & \to & 0 \\ \downarrow 1 & & \downarrow \beta & & \downarrow \alpha & & \downarrow 1 & & \\ K^1(A^*) & \xrightarrow[\delta]{} & K_t^0(A) & \xrightarrow[\eta']{} & K^0(A) & \xrightarrow[\theta]{} & K^0(A^*) & \to & 0. \end{array}$$

In [13] Swan proved the difficult result that α is an epimorphism. Applying the "Five Lemma" to the above diagram, we conclude that also β is an epimorphism, and therefore $\ker\theta = \operatorname{im}\eta\beta$. Next, we note that $K^0(A^*)$ is a free Z-module, and therefore

$$K^0(A) = K^0(A^*) \oplus \ker\theta$$

as additive groups. Furthermore,

$$\ker \theta = \operatorname{im} \eta\beta \cong K_t^0(\mathfrak{O})/\ker \eta\beta.$$

Routine diagram-chasing yields

$$\ker \eta\beta = \ker \beta + \operatorname{im} \delta'$$

and consequently

$$K^0(A) \cong K^0(A^*) \quad \oplus \quad K_t^0(\mathfrak{O})/(\ker \beta + \operatorname{im} \delta').$$

Let $A^* = A_1^* \oplus \cdots \oplus A_n^*$ be the decomposition of A^* into simple rings A_i^*, and let M_i^* be an irreducible A_i^*-module. Set

$$D_i = \operatorname{Hom}_{A_i^*}(M_i^*, M_i^*),$$

so that D_i is a division algebra over F, and A_i^* is a full matrix algebra over D_i. Of course, $K^0(A^*)$ is the free Z-module with Z-basis $[M_1^*], \cdots, [M_n^*]$. Furthermore,

$$K^1(A^*) \cong \textstyle\sum_i K^1(A_i^*) \cong \prod_i D_i^\#,$$

the latter isomorphism determined as in §2.

Let F_i denote the center of D_i, and let $R_i = \text{alg. int.}\{F_i\}$. Each field F_i is then a finite extension of F, and each A_i^* is a central simple algebra over F_i.

Since $\mathfrak{O}$ is a maximal order, we may write

$$\mathfrak{O} = \mathfrak{O}_1 \oplus \cdots \oplus \mathfrak{O}_n,$$

where each $\mathfrak{O}_i$ is a maximal R-order in A_i^*. However R_i is finitely generated over R, and thus $\mathfrak{O}_i$ is also a maximal R_i-order in A_i^*. We may therefore apply the results of the preceding section.

To begin with, we deduce that for each i, there exists a maximal R_i-order $\mathfrak{o}_i$ in D_i, and a finitely generated projective right $\mathfrak{o}_i$-module M_i, such that $F_i M_i = M_i^*$, and

$$\mathfrak{O}_i = \operatorname{Hom}_{\mathfrak{o}_i}(M_i, M_i), \qquad \mathfrak{o}_i = \operatorname{Hom}_{\mathfrak{O}_i}(M_i, M_i).$$

Clearly $FM_i = F_i M_i = M_i^*$. The isomorphism between the categories of left $\mathfrak{o}_i$-modules and left $\mathfrak{O}_i$-modules is given by $X \to M_i \otimes_{\mathfrak{o}_i} X$, where X ranges over all left $\mathfrak{o}_i$-modules.

Next we have

$$K^0(\mathfrak{O}) \cong \textstyle\sum_i K^0(\mathfrak{O}_i) \cong \sum_i K^0(\mathfrak{o}_i),$$

and

$$K_t^0(\mathfrak{O}) \cong \textstyle\sum_i K_t^0(\mathfrak{O}_i) \cong \sum_i K_t^0(\mathfrak{o}_i).$$

Furthermore, R-torsion and R_i-torsion are equivalent concepts, and we need not distinguish between them. The results of §3 are thus directly applicable, and we deduce that

$$K_t^0(\mathfrak{O}_i) \cong K_t^0(\mathfrak{o}_i) \cong I(R_i),$$

with the isomorphisms given as in §3.

The map $\delta' : K^1(A^*) \to K_t^0(\mathfrak{O})$ induces maps $\delta_i' : K^1(A_i^*) \to I(R_i)$, and we have seen that the image of δ_i' is precisely $J(R_i)$.

Our next task is the consideration of the epimorphism $\beta : K_t^0(\mathfrak{O}) \to K_t^0(A)$. For each prime ideal P of R, β maps $K^0(\mathfrak{O}/P\mathfrak{O})$ onto $K^0(A/PA)$. Calling this map β_P, we have

$$\beta = \sum_P \beta_P, \qquad \ker \beta = \sum_P \ker \beta_P.$$

Let us show at once that β_P is an isomorphism whenever $P \nmid g$, where g is the order of G. For suppose that g is a unit in R_P, the localization of R at P. As shown in [13, Lemma 5.1], there are inclusions

$$A \subset \mathfrak{O} \subset g^{-1}A.$$

Therefore $A_P = \mathfrak{O}_P$, and so

$$A/PA \cong A_P/PA_P \cong \mathfrak{O}_P/P\mathfrak{O}_P \cong \mathfrak{O}/P\mathfrak{O}.$$

This implies that β_P is an isomorphism, as claimed. We have thus shown that

$$\ker \beta = \sum_{P|g} \ker \beta_P.$$

In order to investigate the map $\beta_P : K^0(\mathfrak{O}/P\mathfrak{O}) \to K^0(A/PA)$ for an arbitrary prime ideal P of R, we shall make use of the fact that

$$K^0(\mathfrak{O}/P\mathfrak{O}) \cong \sum_{i=1}^n K^0(\mathfrak{O}_i/P\mathfrak{O}_i).$$

Now we have seen that $I(R_i) \cong K_t^0(\mathfrak{O}_i)$, and in this isomorphism an element J of $I(R_i)$ maps onto an element of $K^0(\mathfrak{O}_i/P\mathfrak{O}_i)$ if and only if J is expressible as a product of powers of prime ideals of R_i which divide P. Let us denote by $I^{(P)}(R_i)$ the subgroup of $I(R_i)$ consisting of all such ideals J; then we have

$$I^{(P)}(R_i) \cong K^0(\mathfrak{O}_i/P\mathfrak{O}_i).$$

Let us specify this isomorphism explicitly. For a fixed prime ideal P of R, let P_{ij} range over the prime ideals of R_i which contain P. Then each P_{ij} is given by $P_{ij} = R_i \cap \mathfrak{p}_{ij}$ for some uniquely determined maximal two-sided ideal $\mathfrak{p}_{ij}$ of $\mathfrak{o}_i$. Let $V(\mathfrak{p}_{ij})$ denote an irreducible module over the simple ring $\mathfrak{o}_i/\mathfrak{p}_{ij}$. Then in the isomorphism $I(R_i) \cong K_t^0(\mathfrak{o}_i)$, the ideal P_{ij} maps onto $[V(\mathfrak{p}_{ij})]$. In the isomorphism $K_t^0(\mathfrak{o}_i) \cong K_t^0(\mathfrak{O}_i)$, the latter symbol $[V(\mathfrak{p}_{ij})]$ is mapped onto $[M_i \otimes_{\mathfrak{o}_i} V(\mathfrak{p}_{ij})]$. Summarizing our results, we have

$$\prod_{i=1}^n I^{(P)}(R_i) \cong K^0(\mathfrak{O}/P\mathfrak{O}), \tag{4.1}$$

with

$$P_{ij} \to [M_i \otimes_{\mathfrak{o}_i} V(\mathfrak{p}_{ij})], \qquad i \leq i \leq n,$$

where P_{ij} ranges over the prime ideals of R_i which divide P.

We have seen in §3 that the $(\mathfrak{o}_i/\mathfrak{p}_{ij})$-module $\mathfrak{o}_i/P_{ij}\,\mathfrak{o}_i$ has n_i composition factors $V(\mathfrak{p}_{ij})$, where $n_i^2 = (D_i{:}F_i)$. Hence

$$n_i[M_i \otimes_{\mathfrak{o}_i} V(\mathfrak{p}_{ij})] = [M_i \otimes_{\mathfrak{o}_i} (\mathfrak{o}_i/P_{ij}\,\mathfrak{o}_i)] = [M_i/P_{ij}\,M_i].$$

Thus, the isomorphism (4.1) is given by

$$\prod_{i,j} P_{ij}^{a_{ij}} \to \sum_{i,j} a_{ij} n_i^{-1}[M_i/P_{ij} M_i].$$

Now each M_i is an $\mathfrak{O}_i$-module, hence is an $\mathfrak{O}$-module annihilated by $\{\mathfrak{O}_l : 1 \leq l \leq n,\ l \neq i\}$. Then each $M_i/P_{ij} M_i$ is an $(\mathfrak{O}/P\mathfrak{O})$-module, hence by restriction of operators is also an (A/PA)-module. We may therefore conclude that the additive group $K^0(\mathfrak{O}/P\mathfrak{O})$ has Z-basis

$$\{n_i^{-1}[M_i/P_{ij} M_i] : P_{ij} \supset P,\ 1 \leq i \leq n\},$$

and the map β_P is obtained by viewing each $M_i/P_{ij} M_i$ as (A/PA)-module.

For fixed P, suppose that $\{Y_1, \cdots, Y_s\}$ is a full set of irreducible (A/PA)-modules. Then for each prime ideal P_{ij} of R_i which divides P, we may write

$$[M_i/P_{ij} M_i] = \sum_{k=1}^{s} d_{ij}^{(k)}[Y_k] \in K^0(A/PA),$$

where the $\{d_{ij}^{(k)}\}$ are non-negative integers. These integers may be regarded as a generalization of the decomposition numbers which occur in the theory of modular group representations. In terms of these $\{d_{ij}^{(k)}\}$, we have

$$\prod_{i=1}^{n} I^{(P)}(R_i) \cong K^0(\mathfrak{O}/P\mathfrak{O}) \xrightarrow{\beta_P} K^0(A/PA),$$

with

$$\prod_{i,j} P_{ij}^{a_{ij}} \to \sum_{i,j,k} a_{ij} n_i^{-1} d_{ij}^{(k)}[Y_k] \in K^0(A/PA).$$

Since β is an epimorphism, so is each map β_P.

In the special case where F is a splitting field for G, great simplifications occur. For each i, $1 \leq i \leq n$, the division algebra D_i coincides with F, and then also $F_i = F$. Furthermore, $\mathfrak{o}_i = R_i = R$, and each $n_i = 1$. Each $\mathfrak{O}_i$-module M_i is also an A-module, and $FM_i = M_i^*$, where $M_1^*, \cdots, M_n^*$ are a full set of irreducible A^*-modules. Then each P_{ij} coincides with P, and

$$[M_i/M_i P] = \sum_{k=1}^{s} d_i^{(k)}[Y_k] \in K^0(A/PA),$$

where the $\{d_i^{(k)}\}$ are now the ordinary decomposition numbers. The map β_P is then determined by

$$(P^{a_1}, \cdots, P^{a_n}) \to \sum_{i,k} a_i d_i^{(k)}[Y_k].$$

The statement that β_P is an epimorphism is easily seen to be equivalent to Brauer's Theorem 1.

Collecting our results in the general case, we have thus established the following theorem:

Let G be a finite group, F an algebraic number field, R = alg. int. $\{F\}$, and set $A = RG$, $A^* = FG$. Write $A^* = \sum_{i=1}^{n} A_i^*$, where A_i^* is isomorphic to a full matrix algebra over a division algebra D_i with center F_i; set $n_i^2 = (D_i:F_i)$. Define R_i = alg. int. $\{F_i\}$, and let $I(R_i)$ denote the multiplicative group of R_i-ideals in F_i. For each i let U_i be the divisor of R_i consisting of all infinite primes of R_i at which D_i is ramified. Set

$$J(R_i) = \{xR_i : x \in F_i, x > 0 \text{ at each prime in } U_i\}.$$

Choose any maximal R-order $\mathfrak{O}$ in A^* containing A, and write $\mathfrak{O} = \sum_{i=1}^n \mathfrak{O}_i$, with each $\mathfrak{O}_i$ a maximal R_i-order in A_i^*. For each i, there exists a maximal R_i-order $\mathfrak{o}_i$ in D_i, and a projective right $\mathfrak{o}_i$-module M_i, such that $\mathfrak{O}_i = \operatorname{Hom}_{\mathfrak{o}_i}(M_i, M_i)$. The modules $FM_1, \cdots, FM_n$ form a full set of irreducible A^*-modules.

For P a fixed prime ideal of R, define $I^{(P)}(R_i)$ as the subgroup of $I(R_i)$ generated by the prime ideals P_{ij} of R_i which contain P. Each $M_i/P_{ij}M_i$ may be viewed as an (A/PA)-module, and there is an epimorphism

$$\beta_P : \prod_{i=1}^n I^{(P)}(R_i) \to K^0(A/PA)$$

given by

$$\beta_P : \prod_{i,j} P_{ij}^{a_{ij}} \to \sum_{i,j} a_{ij} n_i^{-1}[M_i/P_{ij}M_i] \in K_0(A/PA).$$

The map β_P may be regarded as a generalization of the decomposition map, and is an isomorphism when P does not divide the order of G.

The additive structure of the Grothendieck group $K^0(A)$ is given by

$$K^0(A) \cong K^0(A^*) \oplus \frac{\prod_{i=1}^n I(R_i)}{\left\{\prod_{i=1}^n J(R_i)\right\}\left\{\prod_{P\mid[G:1]} \ker \beta_P\right\}}.$$

The Grothendieck group $K^0(A^*)$ is a free Z-module on the n generators $[M_1^*], \cdots, [M_n^*]$. The second summand on the right hand side is a finite abelian group, written multiplicatively, the determination of which depends on the ideal theory of each of the rings R_i, as well as the knowledge of the maps β_P.

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