NON-ERGODIC TRANSFORMATIONS WITH DISCRETE SPECTRUM

BY

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1. Let $(X, \mathbf{M}, m)$ be a totally finite, separable, non-atomic measure space, and T an invertible measure-preserving transformation on this space. We shall only be concerned with measure-preserving transformations modulo sets of measure zero, that is, effectively, with measure-preserving automorphisms of the measure algebra of $(X, \mathbf{M}, m)$. There is therefore no loss of generality in assuming that X is the unit interval, $\mathbf{M}$ the class of Lebesgue measurable sets, and m Lebesgue measure (cf. [3, pages 171–174], [4, pages 42–44]). Further, since every automorphism of the measure algebra of the unit interval is induced by a measure-preserving point transformation of the unit interval [12, pages 582–584] we shall speak of "transformations" when, in fact, we mean automorphisms.

The transformation (automorphism) T defines a unitary operator U on $L^2(X, \mathbf{M}, m)$ given by $(Uf)(x) = f(Tx)$ for $f \in L^2(X, \mathbf{M}, m)$. Let T and S be transformations with corresponding unitary operators U and V. T and S are said to be *conjugate* if there exists an invertible measure-preserving transformation (automorphism) P such that $S = P^{-1}TP$. They are said to be *equivalent* if U and V are unitarily equivalent. Conjugacy obviously implies equivalence. The problem of deciding when the converse of this statement holds is one of the most important in ergodic theory. For ergodic transformations with discrete spectrum (i.e. those for which there exists a basis of $L^2(m)$ consisting of eigen functions of the induced unitary operator U) von Neumann [11] showed, in 1932, that equivalence implies conjugacy. That this is false for arbitrary transformations was proved by Halmos and von Neumann (see [4, pages 57–60]). A counterexample to the converse problem in the case of transformations with continuous spectrum was given by Kolmogorov in 1958, thus solving one of the famous problems in ergodic theory (see [9] and also Halmos [4] and [5], Rohlin [14]).

For ergodic transformations with discrete spectrum, very complete results have been obtained. The spectrum forms a group (a countable subgroup of the circle group) and every eigenvalue is simple. Further, to every countable subgroup of the circle group there corresponds an ergodic measure-preserving transformation with discrete spectrum, with the given group as spectrum, which is uniquely determined up to conjugacy. There is thus a (1-1) correspondence between countable subgroups of the circle group and conjugacy classes of ergodic measure-preserving transformations with discrete spectrum. See [11, pages 624–631], [7, pages 346–348] and [4, pages 46–50].

A non-ergodic measure-preserving transformation is, in a certain sense,

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decomposable into ergodic pieces, ([1], [6], [11] and [13]). It is assumed in Halmos-von Neumann [7] that, because of this theorem, the conjugacy problem for non-ergodic transformations with discrete spectrum can be resolved using the results for ergodic transformations. It is explicitly stated in [7, page 346] that equivalence implies conjugacy for *all* transformations with discrete spectrum. In this paper we show that this is almost never the case, i.e. with certain very few exceptions, which we at least partly classify, given a non-ergodic transformation with discrete spectrum, there exists an equivalent non-conjugate transformation.

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2. The definition of conjugacy that we shall use is a little more general than the one given in §1. To motivate this definition we begin with an example.

Let G be an arbitrary countable subgroup of the circle group. Let T_1, T_2, S_1, S_2 be ergodic measure-preserving transformations on $[0, \frac{1}{2})$, $[\frac{1}{2}, 1]$, $[0, \frac{1}{4})$, $[\frac{1}{4}, 1]$ respectively such that the unitary operator of each of them has discrete spectrum G. Let

$$T = T_1 \quad \text{on} \quad [0, \tfrac{1}{2}), \qquad T = T_2 \quad \text{on} \quad [\tfrac{1}{2}, 1];$$

$$S = S_1 \quad \text{on} \quad [0, \tfrac{1}{4}), \qquad S = S_2 \quad \text{on} \quad [\tfrac{1}{4}, 1].$$

Then T and S are equivalent, but are trivially not conjugate. However, as measure-preserving transformations, T and S have a very similar structure; they have isomorphic invariant subalgebras and on the atoms of the invariant subalgebras, their respective restrictions are conjugate but for a change of scale. (This example is mentioned in [14, §1.4].) It is clear that our definition of conjugacy is not wide enough and we therefore introduce the following

Definition. T and S are said to be *conjugate by partition* or *weakly conjugate* if there exist two partitions, $\{A_i\}$ and $\{B_i\}$, $i = 1, 2, \cdots$, of X into disjoint measurable sets of positive measure, such that

(i) the A_i are invariant under T and the B_i under S;

(ii) there exists an invertible measurable transformation P_i of A_i onto B_i for which, for each $C \subseteq A_i$,

$$m(P_i\, C) = \frac{m(B_i)}{m(A_i)}\, m(C),$$

(such a transformation always exists, see Lemma 3 below), and $S \mid B_i$ is conjugate to $P_i(T \mid A_i)P_i^{-1}$.

It is obvious that in the above example T and S are weakly conjugate. If two transformations are conjugate, they are trivially weakly conjugate. If they are weakly conjugate, they are trivially equivalent. *From now on the word "conjugate" will always mean "weakly conjugate".* For any set A invariant under T, we shall write T_A for $T \mid A$. Further, for any transformation T, we shall denote also the induced unitary operator by T. We now prove some preliminary results.

LEMMA 1. *If T has discrete spectrum and A is an invariant subset of X, then T_A has discrete spectrum.*

Proof. If χ_A is the characteristic function of A, χ_A is an eigen function of T and of T_A. If f is any eigen function of T, $\chi_A f$ is an eigen function of T and of T_A. If $g \in L^2(A)$ then g can be extended to a function $\tilde{g}$ on X by putting $\tilde{g}(x) = 0$ for $x \notin A$. Then $\tilde{g} \in L^2(X)$, so that $\tilde{g} = \sum c_i f_i$, where the f_i are eigen functions of T. So $g = \tilde{g}\chi_A = \sum c_i f_i \chi_A$, the eigenfunctions $f_i \chi_A$ of T_A span $L^2(A)$, that is T_A has discrete spectrum.

LEMMA 2. *If T has discrete spectrum and A is invariant, T_A is the identity transformation if and only if T_A has no eigenvalues other than* 1.

Proof. The "only if" part is clear. To prove the converse, note that by Lemma 1, T_A has discrete spectrum, so $f(T_A x) = f(x)$ for every $f \in L^2(A)$. Hence T_A is the identity.

LEMMA 3. *If A and B are any two measurable subsets of X such that $m(B) = \alpha m(A)$, $\alpha > 0$, then there exists an invertible measurable transformation P of A onto B, such that $m(PC) = \alpha m(C)$ for $C \subseteq A$.*

Proof. For $\alpha = 1$, the result is proved in [4, page 74]. Hence there exist S_A, S_B, invertible measure-preserving, mapping A onto $[0, m(A)]$, and $[0, m(B)]$ onto B respectively. Let S_α denote the transformation $x \rightarrow x\alpha$ on $[0, m(A)]$. Put $P = S_B S_\alpha S_A$.

LEMMA 4. *Given any measure-preserving transformation T on X, there exists a decomposition of X into three invariant sets $X_a(T)$, $X_i(T)$ and $X_s(T)$ such that*

(i) *T_{X_a} has a purely atomic invariant subalgebra, i.e. $X_a = \bigcup_{i=1}^{\infty} A_i$ where the A_i (finite or countable in number) are invariant and of positive measure, and T_{A_i} is ergodic. We shall refer to the A_i as the ergodic sets of T.*

(ii) *T_{X_i} is the identity transformation on X_i.*

(iii) *T_{X_s} is not the identity on any set of positive measure, and has a purely non-atomic invariant subalgebra, which is a proper subalgebra of the algebra of measurable subsets of X_s.*

This decomposition is unique modulo null sets. If one of the sets X_i, X_a or X_s is null, the same is true for every transformation conjugate to T.

Proof. Since the measure algebra of X is complete, there is a maximal set X_i (modulo null sets) on which T is the identity. Consider the subalgebra of invariant sets of T_{X-X_i}. It decomposes into a purely atomic part and a non-atomic part, uniquely modulo null sets. Let X_a be the union of the atoms of the atomic part, and put $X_s = X - (X_i \cup X_a)$. The uniqueness of the decomposition is immediate.

The last statement of the lemma is obvious from the definition of conjugacy.

If λ is an eigenvalue of T we write $\text{mult}_T(\lambda)$ or just mult (λ) for the multiplicity of λ, i.e. the dimension of the eigen space of λ. If $\mathcal{A}$ is a subalgebra of $\mathbf{M}$ we write $\mathfrak{N}_{\mathcal{A}}$ for the σ-ideal of null sets in $\mathcal{A}$.

LEMMA 5. *If A is invariant under T, and λ is an eigenvalue of T, then λ is an eigenvalue of at least one of T_A and T_{X-A} and*

$$\text{mult}_T(\lambda) = \text{mult}_{T_A}(\lambda) + \text{mult}_{T_{X-A}}(\lambda).$$

Proof. If f_λ is an eigen function with eigenvalue λ, then

$$f_\lambda = \chi_A f_\lambda + \chi_{X-A} f_\lambda .$$

Further $\chi_A f_\lambda$ is either null or is an eigen function with eigenvalue λ. The same is true of $\chi_{X-A} f_\lambda$, and if both are not null, they are orthogonal, and so linearly independent. Further $\chi_A f_\lambda$ is an eigen function of T_A, $\chi_{X-A} f_\lambda$ of T_{X-A}. Any f and g with supports in A and $X - A$ respectively are orthogonal. Thus if ϕ_i is a basis of the eigen space of λ relative to T_A, ψ_j of the eigen space relative to T_{X-A}, the set of functions ϕ_i, ψ_j (extended to X by putting $\phi_i = 0$ on $X - A$, $\psi_j = 0$ on A) together give a basis of the eigen space relative to T. It follows that

$$\text{mult}_T(\lambda) = \text{mult}_{T_A}(\lambda) + \text{mult}_{T_{X-A}}(\lambda).$$

LEMMA 6. *If $\mathcal{I}$ is the invariant subalgebra of T then*

$$(*) \quad \text{mult}_T(1) = \textit{minimal number of generators of } \mathcal{I}/\mathfrak{N}_{\mathcal{I}}$$

$$= \textit{number of atoms of } \mathcal{I}/\mathfrak{N}_{\mathcal{I}} \ (\textit{if this algebra is atomic})$$

$$= \textit{number of ergodic sets of } X_a(T) \ (\textit{if, again, } \mathcal{I}/\mathfrak{N}_{\mathcal{I}} \textit{ is atomic}).$$

(*Note.* $\mathcal{I}/\mathfrak{N}_{\mathcal{I}}$ is always atomic if the number of generators is finite.)

Proof. Both $\text{mult}_T(1)$ and the number of generators can be at most $\aleph_0$. Further, in an atomic algebra, the atoms are the unique, minimal set of generators. It is therefore enough to show that if one side of $(*)$ is finite, so is the other, and the two are equal.

If the number of generators is finite, say $A_1, \cdots, A_N$ are the generating atoms, then the characteristic functions χ_{A_i} are orthogonal and so linearly independent eigen functions with eigenvalue 1. Further, any such function, being invariant, must be constant on the atoms of the invariant subalgebra, and so must belong to the subspace spanned by the χ_{A_i}. Hence $\text{mult}_T(1) = N$.

If the number of generators is infinite, then there exist disjoint invariant sets $A_1, A_2, \cdots$ of positive measure. χ_{A_i} are orthogonal eigen functions with eigenvalue 1, so $\text{mult}_T(1) = \aleph_0$. This completes the proof.

LEMMA 7. *If λ is an eigenvalue of T, then*

$$\text{mult}_T(\lambda) \leq \text{mult}_T(1).$$

Proof. The result is trivial if $\text{mult}_T(1) = \infty$. If $\text{mult}_T(1) = N < \infty$, then by Lemma 6, X is the union of N invariants sets $A_1, \cdots, A_N$ such that T_{A_i} is ergodic. λ can be at most a simple eigenvalue of T_{A_i} (see [4, page 34]), and so by Lemma 5,

$$\text{mult}_T(\lambda) = \sum_{i=1}^{N} \text{mult}_{T_{A_i}}(\lambda) \leq N.$$

We shall consider the conjugacy problem separately for the three cases

(A) $\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} = \text{mult}_T(1) = \infty$,
(B) $\text{mult}_T(1) = \infty, \quad \sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} < \infty$,
(C) $\text{mult}_T(1) < \infty$.

We first show that in cases (B) and (C), the set $X_s(T)$ must be null.

THEOREM 1. *If T has discrete spectrum and if for each $\lambda \neq 1$ in the spectrum of T, we have* $\text{mult}_T(\lambda) < \infty$, *then $X_s(T)$ is null.*

Proof. Assume the theorem false. Then, since the invariant subalgebra of T_{X_s} is non-atomic, a standard exhaustion argument shows that there exists, in this subalgebra, a collection of sets $\{E_\alpha : 0 < \alpha \leq m(X_s)\}$ with the following properties. $E_\alpha \subseteq E_\beta$ if $\alpha < \beta$, and $m(E_\alpha) = \alpha$. Every eigenvalue of T_{E_α} is an eigenvalue of T; further, by Lemma 1, each T_{E_α} has discrete spectrum and so, by Lemma 2, has an eigenvalue different from 1. Now if f_α and f_β are eigen functions of T_{E_α} and T_{E_β} respectively, and if f_α and f_β are extended to X_s by putting $f_\alpha = 0$ on $X_s - E_\alpha$, $f_\beta = 0$ on $X_s - E_\beta$ then f_α and f_β are linearly independent. So each λ in the spectrum of T can be an eigenvalue of at most a finite number of T_{E_α} and further the spectrum of T contains only a countable number of eigenvalues. Hence the total number of E_α is countable which is a contradiction.

Note. The hypothesis that T has discrete spectrum is only used here to show that each T_{E_α} has an eigenvalue distinct from 1. The theorem is not true without this hypothesis. Let S_1 be a mixing transformation on $[0, \frac{1}{2}]$, S_2 the same transformation translated by $\frac{1}{2}$ acting on $[\frac{1}{2}, 1]$, S the transformation on $[0, 1]$ equal to S_1 on $[0, \frac{1}{2}]$, S_2 on $[\frac{1}{2}, 1]$. Let T be the cartesian product of countably many copies of S. T has no eigenvalue other than 1, but $X_a(T)$ and $X_i(T)$ are null, $X_s(T)$ is not.

3. In this section we consider the conjugacy problem in the simplest cases. For all the transformations considered here there always exist equivalent, non-conjugate transformations. We begin with case (A) above.

THEOREM 2. *If T is a measure-preserving transformation with discrete spectrum, and if*

$$\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} = \text{mult}_T(1) = \infty,$$

then there exists a transformation S equivalent to, but not conjugate to, T.

Proof. By Lemma 4, X can be decomposed into $X_a(T)$, $X_i(T)$ and $X_s(T)$. We first suppose that $X_i(T)$ is null. Let A be any subset of X of positive measure (e.g. $[0, \frac{1}{2}]$). By Lemma 3 there exists an invertible transformation P of X onto A such that $m(PC) = m(A)m(C)$ for all $C \subseteq X$. Put $S_1 = PTP^{-1}$. S_1 is an invertible measure-preserving transformation of A onto itself, and is unitarily equivalent to T. Put $Sx = S_1 x$ for $x \in A$, $Sx = x$ for $x \notin A$. S has discrete spectrum. S_{X-A} has no eigenvalue other than 1, which has infinite multiplicity. But $S_A = S_1$, so S_A is equivalent to T. Hence $\text{mult}_S(\lambda) = \text{mult}_{S_1}(\lambda) = \text{mult}_T(\lambda)$ if $\lambda \neq 1$, and $\text{mult}_S(1) = \infty = \text{mult}_T(1)$. Hence S and T are unitarily equivalent, but they are clearly not conjugate, for $X_i(T)$ is null, but $X_i(S)$ is not.

We now consider the other case when $X_i(T)$ is not null. Put $B = X - X_i(T)$. Since $T_{X_i(T)}$ has no eigenvalues other than 1, it follows by Lemma 5 that for $\lambda \neq 1$, $\text{mult}_{T_B}(\lambda) = \text{mult}_T(\lambda)$ and hence by Lemma 7 that

$$\text{mult}_{T_B}(1) \geq \sup \{\text{mult}_{T_B}(\lambda) : \lambda \neq 1\} = \sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} = \infty.$$

So T and T_B are unitarily equivalent. By Lemma 3 there exists an invertible transformation Q of B onto X such that

$$m(QC) = \frac{1}{m(B)} m(C)$$

for $C \subseteq B$. Put $S = QT_B Q^{-1}$; then S and T_B are unitarily equivalent, so S and T are unitarily equivalent. But S and T are not conjugate as $X_i(S)$ is null, but $X_i(T)$ is not. This completes the proof of Theorem 2.

In some examples of case (B), the same method can be applied.

THEOREM 3. *If T is a measure-preserving transformation with discrete spectrum, if* $\text{mult}_T(1) = \infty$, *and $X_i(T)$ is null then there exists a transformation S equivalent to but not conjugate to T.*

Proof. The proof of Theorem 2 for the case when $X_i(T)$ is null does not make use of the hypothesis that $\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} = \infty$.

Since transformations of type (A) have been dealt with in Theorem 2, and since by Theorem 1 all transformations of types (B) and (C) have X_s null, it follows that the only transformations T, which remain for us to consider, are those in which $X = X_i \cup X_a$, where T_{X_i} is the identity, and X_a is the union of a finite or countable sequence of ergodic sets A_i, i.e. invariant sets A_i such that T_{A_i} is ergodic (with discrete spectrum). The eigenvalues of T_{A_i} are all simple, and form a group which will henceforth be denoted by G_i. The case when $X_a(T)$ has a countable number of atoms has already been dealt with implicity.

THEOREM 4. *If T is a measure-preserving transformation with discrete spectrum such that $X_s(T)$ is null and $X_a(T)$ has a countably infinite number of ergodic sets, then there exists a transformation S equivalent to but not conjugate to T.*

Proof. By Lemma 6, $\text{mult}_T(1) = \infty$. If $X_i(T)$ is null, the result follows by Theorem 3. If $X_i(T)$ is not null, we note that $X_a = X - X_i$, and $\text{mult}_{T_{X_a}}(1) = \infty$; we can now proceed as in the second half of the proof of Theorem 2, where we only used the fact that $\text{mult}_{T_{X-X_i}}(1) = \infty$.

4. The only cases which therefore remain are those in which $X_s(T)$ is null, and $X_a(T)$ has a finite number of ergodic sets $A_1, \cdots, A_N$. The transformations for which $X_i(T)$ is not null come under case (B), those for which $X_i(T)$ is null under case (C). For these cases the crude methods of §3 do not suffice. More delicate group-theoretic methods are needed; in fact all our subsequent proofs depend solely on properties of the groups G_i, the spectra of the ergodic transformations T_{A_i}. We therefore state certain results on infinite abelian groups which we shall require in the sequel. They may be found in Kaplansky [8] especially pages 1–12, or Kurosh [10], especially Chapter VI or Fuchs [2] especially Chapters I and III.

(a) *Every countable divisible (abelian) group is the direct sum of an (at most) countable number of groups each isomorphic either to the additive group of rational numbers R or to some p^∞ group (where p is a prime).*

(b) *Every abelian group can be embedded in a divisible group.*

(c) *The rank of an abelian group is the maximal number (possibly infinite) of elements which are independent over the integers* (see [10, Chapter VI §19] or [2, Chapter I, §8]).

It is possible that a transformation T for which $X_i(T)$ is not null and $X_a(T)$ has a finite number of ergodic sets is equivalent to (but not conjugate to !) a transformation S for which $X_a(S)$ has an infinite number of ergodic sets. We shall first therefore completely classify those T for which this happens, and then confine our attention solely to those transformations for which all equivalent transformations have only a finite number of ergodic sets. We note first that two transformations S and T, of the type we are now considering, are conjugate if and only if (I) $X_i(T)$ and $X_i(S)$ are either both null or both not null, (II) there is a (1-1) correspondence between the ergodic sets A_j of T and B_j of S such that the groups of T_{A_j} and S_{B_j} are *identical.*

LEMMA 8. *A direct sum of N copies of the additive group of rationals, $(2 \le N \le \aleph_0)$, $R_1 \oplus R_2 \oplus \cdots$ is the union of a countably infinite sequence of subgroups $Q_1, Q_2, \cdots$ such that $Q_i \cap Q_j = \{0\}$ for $i \neq j$, and each Q_i is of rank* 1 *(in fact isomorphic to the rationals).*

Proof. A typical element of $R_1 \oplus R_2 \oplus \cdots$ may be written in the form $(r_{i_1}, r_{i_2}, \cdots, r_{i_n})$ where $r_{i_j} \neq 0$ and $r_{i_j} \in R_{i_j}$. Consider the set of all elements $(s_{i_1}, s_{i_2}, \cdots, s_{i_n})$ such that $s_{i_1}/r_{i_1} = s_{i_2}/r_{i_2} = \cdots = s_{i_n}/r_{i_n}$, (i.e., if $R_1 \oplus R_2 \oplus \cdots$ is considered as a vector space over R, this is the line joining the origin to $(r_{i_1}, \cdots, r_{i_n})$). Such elements form a group isomorphic to the rationals. Moreover no non-zero element of $R_1 \oplus R_2 \oplus \cdots$ belongs to more than one of these groups. Since $R_1 \oplus R_2 \oplus \cdots$ is countable the number of such groups is countable. These form the required groups Q_i.

LEMMA 9. *Any countable torsion-free abelian group H of rank ≥ 2 (possibly infinite), is the union of a countably infinite sequence of subgroups $H_1, H_2, \cdots$ such that $H_i \cap H_j = \{0\}$ for $i \neq j$, and each H_i is of rank 1.*

Proof. By (a) and (b) quoted above, H can be embedded in a direct sum of a countable number of groups each isomorphic to the additive group of rationals. Now by Lemma 8, such a direct sum is the union of groups Q_i of rank 1, $i = 1, 2, \cdots$ such that $Q_i \cap Q_j = \{0\}$ if $i \neq j$. So $H = \bigcup_{i=1}^{\infty} H \cap Q_i$. Now $H \not\subseteq Q_i$ for any i, as Q_i is of rank 1 and H is of rank ≥ 2. Further

$$H \neq \bigcup_{r=1}^{N} H \cap Q_{i_r}$$

for any finite number of groups $Q_{i_1}, \cdots, Q_{i_r}$. For suppose g_1, g_2 are non-zero, $g_1 \in H \cap Q_{i_1}$, $g_2 \in H \cap Q_{i_2}$. Since g_1 and g_2 are of infinite order, the elements $kg_1 + g_2$ ($k = 1, 2, \cdots$) must all belong to different groups $H \cap Q_{i_r}$.

Putting $H_i = H \cap Q_i$, we get the required result.

THEOREM 5. *Let T be a measure-preserving transformation with discrete spectrum satisfying the following conditions*: (i) $\text{mult}_T(1) = \infty$,

$$\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} < \infty,$$

(ii) *$X_a(T)$ has a finite number of ergodic sets $A_1, \cdots, A_N$, and* (iii) *for at least one n, the spectrum of T_{A_n} is a torsion-free group of rank ≥ 2. Then, there exists a transformation S for which $X_a(S)$ has an infinite number of ergodic sets, and which is equivalent to, but not conjugate to, T.*

Proof. Let G_i denote the spectrum of T_{A_i}, $i = 1, \cdots, N$. We may suppose that (iii) is satisfied for $n = N$. By Lemma 9, $G_N = \bigcup_{j=1}^{\infty} H_j$, where $H_j \cap H_k = \{1\}$ for $j \neq k$. (We write the groups multiplicatively.)

Let $B, B_1, B_2, \cdots$ be an arbitrary partition of X into sets of positive measure. Let S_0 be the identity on B; S_i an ergodic transformation on B_i with discrete spectrum G_i for $i = 1, \cdots, N - 1$; S_j' an ergodic transformation with discrete spectrum H_j on B_{N+j-1} for $j = 1, 2, 3, \cdots$. Define $S = S_0$ on B, $S = S_i$ on B_i, $1 \leq i \leq N - 1$, $S = S_j'$ on B_{N+j-1}, $j \geq 1$. Then $X_a(S)$ has an infinite number of ergodic sets $B_1, B_2, \cdots$; S is equivalent to T as it has the same eigenvalues with the same multiplicities, but it is obviously not conjugate to T.

We shall show later (Theorem 7) that the condition that at least one of the G_i be torsion-free of rank ≥ 2 is also necessary for the conclusion of Theorem 5. We first prove some useful but elementary lemmas and incidentally show that equivalence implies conjugacy if all the groups $G_1, \cdots, G_N$ are torsion groups.

LEMMA 10. *Any finitely generated subgroup of the additive group of rationals is cyclic.*

The proof is both simple and well known.

If T is a transformation with $X_s(T)$ null, then the *joint multiplicity relative to* T of any r eigenvalues $\lambda_1, \cdots, \lambda_r$ of T is the number of groups G_i associated with T such that $\lambda_j \in G_i$ for all $1 \leq j \leq r$.

LEMMA 11. (a) *Let* $G_1, \cdots, G_N$, *and* $H_1, \cdots, H_M$ *be two sets of groups such that* (i) $\bigcup_{i=1}^{N} G_i = \bigcup_{j=1}^{M} H_j$ *and* (ii) *for any set of* r *elements* $a_1, \cdots, a_r$, *the number of groups* G_i, *such that* $a_k \in G_i$ *for all* $1 \leq k \leq r$ *is the same as the number of groups* H_j *such that* $a_k \in H_j$ *for all* $1 \leq k \leq r$; *then* $N = M$ *and the* G_i *and the* H_j *are pairwise identical.*

(b) *Let* T *and* S *be equivalent transformations with discrete spectrum such that* (i) $\sup \{\text{mult}(\lambda) : \lambda \neq 1\} < \infty$; (ii) $X_a(T)$ *has a finite number of ergodic sets* $A_1, \cdots, A_N$; (iii) $X_a(S)$ *has a finite number of ergodic sets* $B_1, \cdots, B_M$; (iv) *the spectrum of* T_{A_i} *is* G_i, *of* S_{B_j} *is* H_j; *and* (v) *the joint multiplicities of every finite set of elements are the same relative to* T *and* S. *Then the* G_i *and the* H_j *are pairwise identical, and* T *and* S *are conjugate.*

Proof. (a) If the G_i and H_j are not pairwise identical, then either there exists a G_i, say G_1, such that $G_1 \not\subseteq H_j$ for any j, or else there exists an H_j, say H_1, such that $H_1 \not\subseteq G_i$ for any i. Suppose that the first alternative holds. Then there exist $a_1, a_2, \cdots, a_M$ such that $a_j \in G_1$ for $j = 1, \cdots, M$ and $a_j \notin H_j$. The joint multiplicity of $a_1, \cdots, a_M$ is 0 for the H_j and ≥ 1 for the G_i, contradiction. Part (b) of the Lemma then follows from the conjugacy theorem for ergodic transformations.

LEMMA 12. (a) *Let* $G_1, \cdots, G_N$ *and* $H_1, \cdots, H_M$, *be torsion subgroups of the circle group such that* (i) $\bigcup_{i=1}^{N} G_i = \bigcup_{j=1}^{M} H_j$ *and* (ii) *for any element* a, *the number of groups* G_i *such that* $a \in G_i$ *is the same as the number of groups* H_j *such that* $a \in H_j$; *then given any set of* r *elements* $a_1, \cdots, a_r$, *the number of groups* G_i *such that* $a_k \in G_i$ *for all* $1 \leq k \leq r$, *is the same as the number of groups* H_j *such that* $a_k \in H_j$ *for all* $1 \leq k \leq r$. M *and* N *may be infinite.*

(b) *If* T *and* S *are equivalent and have discrete spectrum and if*

$$\sup \{\text{mult}(\lambda) : \lambda \neq 1\} < \infty;$$

if $G_1, G_2, \cdots$ *are the groups associated with the ergodic sets of* T, $H_1, H_2, \cdots$ *with those of* S; *and if further the* G_i *and* H_j *are all torsion groups, then the joint multiplicity of any finite set of elements is the same relative to* T *and to* S.

Proof. (a) The elements of G_i and H_j are all roots of unity and may be expressed as primitive roots of unity. Given any numbers

$$\omega^{(1)} = \exp(2\pi i l_1/m_1), \cdots, \omega^{(n)} = \exp(2\pi i l_n/m_n)$$

with $(l_i, m_i) = 1$, there exists by Lemma 10, a primitive root of unity $\omega = \exp(2\pi i p/q)$ such that the cyclic group generated by ω is the group generated by $\omega^{(1)}, \cdots, \omega^{(n)}$. Hence

the number of groups G_i such that $\omega^{(r)} \in G_i$ for $1 \leq r \leq n$

$$= \text{the number of groups } G_i \text{ such that } \omega \in G_i$$

$=$ the number of groups H_j such that $\omega \in H_j$

$=$ the number of groups H_j such that $\omega^{(r)} \in H_j$ for $1 \leq r \leq n$.

Part (b) of the lemma follows since the equivalence of T and S implies that G_i and H_j satisfy the conditions (i) and (ii) of part (a).

THEOREM 6. (a) *Let T be a measure-preserving transformation with discrete spectrum such that* (i) $\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} < \infty$, (ii) $X_a(T)$ *has only finitely many ergodic sets* $A_1, \cdots, A_N$, *and* (iii) *the spectrum G_i of T_{A_i} is a torsion group for each $i = 1, \cdots, N$. Then every S, equivalent to T, is conjugate to T.*

(b) *Let T be a measure-preserving transformation with discrete spectrum such that* (i) $\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} < \infty$, (ii) $X_a(T)$ *has only finitely many ergodic sets* $A_1, \cdots, A_N$, (iii) *the spectrum of T_{A_i} is G_i, $i = 1, \cdots, N$; let S be any transformation equivalent to T, with possibly an infinite number of ergodic sets, such that the groups associated with the ergodic sets are $H_1, H_2, \cdots$; then there exist $j_1, \cdots, j_N$ such that the torsion subgroups of G_i and H_{j_i} are identical for $i = 1, \cdots, N$. The remaining H_j are torsion free.*

Proof. (a) We first show that the number of ergodic sets of $X_a(S)$ must be finite. Else let $B_1, B_2, \cdots$ be the ergodic sets of S and let H_j be the spectrum of S_{B_j}. H_j is a torsion group. We rearrange the groups G_i and H_j in blocks $\mathcal{G}_r$ and $\mathcal{H}_s$ as follows: Let p_r denote the r-th prime. The r-th block $\mathcal{G}_r$ consists of all those groups G_i, which contain a p_r-th root of unity, but contain no p_t-th root for $t < r$. Similarly we define the blocks $\mathcal{H}_s$. Note that if a group contains a (pn)-th root of unity, it contains a p-th root of unity. Each block has at most N groups in it. (This is immediate for the G_i, and it follows for the H_j from the equivalence of T and S.) There are a finite number of blocks $\mathcal{G}_r$, say $\mathcal{G}_1, \cdots, \mathcal{G}_R$, but an infinite number of $\mathcal{H}_s$. So there exists $k > R$, such that $\mathcal{H}_k$ is non-empty. We shall write ω_p for a p-th root of unity. By Lemma 12, the joint multiplicity relative to T and S is the same for ω_2 and ω_{p_k}; hence ω_{p_k} belongs to the same number of groups of $\mathcal{G}_1$ as of $\mathcal{H}_1$. Again ω_{p_k} belongs to the same number of groups of $\mathcal{G}_2$ as of $\mathcal{H}_2$, this number being

$$\text{jt mult}\ (\omega_3, \omega_{p_k}) - \text{jt mult}\ (\omega_2, \omega_3, \omega_{p_k});$$

(the joint multiplicities relative to T and S again being the same). More generally for $r \leq k$, it belongs to the same number of groups of $\mathcal{G}_r$ as of $\mathcal{H}_r$ this number being

$$\begin{aligned}\text{jt mult}\ (\omega_{p_r}, \omega_{p_k}) &- \text{jt mult}\ (\omega_2, \omega_{p_r}, \omega_{p_k}) \\ &- (\text{jt mult}\ (\omega_3, \omega_{p_r}, \omega_{p_k}) - \text{jt mult}\ (\omega_2, \omega_3, \omega_{p_r}, \omega_{p_k})) \\ &- (\text{jt mult}\ (\omega_5, \omega_{p_r}, \omega_{p_k}) - \cdots,\end{aligned}$$

in fact a certain linear combination of joint multiplicities (the same relative to T as to S). Hence the number of groups to which ω_{p_k} belongs in $\bigcup_{r=1}^{R} \mathcal{G}_r$ is the same as in $\bigcup_{r=1}^{R} \mathcal{H}_r$. But $\mathcal{G}_1, \cdots, \mathcal{G}_R$ exhaust the G_i whereas there exists an $H_j \notin \bigcup_{r=1}^{R} \mathcal{H}_r$ such that $\omega_{p_k} \in H_j$. So ω_{p_k} does not have the same multiplicity relative to T as to S contradicting the equivalence of T and S. Hence $X_{\mathfrak{a}}(S)$ has only a finite number of ergodic sets. The conjugacy of S and T now follows from Lemmas 11 and 12.

The proof of (b) is almost identical with that of (a).

THEOREM 7. *Let T be a measure-preserving transformation with discrete spectrum such that* (i) $\text{mult}_T(1) = \infty$, $\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} < \infty$; (ii) *$X_{\mathfrak{a}}(T)$ has only a finite number of ergodic sets $A_1, \cdots, A_N$, the spectrum of T_{A_i} being G_i, and* (iii) *none of the G_i is both torsion free and of rank ≥ 2. Then for every transformation S equivalent to T, $X_{\mathfrak{a}}(S)$ has only a finite number of ergodic sets.*

Proof. Let $\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} = M$. Suppose that the theorem is false, i.e. that there exists an S equivalent to T such that $X_{\mathfrak{a}}(S)$ has an infinite number of ergodic sets $B_1, B_2, \cdots$. Denote by H_j the spectrum of S_{B_j} for $j = 1, 2, \cdots$. If the G_i are all torsion groups the result follows by Theorem 6(a).

We first consider the case when all the G_i are torsion free of rank 1. For some k, $G_k \cap H_j \neq \{1\}$ for infinitely many j; let $j = j_1, j_2, \cdots, j_{M+1}$ be $M + 1$ distinct values of j for which this is true. Let $\alpha \neq 1$ be an element of infinite order in G_k. Since G_k has rank 1, there is an integer r_l, with $\alpha^{r_l} \in H_{j_l}$ $(l = 1, 2, \cdots, M + 1)$. Then, if $r = \prod_{l=1}^{M+1} r_l$, α^r belongs to at least $M + 1$ of the groups H_j, i.e. $\text{mult}_S(\alpha^r) \geq M + 1$, contradiction.

Now we consider the general case. Suppose that $G_1, \cdots, G_n$ are groups with torsion, $G_{n+1}, \cdots, G_N$ are torsion free. By Theorem 6(b), there exist n values of j, say $j = 1, \cdots, n$, such that the torsion subgroups of G_i and H_i are identical for $i = 1, \cdots, n$; $H_{n+1}, H_{n+2}, \cdots$ are then torsion free. Now $G_{n+1}, \cdots, G_N$ being torsion free, are of rank 1. Hence, applying to the groups H_j, G_i with $i > n$ the argument given above in the special case of groups of rank 1, we see that there exists an integer k, which we may suppose $>n$, such that for $j \geq k$, and $n + 1 \leq i \leq N$, $G_i \cap H_j = \{1\}$. Let a be an element in H_{k+1}, with $a \neq 1$ and $\text{mult}_S(a) = r$ say. Since $\text{mult}_S(b) \leq M$ for all eigenvalues b, and since $\text{mult}_S(a^m) \geq \text{mult}_S(a)$, we may assume, by replacing a by a suitable power of a, that $\text{mult}_S(a^m) = \text{mult}_S(a)$ for $m = 2, 3, \cdots$. Now a belongs to r of the groups $G_1, \cdots, G_n$; let $G_1, \cdots, G_r$ be these groups. Let us suppose that the r groups H_j to which a belongs are $H_{j_1}, \cdots, H_{j_s}$, with $j_1, \cdots, j_s$ all $\leq k$, and $H_{k+1}, \cdots, H_{k+r-s}$. Clearly $s \leq r - 1$. Let $\tilde{G}_i$ denote the torsion subgroup of G_i, $i = 1, \cdots, r$ and $\tilde{H}_{j_i}$ denote the torsion subgroup of H_{j_i}, $i = 1, \cdots, s$.

If $t \in \tilde{G}_i$ for some i, $t \neq 1$, then ta cannot belong to any H_j other than $H_{j_1}, \cdots, H_{j_s}$. For if ta belongs to any of $H_{k+1}, \cdots, H_{k+r-s}$, then so does t, which is impossible, since these groups are torsion free. If ta belongs to any other H_j, then $a^{\text{order}(t)}$ has a multiplicity greater than that of a. Hence it can only belong to $H_{j_1}, \cdots, H_{j_s}$. Similarly if $t \in \tilde{H}_{j_i}$, then ta cannot belong to any G_j other than $G_1, \cdots, G_r$ and so t must belong to one of $\tilde{G}_1, \cdots, \tilde{G}_r$. Further, if t belongs to exactly l of $\tilde{G}_1, \cdots, \tilde{G}_r$, it belongs to exactly l of $\tilde{H}_{j_1}, \cdots, \tilde{H}_{j_s}$, for l is then the multiplicity of ta. Hence, applying Lemmas 11(a) and 12(a) to $\tilde{G}_1, \cdots, \tilde{G}_r$ and $\tilde{H}_{j_1}, \cdots, \tilde{H}_{j_s}$, we see that these groups are pairwise identical. It follows that $r = s$, which is impossible as $s \leq r - 1$. Hence the number of groups H_j must be finite.

THEOREM 8. *Let T be a measure-preserving transformation with discrete spectrum such that* (i) $\sup \{\text{mult}_T(\lambda) : \lambda \neq 1\} < \infty$, (ii) $X_a(T)$ *has a finite number of ergodic sets $A_1, \cdots, A_N$,* (iii) *the spectrum of T_{A_i} is G_i, and* (iv) *each group G_i is either a pure torsion group or is torsion free of rank* 1; *then every S equivalent to T is conjugate to T.*

Proof. The groups G_i can be divided into blocks, $\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_r$ $(r \leq N)$, $\mathcal{G}_0$ consisting of the torsion groups, and each of the other blocks $\mathcal{G}_i$ containing all groups whose elements are integrally dependent on some fixed element a_i of infinite order. Let S be equivalent to T, $X_a(S)$ have ergodic sets $B_1, \cdots, B_M$ (finite in number by Theorem 7) with spectra $H_1, \cdots, H_M$. None of the H_j can have rank greater than 1. For if a, b were integrally independent elements in H_j say, then $a^m b^n$ would have to belong to distinct G_i for all distinct coprime pairs of integers (m, n), contradicting the fact that there are only finitely many G_i. Let $G_1, \cdots, G_k$ be the torsion groups among the G_i. Then, by Theorem 6(b), there exist k groups H_j, say $H_1, \cdots, H_k$, such that the torsion subgroup of H_i is identical with G_i, $i = 1, 2, \cdots, k$. H_j will be torsion free for $j > k$. We show that $H_1, \cdots, H_k$ are themselves torsion. For suppose $a \in H_1$, a of infinite order. By replacing a by a suitable power of a, we may suppose that mult (a^m) = mult (a) for $m > 1$. Let t be a torsion element of H_1. Then $ta \in H_1$ and so $ta \in G_i$ for some $i > K$. Since this G_i is torsion free $a \notin G_i$, and so mult $(a^{\text{order}(t)}) >$ mult (a) which is a contradiction. Thus $H_1, \cdots, H_k$ are torsion and hence identical with $G_1, \cdots, G_k$. The remaining H_j $(j > k)$ are all torsion free of rank 1, and so can also be divided into blocks, each block containing all groups whose elements are integrally dependent. Elements in groups of the same block of groups G_i, belong to groups of the same block of groups H_j, and vice versa. Consequently we may assume that there is just one block of G_i, and one of H_j, i.e. that all elements of all the G_i are dependent on just one element a say. Each G_i is isomorphic to a subgroup of the additive group of rationals; it follows by Lemma 10 that the joint multiplicity of any finite number of elements $a_1, \cdots, a_n$ is the same as the multiplicity of a certain single element a_0, the generator of the group generated by $a_1, \cdots, a_n$. The result now follows from Lemma 11.

We note that Theorems 6 and 8 are true in both cases (B) and (C), i.e. whether $\text{mult}_T(1) = \infty$, or $< \infty$. In the latter case, the proofs could be simplified, as no equivalent transformation can have an infinite number of ergodic sets, in fact by Lemma 6, $\text{mult}_T(1)$ = number of ergodic sets of $X_a(T)$.

5. We conclude with some examples showing that nothing can be said in the mixed groups case, nor even in the case when $\text{mult}_T(1) < \infty$, and all the groups are torsion free of arbitrary rank. We write $\text{Gp}(a_1, \cdots, a_n)$ for the group generated by $a_1, \cdots, a_n$.

(i) Let a be any element of infinite order of the circle group, A_1, A_2, A_3 a partition of X into 3 sets of positive measure. Put

$$G_1 = \text{Gp}(a, -1), \qquad G_2 = \text{Gp}(a^2), \qquad G_3 = \text{Gp}(a^2)$$
$$H_1 = \text{Gp}(a^2, -1), \qquad H_2 = \text{Gp}(a), \qquad H_3 = \text{Gp}(-a).$$

Let T, S both have A_1, A_2, A_3 as ergodic sets, the spectrum of T_{A_i} being G_i, of S_{A_i} being H_i, $i = 1, 2, 3$. Then T and S are equivalent but not conjugate.

(ii) If, in the above example, G_3 is replaced by $G_3' = \text{Gp}(a)$, and T' is the corresponding transformation with spectrum of T'_{A_3} being G_3', then every S equivalent to T' is in fact conjugate to T'.

(iii) Let a, A_1, A_2, A_3 be as before, b integrally independent of a. Put

$$G_1 = \text{Gp}(a, b), \qquad G_2 = \text{Gp}(a^2, b^2), \qquad G_3 = \text{Gp}(a^2, b^2)$$
$$H_1 = \text{Gp}(a, b^2), \qquad H_2 = \text{Gp}(a^2, b), \qquad G_3 = \text{Gp}(ab, a^2, b^2)$$

All the above groups are torsion free. Define T and S so that the spectrum of T_{A_i} is G_i, of S_{A_i} is H_i for $i = 1, 2, 3$. Then T and S are equivalent but not conjugate.

We remark however that if $\text{mult}_T(1) = 2$, then equivalence implies conjugacy.

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