# ON THE SEMISIMPLICITY OF THE MODULAR REPRESENTATION ALGEBRA OF A FINITE GROUP

BY

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## 1. Introduction

1.1. Let $G$ be a finite group with unit element $e$ and $K$ a field of prime characteristic $p$. By a $G$-module $M$ we mean a $(K, G)$-module (elements of $G$ act on the right). Denote by dim $M$ the dimension of $M$ as a $K$-module; we shall assume dim $M$ is finite.

$\{M\}$ denotes the $(K, G)$-isomorphism class of $M$.

The modular representation algebra $A(K, G)$ is the linear algebra over the complex field $\boldsymbol{C}$ defined as follows:

The elements of $A(K, G)$ are the finite linear combinations over $\boldsymbol{C}$ of the $G$-module classes $\{M\}$, subject to the relations

$$\{M_1 + M_2\} = \{M_1\} + \{M_2\}$$

for all $G$-modules $M_1$, $M_2$. Here $M_1 + M_2$ denotes the direct sum $M_1 \oplus M_2$. Multiplication in $A(K, G)$ will be denoted by $\otimes$ and is defined by

$$\{M_1\} \otimes \{M_2\} = \{M_1 \otimes M_2\}$$

where $M_1 \otimes M_2$ is the tensor product over $K$, considered as a $G$-module by the rule $(m_1 \otimes m_2)x = m_1 x \otimes m_2 x$ $(m_1 \in M_1, m_2 \in M_2, x \in G)$.

By the Krull-Schmidt theorem for $G$-modules, $A(K, G)$ has as a basis (over $\boldsymbol{C}$) the classes of the indecomposable $G$-modules. By a theorem of D. G. Higman [5], the number of indecomposable classes is finite if and only if the Sylow $p$-subgroups of $G$ are cyclic.

Let $H$ be a subgroup of $G$. For any $G$-module $M$ let $M_H$ be the $H$-module formed by restriction of $M$ to $H$; for any $H$-module $L$ let $L^G$ be the $G$-module induced from $L$. A $G$-module $M$ is $H$-projective if there exists an $H$-module $L$ such that $M$ is isomorphic to a direct summand of $L^G$.

Denote by $A_H(K, G)$ the subspace of $A(K, G)$ spanned by the classes of $H$-projective $G$-modules. From the identity

$$L^G \otimes M \cong (L \otimes M_H)^G$$

which holds for any $H$-module $L$ and $G$-module $M$ it follows that $A_H(K, G)$ is an ideal of $A(K, G)$.

Let $F$ be a subgroup of $H$. We shall write $F \leq H$. Let $N$ be an $F$-module. Since $(N^H)^G \cong N^G$, a $G$-module which is $F$-projective is $H$-projective, i.e.

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$A_F(K, G) \leq A_H(K, G)$. Write

$$A'_H(K, G) = \sum A_F(K, G)$$

where the sum is taken over all proper subgroups $F$ of $H$; define $A'_{\{e\}}(K, G) = 0$. By the remark above, $A'_H(K, G)$ is an ideal of $A(K, G)$ and so of $A_H(K, G)$.

Let $D_H(K, G) = A_H(K, G)/A'_H(K, G)$.

It is to be noted that if $Q$ is a cyclic $p$-subgroup of $G$ then since the number of $Q$-modules and so the number of $Q$-projective modules is finite, $A_Q(K, G)$ and $D_Q(K, G)$ are both finite-dimensional algebras.

The aim of this paper is to prove the following result:

(i) *If $G$ is a finite group and $Q$ a cyclic normal $p$-subgroup of $G$ then the algebra $D_Q(K, G)$ is semisimple.*

Now Green [2] has shown that

(ii) *if $Q$ is any $p$-subgroup of a finite group $G$ then*

$$D_Q(K, G) \cong D_Q(K, N_G(Q))$$

*where $N_G(Q)$ is the normalizer of $Q$ in $G$.*

These two results combine to give immediately that

(iii) *If $Q$ is any cyclic $p$-subgroup of a finite group $G$ then $D_Q(K, G)$ is semisimple.*

For any group $G$, if $S$ is a Sylow $p$-subgroup of $G$, $A(K, G) = A_S(K, G)$ since every indecomposable $G$-module is $S$-projective (see for example Green [4, Theorem 2]). Then it is easily shown (Green [2]) that $A(K, G)$ is semisimple if and only if $D_Q(K, G)$ is semisimple for all $p$-subgroups $Q$ of $G$. This gives the following as an immediate corollary to (iii):

(iv) *If $G$ is a finite group with a cyclic Sylow $p$-subgroup then $A(K, G)$ is semisimple.*

## 2. Representation algebra

2.1. By a character of a commutative algebra $A$ with identity over $\boldsymbol{C}$ is meant a non-zero algebra homomorphism $\varphi : A \rightarrow \boldsymbol{C}$. A finite-dimensional algebra $A$ will be semisimple if and only if the number of characters of $A$ equals the dimension of $A$.

From now on $G$ is a finite group and $Q$ a cyclic normal $p$-subgroup of $G$ with $| Q | = h$.

We shall prove $D_Q(K, G)$ is semisimple by induction on $| Q |$ (Theorem 1). When $Q = (e)$ this follows from Lemma 12. If $| Q | > 1$ let $P$ be the cyclic subgroup of $Q$ of order $p$. Define $\bar{G} = G/P$, $\bar{Q} = Q/P$. Then by the induction hypothesis $D_{\bar{Q}}(K, \bar{G})$ is semisimple.

In §3, $D_Q(K, G)$ is expressed as an extension of $D_{\bar{Q}}(K, \bar{G})$ by the relations (3.11a, b, c).

In §4 we show a sufficient number of characters of $D_Q(K, G)$ can be obtained by extending the characters of $D_{\bar{Q}}(K, \bar{G})$ to ensure the semisimplicity of $D_Q(K, G)$ (Lemma 13).

The remainder of §2 will be given over to describing the algebra $A_Q(K, G)$.

2 2. By the proof of Theorem 1 (Green [3]) we may assume $K$ to be algebraically closed.

Let $\Gamma(K, Q)$ be the group algebra of $Q$ over $K$; write $w = x - e$ where $x$ generates $Q$. Then

$$V_i = \Gamma(K, Q)w^{h-i} \qquad (i = 1, \cdots, h)$$

is an ideal of $\Gamma(K, Q)$ and $V_1, \cdots, V_h$ form a set of representatives of the indecomposable classes of $Q$-modules (see for example D. G. Higman [5]).

Since each $V_i$ $(i = 1, \cdots, h)$ is an ideal of $\Gamma(K, Q)$ we may interpret $V_i^G$ as $V_i\Gamma(K, G)$ (see for example Green [4, 2.6, p. 431]). From the same identity it can be seen that $(V_i^G)_Q \cong (G:Q)V_i$.

Lemma 1. *Let $|Q| > 1$. A necessary and sufficient condition that $V_i$ is not $F$-projective for any proper subgroup $F$ of $Q$ is that $(p, i) = 1$.*

*Proof.* Let $Q'$ be the maximal proper subgroup of $Q$, viz., the subgroup generated by $x^p$. Let $\bar{h} = h/p$. Then since any proper subgroup $F$ of $Q$ is contained in $Q'$, every $F$-projective $Q$-module is $Q'$-projective.

Let $\bar{w} = x^p - e$. Then $V_j' \cong \Gamma(K, Q')\bar{w}^{\bar{h}-j}$ $(j = 1, \cdots, \bar{h})$ represent the indecomposable classes of $Q'$-modules. Further

$$(V_j')^Q \cong V_j' \, \Gamma(K, Q) \cong \Gamma(K, Q')\bar{w}^{\bar{h}-j}\Gamma(K, Q)$$
$$= \Gamma(K, Q)w^{h-pj}$$

since $Q$ and hence $\Gamma(K, Q)$ is abelian. That is

$$(V_j')^Q \cong V_{pj}.$$

Thus the $Q$-module $V_j$ will be $Q'$-projective if and only if $p$ divides $j$, proving the lemma.

Lemma 2. (a) *For any $G$-module $M$ and any $i > 0$, $Mw^i$ is a $G$-submodule of $M$.*

(b) *If $M_1$, $M_2$ are $G$-modules then for any $i \geq 0$,*

$$(M_1 + M_2)w^i \cong M_1 w^i + M_2 w^i.$$

(c) *For any $Q$-module $L$ and $i > 0$, $(Lw^i)^G \cong L^G w^i$.*

(d) *Let $M$ be any $G$-module and $S$ a $G$-module on which $Q$ acts trivially. Then*

$$(M \otimes S)w \cong Mw \otimes S.$$

*Proof.* (a) It is trivial that $Mw^i$ is a $K$-subspace of $M$. Let $g \in G$; then $g^{-1}xg = x^t$ for some $t$ $(1 \leq t \leq h)$ and so

$$g^{-1}wg = g^{-1}xg - e = x^t - e = bw = wb$$

where

$$b = (e + x + \cdots + x^{t-1}) \in \Gamma(K, G).$$

Let $m \in M$; then

$$mw^i g = mgg^{-1}w^i g = mgb^i w^i \in Mw^i.$$

Hence $Mw^i$ is a $G$-submodule of $M$.

(b) The proof here is trivial.

(c) By (b) it suffices to prove (c) for $L$ indecomposable, i.e., for $L = V_j$ $(j = 1, \cdots, h)$. Then as in (a) if $g \in G$, $g^{-1}w^i g = b^i w^i$, i.e.

$$w^i g = gb^i w^i \in \Gamma(K, G)w^i.$$

If $y \in \Gamma(K, G)$, $y = \sum_{g \in G} a_g\, g$ $(a_g \in K)$ and so

$$w^i y = \sum_{g \in G} a_g\, gw^i \in \Gamma(K, G)w^i.$$

Similarly $yw^i \in w^i\Gamma(K, G)$ whence $w^i\Gamma(K, G) = \Gamma(K, G)w^i$. Thus

$$(V_j w^i)^G \cong V_j w^i \Gamma(K, G) = V_j \Gamma(K, G)w^i \cong V_j^G w^i.$$

(d) Let $u \in (M \otimes S)w$; then $u$ is a linear combination of terms of the form $m \otimes s$ $(m \in M, s \in S)$. Since

$$\begin{aligned}(m \otimes s)w &= (m \otimes s)(x - e)\\ &= mx \otimes sx - m \otimes s\\ &= mx \otimes s - m \otimes s\\ &= m(x - e) \otimes s\\ &= mw \otimes s,\end{aligned}$$

the mapping $\theta : (M \otimes S)w \to Mw \otimes S$ defined by

$$(m \otimes s)w\theta = mw \otimes s$$

is a $(K, G)$-isomorphism.

Let $R_1, \cdots, R_s$ be the irreducible $G$-modules and $U_1, \cdots, U_s$ the principal indecomposable $G$-modules such that $R_\tau$ is the unique minimal $G$-submodule of $U_\tau$ (Artin, Nesbitt and Thrall, [1, pp. 99, 111]. We choose $R_1$ to be the unit $G$-module. Write $M_i^\tau = U_\tau w^{h-i}$ $(i = 1, \cdots, h;\ \tau = 1, \cdots, s)$. By Lemma 2(a), $M_i^\tau$ is a $G$-submodule of $U_\tau$.

LEMMA 3. *The classes* $\{M_i^\tau\}$ $(i = 1, \cdots, h;\ \tau = 1, \cdots, s)$ *form a basis for* $A_Q(K, G)$.

*Proof.* By the Krull-Schmidt theorem for $G$-modules, $A_Q(K, G)$ will have as basis the set of indecomposable classes of $Q$-projective $G$-modules, viz., the classes of indecomposable components of $V_i^G$ ($i = 1, \cdots, h$).

Let $f_\tau = \dim R_\tau$. Then

$$V_h^G \cong \Gamma(K, Q)\Gamma(K, G) = \Gamma(K, G).$$

Since $\Gamma(K, G) \cong \sum_{\tau=1}^{s} f_\tau U_\tau$ (see [1])

$$V_h^G \cong \sum_{\tau=1}^{s} f_\tau U_\tau$$

and so by Lemma 2(b), (c)

$$V_i^G \cong (V_h w^{h-1})^G \cong \sum_{\tau=1}^{s} f_\tau M_i^\tau. \tag{2.21}$$

Since

$$\Gamma(K, G)_Q \cong (G:Q)\Gamma(K, Q),$$

$(U_\tau)_Q$ is isomorphic to the sum of say $n_\tau$ ($n_\tau \neq 0$) copies of $\Gamma(K, Q)$ and so $(M_i^\tau)_Q$ is isomorphic to the sum of $n_\tau$ copies of $\Gamma(K, Q)w^{h-i} \cong V_i$ and hence is a non-zero $G$-module. By the same fact if $i \neq j$, $M_i^\tau$ is not isomorphic to $M_j^\rho$ for any $\tau$ or $\rho$ ($\tau, \rho = 1, \cdots, s$).

As $M_i^\tau$ has the unique minimal $G$-submodule $R_\tau$ it is indecomposable and not isomorphic to $M_i^\rho$ for $\rho \neq \tau$. Hence $\{M_i^\tau\}$ ($i = 1, \cdots, h$; $\tau = 1, \cdots, s$) are distinct indecomposable $Q$-projective classes and by 2.21 are the complete set of such classes, which proves the lemma.

COROLLARY I. *The classes* $\{M_i^\tau\} + A_Q'(K, G)$ ($\tau = 1, \cdots, s$; $i = 1, \cdots, h$; $(i, p) = 1$) *form a basis for* $D_Q(K, G)$.

*Proof.* $M_i^\tau$ is $F$-projective for some subgroup $F$ of $Q$ if and only if $(M_i^\tau)_Q$ is $F$-projective. Since $(M_i^\tau)_Q \cong n_\tau V_i$, by Lemma 1 $M_i^\tau$ is not $F$-projective for a proper subgroup $F$ of $Q$ unless $p$ divides $i$. Thus $\{M_i^\tau\} \in A_Q'(K, G)$ if and only if $p$ divides $i$, proving the result.

COROLLARY II. *Elements of $Q$ act trivially on* $M_1^\tau$ and hence on $R_\tau$ ($\tau = 1, \cdots, s$).

For, $(M_1^\tau)_Q \cong n_\tau V_1 \cong n_\tau K_G$ and $R_\tau \subset M_1^\tau$.

From now on we shall simplify the notation by writing $M$ for the isomorphism class $\{M\}$ in $A(K, G)$ and also for its quotient class $\{M\} + A_Q'(K, G)$ in $D_Q(K, G)$.

2.3 Let $\mathcal{J}$ be the ideal of $A(K, G)$ generated by all elements of $A(K, G)$ of the form $M - M' - M''$ where there exists an exact sequence of $G$-modules and $G$-homomorphisms

$$0 \to M' \to M \to M'' \to 0.$$

The Grothendieck algebra $A^*(K, G)$ is the quotient $A(K, G)/\mathcal{J}$. By the Jordan-Holder theorem for $G$-modules, writing $\mathcal{R}_\tau = R_\tau + \mathcal{J} \in A^*(K, G)$, $\mathcal{R}_1, \cdots, \mathcal{R}_s$ form a basis of $A^*(K, G)$.

Let $m_\sigma(M)$ be the multiplicity of $R_\sigma$ as a composition factor of the $G$-module $M$. We may extend this notation to an arbitrary element $X = \sum_\delta b_\delta M_\delta$ ($b_\delta \in \boldsymbol{C}$) of $A(K, G)$ by putting $m_\sigma(X) = \sum_\delta b_\delta m_\sigma(M_\delta)$. Then again by the Jordan-Holder theorem, $M + \mathcal{J} = \sum_{\sigma=1}^{s} m_\sigma(M)\mathcal{R}_\sigma$.

Let $M'$ be the maximal proper $G$-submodule of the $G$-module $M$; then $M/M' \cong R_\sigma$ for some $\sigma$ $(1 \leq \sigma \leq s)$ and there exists an exact sequence of $G$-modules and $G$-homomorphisms

$$0 \to M' \to M \to R_\sigma \to 0.$$

From this may be formed, by tensor multiplication with $U_\tau$ $(1 \leq \tau \leq s)$, the exact sequence

$$0 \to M' \otimes U_\tau \to M \otimes U_\tau \to R_\sigma \otimes U_\tau \to 0.$$

Since $U_\tau$, and thus $U_\tau \otimes R_\sigma$, is projective, this sequence splits and so $M \otimes U_\tau \cong M' \otimes U_\tau + R_\sigma \otimes U_\tau$. By repeated application we find

$$U_\tau \otimes M \cong \textstyle\sum_\sigma U_\tau \otimes m_\sigma(M) R_\sigma .$$

Thus

$$U_\tau \otimes X = \textstyle\sum_\sigma U_\tau \otimes m_\sigma(X)\mathcal{R}_\sigma . \tag{2.31}$$

Let $B_i$ be the subspace of $A(K, G)$ spanned by the indecomposable classes $M_i^1, \cdots, M_i^s$.

LEMMA 4. *For each $i$ $(1 \leq i \leq h)$, $B_i$ is an $A^*(K, G)$-module with elements of $A^*(K, G)$ acting on the right according to the rule*

$$M_i^\tau \cdot \mathcal{R}_\sigma = M_i^\tau \otimes R_\sigma \quad (\tau, \sigma = 1, \cdots, s)\ (M_i^\tau \in B_i,\ \mathcal{R}_\sigma \in A^*(K, G)).$$

*Proof.* Any element of $\mathcal{J}$ is a linear combination of elements of the form $Z = N - N' - N''$ where there is an exact sequence of $G$-modules and $G$-homomorphisms

$$0 \to N' \to N \to N'' \to 0.$$

Since $m_\sigma(Z) = 0$ $(\sigma = 1, \cdots, s)$, $M_h^\tau \otimes Z = U_\tau \otimes Z = 0$ by (2.31) and so $B_h$ is made an $A^*(K, G)$-module by the rule defined above.

Further by Lemma 2(d),

$$M_i^\tau \otimes R_\sigma = (M_h^\tau \otimes R_\sigma)w^{h-i} \in B_i .$$

Thus $B_i$ is also an $A^*(K, G)$-module by this rule.

We now define the mapping $\chi : A_Q(K, G) \to A^*(K, G)$ by

$$M_i^\tau \chi = \textstyle\sum_\sigma m_\sigma(M_i^\tau)\mathcal{R}_\sigma \qquad (\tau = 1, \cdots, s; i = 1, \cdots, h).$$

$\chi$ is obviously an algebra homomorphism. Further,

$$U_\tau \chi = \textstyle\sum_\sigma c_{\sigma\tau} \mathcal{R}_\sigma$$

where $c_{\sigma\tau} = m_\sigma(U_\tau)$ and $(c_{\sigma\tau})$ is the Cartan matrix, which is non-singular. Also for any $\rho$ $(\rho = 1, \cdots, s)$ and $\tau$ $(\tau = 1, \cdots, s)$

$$(U_\tau \cdot \mathcal{R}_\rho)\chi = \textstyle\sum_\sigma c_{\sigma\tau} \mathcal{R}_\sigma \otimes \mathcal{R}_\rho = U_\tau \chi \otimes \mathcal{R}_\rho .$$

Thus we have

LEMMA 5. *$\chi$ is an $A^*(K, G)$-isomorphism when restricted to $B_h$.*

For any element $X = \sum_{j,\tau} u_{j\tau} M_j^\tau$ ($u_{j\tau} \in \mathbf{C}$) of $A_Q(K, G)$ write $Xw^i = \sum_{j,\tau} u_{j\tau} M_j^\tau w^i$. Also given an element $X_i = \sum_{\tau=1}^s u_\tau M_i^\tau$ ($u_\tau \in \mathbf{C}$) of $B_i$ define $X_j = \sum_{\tau=1}^s u_\tau M_j^\tau$; then $X_i = X_h w^{h-i}$. Define

$$Y_h = R_1 \chi^{-1} = \sum b_\tau M_h^\tau \quad \text{(say)}$$

and $Y_i = Y_h w^{h-i}$.

LEMMA 6. *Let $X_i \in B_i$; then $X_i = Y_i \cdot (X_h)\chi$.*

*Proof.* Since $\chi$ restricted to $B_h$ is an isomorphism $Y_h$ is the identity of $B_h$ and thus

$$X_h = X_h \otimes Y_h = S(X_h) \otimes Y_h$$

where $S(X) = \sum_\sigma m_\sigma(X) R_\sigma$, by (2.31). Then for $1 \le i \le h$,

$$X_i = X_h w^{h-i} = (S(X_h) \otimes Y_h) w^{h-i}$$
$$= S(X_h) \otimes Y_h w^{h-i}$$

by Lemma 2(d).

By Lemma 4, $S(X_h) \otimes Y_i = Y_i \cdot (X_h)\chi$.

From Lemma 6 we can deduce immediately that $Y_1, \cdots, Y_h$ form an $A^*(K, G)$-basis for $A_Q(K, G)$, since $M_i^\tau = Y_i \cdot (M_h^\tau)\chi$. Similarly the set

$$\{Y_i;\ i = 1, \cdots, h;\ (i, p) = 1\}$$

forms a free $A^*(K, G)$-basis for $D_Q(K, G)$. In each case the basis is a free basis since, from the non-singularity of the Cartan matrix, $M_h^\tau \chi$ is distinct for distinct $\tau$.

Given $X_h = \sum_\tau u_\tau M_h^\tau \in B_h$ we define for $j < i$

$$X_i/X_j = \sum_\tau u_\tau M_i^\tau / M_j^\tau .$$

LEMMA 7. *Let $R$ be any element of $A^*(K, G)$; then for $X_h \in B_h$, and $j < i$, $(X_i \cdot \mathfrak{R})/(X_j \cdot \mathfrak{R}) = (X_i/X_j) \cdot \mathfrak{R}$.*

*Proof.* From the definition of $X_i/X_j$ it is sufficient to prove the result in the case where $X_i = M_i^\tau$ ($\tau = 1, \cdots, s$) and $\mathfrak{R} = \mathfrak{R}_\sigma$ ($\sigma = 1, \cdots, s$). In this case it is easily verified that the mapping

$$\theta : (M_i^\tau \otimes S)/(M_j^\tau \otimes S) \to (M_i^\tau / M_j^\tau) \otimes S$$

defined by

$$\{m \otimes n + M_j^\tau \otimes S\}\theta = (m + M_j^\tau) \otimes n, \qquad (m \in M_j^\tau,\ n \in S)$$

is a $G$-isomorphism where $S$ is $Q$-trivial. Since $M_i^\tau \otimes R_\sigma = M_i^\tau \cdot \mathfrak{R}_\sigma$, the result is proved.

LEMMA 8. *There exists an element $\mathfrak{R}$ of $A^*(K, G)$ such that, for any $X_h \in B_h$,*

$$X_i/X_j = X_{i-j} \cdot \mathfrak{R}^j$$

*for all $j$ and $i$ ($j = 1, \cdots, h; j < i \leq h$) where $\mathcal{R}^j$ denotes the product of $j$ copies of $\mathcal{R}$.*

*Proof.* Given $j < i$,

$$V_{i-j}^G \cong (V_i/V_j)^G \cong V_i^G/V_j^G ,$$

i.e.,

$$\sum_{\tau=1}^s f_\tau M_{i-j}^\tau \cong \sum_{\tau=1}^s f_\tau M_i^\tau/M_j^\tau \qquad \text{(see (2.21))}.$$

Since there are the same number of distinct summands on the right as there are distinct indecomposable $G$-modules on the left, by the Krull-Schmidt theorem for $G$-modules, there exists a permutation $\xi(i, j): \tau \rightarrow \tau'$ such that

$$M_i^\tau/M_j^\tau \cong M_{i-j}^{\tau'} .$$

Thus

$$M_2^\tau/M_1^\tau \cong M_1^{\tau'} = Y_1 \cdot (M_h^{\tau'})\chi.$$

Hence

$$Y_2/Y_1 = \sum_{\tau=1}^s b_\tau M_2^\tau/M_1^\tau = \sum_{\tau=1}^s b_\tau Y_1 \cdot (M_h^{\tau'})\chi.$$

Write $\mathcal{R} = \sum_{\tau=1}^s b_\tau(M_h^{\tau'})\chi$ where $\tau' = \tau\xi(2, 1)$.
Then

$$\begin{aligned} X_2/X_1 &= Y_2 \cdot (X_h)\chi/Y_1 \cdot (X_1)\chi \\ &= (Y_2/Y_1)\cdot(X_h)\chi, && \text{by Lemma 7,} \\ &= (Y_1 \cdot \mathcal{R})\cdot(X_h)\chi, && \text{by above} \\ &= X_1 \cdot \mathcal{R}. \end{aligned}$$

Further since for $1 \leq i \leq h$, $M_i^\tau/M_1^\tau$ is indecomposable, contains $M_2^\tau/M_1^\tau$ and is isomorphic to a summand of $V_{i-1}^G$, $M_i^\tau/M_1^\tau \cong M_{i-1}^{\tau'}$ where again $\tau' = \tau\xi(2, 1)$. Hence

$$M_i^\tau/M_1^\tau \cong M_{i-1}^\tau \cdot \mathcal{R}.$$

Thus the lemma is true for $j = 1$. Assume that $j > 1$ and that it is true for $j - 1$. Then for $i > j$

$$\begin{aligned} X_i/X_j &= (X_i/X_1)/(X_j/X_1) \\ &= X_{i-1} \cdot \mathcal{R}/X_{j-1} \cdot \mathcal{R}, && \text{by the case } j = 1 \\ &= (X_{i-1}/X_{j-1}) \cdot \mathcal{R}, && \text{by Lemma 7} \\ &= X_{i-j} \cdot \mathcal{R}^{j-1} \cdot \mathcal{R}, && \text{by the hypothesis} \\ &= X_{i-j} \cdot \mathcal{R}^j. \end{aligned}$$

Thus the lemma is proved by induction.

We note that $\mathcal{R}$ operating on elements of $A_Q(K, G)$ induces the permutation $\xi(2, 1)$ of the superscripts $\tau$. Further, since $V_1^G \cdot \mathcal{R} = V_1^G$, dim $\mathcal{R}$ = dim $R = 1$.

LEMMA 9. *Let $M$ and $N$ be direct sums of $Q$-projective $G$-modules. Then*

*$M \cong N$ if and only if*

$$\frac{Mw^{j-1}}{Mw^j} \cong \frac{Nw^{n-1}}{Nw^j} \qquad \textit{for all} \quad j = 1, 2, \cdots$$

*Proof.* If $M \cong N$ it is trivial to show that

$$\frac{Mw^{j-1}}{Mw^j} \cong \frac{Nw^{j-1}}{Nw^j} \qquad \text{for all } j.$$

Assume

$$\frac{Mw^{j-1}}{Mw^j} \cong \frac{Nw^{j-1}}{Nw^j} \qquad \text{for all } j.$$

Then, in particular

$$\frac{M}{Mw} \cong \frac{N}{Nw}$$

whence $M$ and $N$ have the same number $t$ of indecomposable components. Further if $M_k^\tau$ is an indecomposable component of $M$ such that $k$ is maximum as a suffix of indecomposable components of $M$, then since $M\omega^j = 0$ for $j \geq k$, $N\omega^j = 0$ for $j \geq k$ and so $k$ is maximum as a suffix of indecomposable components of $N$.

If $M$ is indecomposable, i.e., has only one indecomposable component, $M \cong M_k^\tau$ for some $\tau$ $(1 \leq \tau \leq s)$ and $k$. Thus $N \cong N_k^\sigma$ for some $\sigma$ $(1 \leq \sigma \leq s)$. Further

$$M_1^\tau \cong \frac{Mw^{k-1}}{Mw^k} \cong \frac{Nw^{k-1}}{Nw^k} \cong N_1^\sigma$$

whence $\sigma = \tau$. Thus the result is true for $t = 1$.

Assume the result true for all $G$ modules $M$ and $N$ with less than $t$ indecomposable components. Let $M$ have $t$ indecomposable components. Choose one such component $M'$ of $M$ such that $M' \cong M_k^\tau$ where $k$ is maximum as a suffix of indecomposable components of $M$. Let $M''$ be the complement of $M'$ in $M$, i.e., the $G$-module such that $M' + M'' \cong M$. Then $M_k^\tau$ contributes a component

$$\frac{M_k^\tau\, w^{k-1}}{M_k^\tau\, w^k} \cong M_1^\tau$$

to $\dfrac{Mw^{k-1}}{Mw^k}$ and so there exists an indecomposable component $N'$ of $N$ such that

$$\frac{N'w^{k-1}}{N'w^k} \cong M_1^\tau .$$

Now $N'w^k = 0$; thus $N'w^{k-1} \cong M_1^\tau$ whence $N' \cong M_k^\tau$. Let $N''$ be the complement of $N'$ in $N$. Then $M''$ and $N''$ satisfy the conditions of the hypothesis and so $N'' \cong M''$.

Thus $M \cong M' + M'' \cong N' + N'' \cong N$.

The result thus follows by induction on $t$.

## 3. Extension Relations

3.1. Define $P$, $\bar{G}$, $\bar{Q}$ as in §2 and let $q = |\bar{Q}|$. Then $h = pq$. The epimorphism from $G$ onto $\bar{G}$ which takes an element $g$ of $G$ to $\bar{g} = gP$ induces a monomorphism $\theta : A(K, \bar{G}) \to A(K, G)$ by identifying a $\bar{G}$-module $\bar{M}$ with the $G$-module having the same underlying space as $\bar{M}$ and on which each element $g$ of $G$ acts by $\bar{m}g = \bar{m}\bar{g}$ $(\bar{m} \in \bar{M})$. Since $R_\sigma$ $(\sigma = 1, \cdots, s)$ is $P$-trivial it will be the image under $\theta$ of an irreducible $\bar{G}$-module $\bar{R}_\sigma$ whence $A^*(K, G)$ is isomorphic to $A^*(K, \bar{G})$. $\theta$ also identifies the indecomposable $\bar{G}$-module $\bar{M}_i^\tau$ with the corresponding $G$-module $M_i^\tau$ $(i = 1, \cdots, q; \tau = 1, \cdots, s)$ and so embeds $A_{\bar{Q}}(K, \bar{G})$ in $A_Q(K, G)$ as an algebra over $\mathbf{C}$ and, by the isomorphism of $A^*(K, \bar{G})$ to $A^*(K, G)$, as an algebra over $A^*(K, G)$.

Since $\bar{M}_i^\tau \in A'_Q(K, \bar{G})$ and $M_i^\tau \in A'_Q(K, G)$ if and only if $p$ divides $i$, $D_{\bar{Q}}(K, \bar{G})$ is embedded as an algebra over $A^*(K, G)$ in $D_Q(K, G)$.

Write $M_i^\tau = \bar{M}_i^\tau$. For any integer $r$ such that $0 < r \leq h$ write $r = r_0 q + r_1$ $(0 \leq r_1 < q)$. We shall prove

LEMMA 10. *For any $r$, $0 < r < h$, such that $(p, r) = 1$ one of the following relation holds in $D_Q(K, G)$:*

$$Y_r \otimes (Y_{q+1} - Y_{q-1} \cdot \mathcal{R}) = Y_1 \otimes (Y_{q+r} - Y_{q-r} \cdot \mathcal{R}) \quad \text{if } 1 \leq r < q; \tag{3.11a}$$

$$Y_r \otimes (Y_{q+1} - Y_{q-1} \cdot \mathcal{R}) = Y_1 \otimes (Y_{r+q} - Y_{r-q} \cdot \mathcal{R}^q) \quad \text{if } q < r < (p-1)q; \tag{3.11b}$$

$$Y_r \otimes (Y_{q+1} - Y_{q-1} \cdot \mathcal{R}) = Y_1 \otimes (Y_{r-q} \cdot R^q - Y_{h-r_1} \cdot \mathcal{R}^{r_1}) \quad \text{if } (p-1)q < r < h; \tag{3.11c}$$

*where $\mathcal{R}$ is as defined in Lemma 8.*

Before embarking on the proof of Lemma 9 it is necessary to discuss some of the properties of tensor multiplication of $G$-modules.

Consider the exact sequence of $G$-modules

$$0 \to M_1^\sigma \xrightarrow{\iota} M_i^\sigma \to M_{i-1}^\sigma \cdot \mathcal{R} \to 0$$

where $\iota$ is the inclusion map. By taking the tensor product with $M_r^\tau$ one obtains the exact sequence

$$0 \to M_1^\sigma \otimes M_r^\tau \xrightarrow{\iota} M_i^\sigma \otimes M_r^\tau \to M_{i-1}^\sigma \cdot \mathcal{R} \otimes M_r^\tau \to 0.$$

Writing $L = M_1^\sigma \otimes M_r^\tau$, $M = M_i^\sigma \otimes M_r^\tau$, $N = M_{i-1}^\sigma \cdot \mathcal{R} \otimes M_r^\tau$ we can form the exact sequences

$$0 \to L \cap Mw^j \to Mw^j \xrightarrow{\mathcal{E}_j} Nw^j \to 0$$

for $j = 0, 1, \cdots, |M| - 1$ where $|M|$ is the maximum dimension of the indecomposable components of $M$.

These in turn give rise to the exact sequences

$$0 \longrightarrow \frac{L \cap Mw^{j-1}}{L \cap Mw^j} \xrightarrow{\bar{\iota}} \frac{Mw^{j-1}}{Mw^j} \xrightarrow{\bar{\varepsilon}_j} \frac{Nw^{j-1}}{Nw^j} \longrightarrow 0$$

where $\bar{\iota}$ is the inclusion map ($j = 0, 1, \cdots, |M| - 1$).

Now

$$\left(\frac{Mw^{j-1}}{Mw^j}\right)_Q \cong n_\sigma n_\tau \frac{(V_i \otimes V_r)w^{j-1}}{(V_i \otimes V_r)w^j}$$

and

$$\left(\frac{Nw^{j-1}}{Nw^j}\right)_Q \cong n_\sigma n_\tau \frac{(V_{i-1} \otimes V_r)w^{j-1}}{(V_{i-1} \otimes V_r)w^j} ;$$

further by applying a result of Green ([3, 2.6d]) there exists a set of integers $J = \{i_1, \cdots, i_r\}$ ($i_1 < i_2 < \cdots < i_r$), such that

$$\frac{(V_i \otimes V_r)w^{j-1}}{(V_i \otimes V_r)w^j} \cong \frac{(V_{i-1} \otimes V_r)w^{j-1}}{(v_{i-1} \otimes V_r)w^j} + \delta(j, J)V_1 \tag{3.12}$$

where $\delta(j, J) = 1$ if $j \in J$ and is otherwise zero.

We can now show

$$\textit{if } j_{k-1} \leq j < j_k \textit{ then } \ker \varepsilon_j \cong Lw^{k-1}. \tag{3.13}$$

*Proof.* The result is trivially true for $j = 0$. Assume $j_{k-1} \leq j < j_k$ and that $\ker \varepsilon_j \cong Lw^{k-1}$ ($\cong M_1^\sigma \otimes M_{r-k+1}^\tau$ by Lemma 2(d)). If $j + 1 \neq j_k$ then $\ker \bar{\varepsilon}_{j+1} = 0$ and so $\ker \varepsilon_j \cong \ker \varepsilon_{j+1} \cong Lw^{k-1}$. If $j + 1 = j_k$ then by (3.12), $(\ker \bar{\varepsilon}_j)_Q$ and hence by Lemma 2(c) $\ker \bar{\varepsilon}_j$ is annihilated by $w$. Thus $\ker \varepsilon_j \supset \ker \varepsilon_{j+1} \supset (\ker \varepsilon_j)w$, i.e., $Lw^{k-1} \supset \ker \varepsilon_{j-1} \supset Lw^k$. But again by (3.12),

$$\dim Lw^{k-1} - \dim \ker \varepsilon_{j+1} = n_\sigma n_\tau = \dim Lw^{k-1} - \dim Lw^k.$$

Thus $\ker \varepsilon_{j+1} \cong Lw^k$ and the result follows by induction.

From (3.13) since

$$\begin{aligned} Lw^{k-1}/Lw^k &\cong \frac{M_1^\sigma \otimes M_{r-k+1}^\tau}{M_1^\sigma \otimes M_{r-k}^\tau}, && \text{by Lemma 2(d)} \\ &\cong M_1^\sigma \otimes M_1^\tau \cdot \mathcal{R}^{r-k} && \text{by Lemma 8} \end{aligned}$$

we deduce

$$\textit{if } j = j_k \in J, \ker \bar{\varepsilon}_j \cong M_1^\sigma \otimes M_1^\tau \cdot \mathcal{R}^{r-k}. \tag{3.14}$$

Given integers $r, s$ ($r \leq s$) define an interval $[r, s]$ as the set of integers $j$ such that $r \leq j \leq s$. Then the set $J$ may be uniquely written as the union of intervals

$$J = \bigcup_\rho [a(\rho), b(\rho)]$$

such that $b(\rho - 1) + 1 < a(\rho) < b(\rho) < a(\rho + 1) - 1$ where defined. Let $a(\rho) = j_{k(\rho)}$ and $b(\rho) = j_{l(\rho)}$. Then

$$(3.15) \quad M + \sum_\rho M_1^\sigma \otimes M_{a(\rho)-1}^\tau \cdot \mathcal{R}^{r-k(\rho)+1} = N + \sum_\rho M_1^\sigma \otimes M_{b(\rho)}^\tau \cdot \mathcal{R}^{r-l(\rho)}.$$

*Proof.* By Lemma 2(d), and the proofs of Lemma 7 and Lemma 8,

$$\frac{(M_1^\sigma \otimes M_{a(\rho)-1}^\tau \cdot \mathcal{R}^{r-k(\rho)+1})w^{j-1}}{(M_1^\sigma \otimes M_{a(\rho)-1}^\tau \cdot \mathcal{R}^{r-k(\rho)+1})w^j} = M_1^\sigma \otimes M_1^\tau \cdot \mathcal{R}^{a(\rho)-j+r-k(\rho)}$$

if $a(\rho) > j$ and is otherwise zero. Similarly

$$\frac{(M_1^\sigma \otimes M_{b(\rho)}^\tau \cdot \mathcal{R}^{r-l(\rho)})w^{j-1}}{(M_1^\sigma \otimes M_{b(\rho)}^\tau \cdot \mathcal{R}^{r-l(\rho)})w^j} = M_1^\sigma \otimes M_1^\tau \cdot \mathcal{R}^{b(\rho)-j+r-l(\rho)}$$

if $b(\rho) > j - 1$ and is otherwise zero.

Write

$$[\textstyle\sum_{a(\rho)>j}] = \sum_{a(\rho)>j} M_1^\sigma \otimes M_1^\tau \cdot \mathcal{R}^{a(\rho)-j+r-k(\rho)}$$

and

$$[\textstyle\sum_{b(\rho)>j-1}] = \sum_{b(\rho)>j-1} M_1^\sigma \otimes M_1^\tau \cdot \mathcal{R}^{b(\rho)-j+r-l(\rho)}.$$

Then by Lemma 9 to prove (3.15) it suffices to show that

$$\frac{Mw^{j-1}}{Mw^j} + [\textstyle\sum_{a(\rho)>j}] = \frac{Nw^{j-1}}{Nw^j} + [\textstyle\sum_{b(\rho)>j-1}]$$

for all $j$, $1 \leq j \leq |M|$. Now since

$$\ker \bar{\varepsilon}_j \subset \frac{Mw^{j-1}}{Mw^j}$$

and both are the direct sums of $G$-modules of unit suffix, $\ker \bar{\varepsilon}_j$ is a component of $\dfrac{Mw^{j-1}}{Mw^j}$ and has as complement $\dfrac{Nw^{j-1}}{Nw^j}$, i.e.,

$$\frac{Mw^{j-1}}{Mw^j} = \frac{Nw^{j-1}}{Nw^j} + \ker \bar{\varepsilon}_j .$$

Thus proving (3.15) reduces to showing

$$(3.16) \qquad [\textstyle\sum_{a(\rho)>j}] + \ker \bar{\varepsilon}_j = [\textstyle\sum_{b(\rho)>j-1}]$$

for all $j$, $1 \leq j \leq |M|$.

The proof will be by induction on $|M| - j$. Assume (3.16) true for all $j > d$. This is true for $d = |M|$. Let $d < |M|$. We require to prove that (3.16) is true for $j = d$.

*Case* (i). Let $d$, $d + 1$ be both contained in $J$ or both outside $J$. Then

$d \neq b(\rho)$ for any $\rho$ and $d + 1 \neq a(\rho)$ for any $\rho$. Thus

$$[\textstyle\sum_{a(\rho)>d}] = [\sum_{a(\rho)>d+1}]\cdot\mathcal{R} \quad \text{and} \quad [\sum_{b(\rho)>d-1}] = [\sum_{b(\rho)>d}]\cdot\mathcal{R}.$$

Further ker $\bar{\varepsilon}_d = (\ker \bar{\varepsilon}_{d+1})\cdot\mathcal{R}$ by (3.14). By the induction hypothesis for $j = d + 1$,

$$[\textstyle\sum_{a(\rho)>d+1}] + \ker \bar{\varepsilon}_{d+1} = [\sum_{b(\rho)>d}].$$

Thus

$$\begin{aligned}[\textstyle\sum_{b(\rho)>d-1}] = [\sum_{b(\rho)>d}]\cdot\mathcal{R} &= [\textstyle\sum_{a(\rho)>d+1}]\cdot\mathcal{R} + (\ker \bar{\varepsilon}_{d+1})\cdot\mathcal{R} \\ &= [\textstyle\sum_{a(\rho)>d}] + \ker \bar{\varepsilon}_d\end{aligned}$$

whence (3.16) holds for $j = d$.

*Case* (ii). Let $d \in J$ and $d + 1 \notin J$. Then $d = b(\rho_1)$ for some $\rho = \rho_1$ but $d + 1 \neq a(\rho)$ for any $\rho$. Thus

$$[\textstyle\sum_{a(\rho)>d}] = [\sum_{a(\rho)>d+1}]\cdot\mathcal{R}$$

while

$$[\textstyle\sum_{b(\rho)>d-1}] = [\sum_{b(\rho)>d}]\cdot\mathcal{R} + M_1^\sigma \otimes M_1^\tau\cdot\mathcal{R}^{r-l(\rho_1)}$$

since $d = b(\rho_1)$. Further ker $\bar{\varepsilon}_{d+1} = 0$ while ker $\bar{\varepsilon}_d = M_1^\sigma \otimes M_1^\tau\cdot\mathcal{R}^{r-l(\rho_1)}$ by (3.14). Applying the induction hypothesis for $j = d + 1$,

$$[\textstyle\sum_{a(\rho)>d+1}] = [\sum_{b(\rho)>d}].$$

Thus

$$\begin{aligned}[\textstyle\sum_{b(\rho)>d-1}] &= [\textstyle\sum_{b(\rho)>d}]\cdot\mathcal{R} + M_1^\sigma \otimes M_1^\tau\cdot\mathcal{R}^{r-l(\rho_1)} \\ &= [\textstyle\sum_{a(\rho)>d+1}]\cdot\mathcal{R} + \ker \bar{\varepsilon}_d \\ &= [\textstyle\sum_{a(\rho)>d}] + \ker \bar{\varepsilon}_d\end{aligned}$$

whence (3.16) is satisfied for $j = d$.

*Case* (iii). Let $d \notin J$ and $d + 1 \in J$. Then $d \neq b(\rho)$ for any $\rho$ but $d + 1 = a(\rho_2)$ for some $\rho = \rho_2$. Thus

$$[\textstyle\sum_{b(\rho)>d-1}] = [\sum_{b(\rho)>d}]\cdot\mathcal{R}$$

while

$$[\textstyle\sum_{a(\rho)>d}] = [\sum_{a(\rho)>d+1}]\cdot\mathcal{R} + M_1^\sigma \otimes M_1^\tau\cdot\mathcal{R}^{r-k(\rho_2)+1}$$

since $a(\rho_2) = d + 1$. Further ker $\bar{\varepsilon}_d = 0$ while ker $\bar{\varepsilon}_{d+1} = M_1^\sigma \otimes M_1^\tau\cdot\mathcal{R}^{r-l(\rho_2)}$. Applying the induction hypothesis for $j = d + 1$,

$$[\textstyle\sum_{a(\rho)>d+1}] + M_1^\sigma \otimes M_1^\tau\cdot\mathcal{R}^{r-l(\rho_2)} = [\sum_{b(\rho)>d}].$$

Thus

$$\begin{aligned}[\textstyle\sum_{b(\rho)>d-1}] = [\sum_{b(\rho)>d}]\cdot\mathcal{R} &= [\textstyle\sum_{a(\rho)>d+1}]\cdot\mathcal{R} + M_1^\sigma \otimes M_1^\tau\cdot\mathcal{R}^{r-l(\rho_2)+1} \\ &= [\textstyle\sum_{a(\rho)>d}]\end{aligned}$$

whence (3.16) is satisfied for $j = d$.

*Proof of Lemma* 10. The set $J$ associated with the exact sequence of $G$-modules

$$0 \to M_1^\sigma \otimes M_r^\tau \xrightarrow{\iota} M_q^\sigma \otimes M_r^\tau \to M_{q-1}^\sigma \cdot \mathcal{R} \otimes M_r^\tau \to 0$$

where $\iota$ is the inclusion map is (see Green [3, 2.9c, 2.9d])

$$J = [q - r + 1, q] \qquad \text{for } 0 < r \leq q$$

$$J = [1, r_0 q] \cup [(r_0 + 1) q - r_1 + 1, (r_0 + 1) q]$$
$$\text{for } q < r \leq pq = h.$$

The corresponding set associated with the exact sequence

$$0 \to M_1^\sigma \otimes M_r^\tau \xrightarrow{\iota} M_{q+1}^\sigma \otimes M_r^\tau \to M_q^\sigma \cdot \mathcal{R} \otimes M_r^\tau \to 0$$

is

$$J' = [q + 1, q + r] \qquad \text{for } 0 < r \leq q$$

$$J' = [1, r - q] \cup [r_0 q + 1, (r_0 + 1)q - r_1] \cup [(r_0 + 1)q + 1, q + r]$$
$$\text{for } q < r < (p - 1)q$$

$$J' = [1, r - q] \cup [(p - 1)q + 1, pq] \qquad \text{for } (p - 1)q \leq r \leq pq = h.$$

Hence by (3.15) for $0 < r \leq q$ we have the relations in $A_Q(K, G)$,

$$M_{q-1}^\sigma \cdot \mathcal{R} \otimes M_r^\tau = M_q^\sigma \otimes M_r^\tau + M_1^\sigma \otimes M_{q-r}^\tau \cdot \mathcal{R}^r - M_1^\sigma \otimes M_q^\tau$$

and

$$M_{q+1}^\sigma \otimes M_r^\tau = M_q^\sigma \otimes M_r^\tau \cdot \mathcal{R} + M_1^\sigma \otimes M_{q+r}^\tau - M_1^\sigma \otimes M_q^\tau \cdot \mathcal{R}^r.$$

From these, since $Y_r = \sum_\tau b_\tau M_r^\tau$ we form the relations for $0 < r \leq q$,

$$\{Y_{q-1} \cdot \mathcal{R} - Y_q\} \otimes Y_r = Y_1 \otimes \{Y_{q-r} \cdot \mathcal{R}^r - Y_q\} \tag{3.17a}$$

$$\{Y_{q+1} - Y_q \cdot \mathcal{R}\} \otimes Y_r = Y_1 \otimes \{Y_{q+r} - Y_q \cdot \mathcal{R}^r\}. \tag{3.17b}$$

Now $Y_r = 0 \bmod A'_Q(K, G)$ if and only if $p$ divides $r$. Hence (3.17a, b) give rise in $D_Q(K, G)$ to the relations

$$Y_{q-1} \cdot \mathcal{R} \otimes Y_r = Y_1 \otimes Y_{q-r} \cdot \mathcal{R}^r$$

and

$$Y_{q+1} \otimes Y_r = Y_1 \otimes Y_{q+r} \qquad \text{where } (p, r) = 1.$$

Thus for $0 < r < q$ such that $(p, r) = 1$,

$$Y_r \otimes (Y_{q+1} - Y_{q-1} \cdot R) = Y_1 \otimes (Y_{q+r} - Y_{q-r} \cdot R^r). \tag{3.11a}$$

By a similar procedure (3.11b, c) are derived.

LEMMA 11. *The relations* (3.11a, b, c) *together with the relation* (*derived from Lemma* 6) $M_i^\tau = Y_i \cdot (M_h^\tau)\chi$ $(i = 1, \cdots, h)$ *define* $D_Q(K, G)$ *as an extension of* $D_{\bar{Q}}(K, \bar{G})$ *considered as an algebra over* **C**.

*Proof.* By (2.31) for any $\tau$ and $\rho$, $1 \leq \tau, \rho \leq s$,

$$U_\tau \otimes M_1^\rho = U_\tau \cdot (M_1^\rho)\chi.$$

Hence by Lemma 2(d), for any $i$, $1 \leq i \leq h$,

$$\begin{aligned} M_i^\tau \otimes M_1^\rho = U_\tau w^{h-i} \otimes M_1^\rho &= (U_\tau \otimes M_1^\rho) w^{h-i} \\ &= (U_\tau \cdot (M_1^\rho)\chi) w^{h-i} \\ &= M_i^\tau \cdot (M_1^\rho)\chi. \end{aligned}$$

Thus $Y_i \otimes Y_1 = Y_i \cdot (Y_1)\chi$ and we may substitute this in (3.11). Further by Lemma 7, $(Y_i)\chi = \sum_{j=1}^{i} \mathfrak{R}^{j-1} \cdot (Y_1)\chi$. But $Y_h \chi = \mathfrak{R}_1$ by definition of $\chi$ and so

$$\sum_{j=1}^{h} \mathfrak{R}^{h-1} \cdot (Y_1)\chi = \mathfrak{R}_1$$

whence $(Y_1)\chi$ has an inverse in $A^*(K, G)$. Also since $\mathfrak{R}$ operating on elements of $B_i$ causes a permutation of finite order, $\mathfrak{R}^f = \mathfrak{R}_1$ whence $\mathfrak{R}$ is invertible in $A^*(K, G)$.

With this knowledge we may now use (3.11) to write $Y_i (q < i < h;$ $(i, p) = 1)$ as a polynomial in $W = Y_{q+1} - Y_{q-1} \cdot \mathfrak{R}$ with coefficients in $A^*(K, G)$ and $D_{\bar{Q}}(K, \bar{G})$. Again using (3.11) we may express $Y_j \otimes W$ $(1 \leq j < h;$ $(j, p) = 1)$ as a linear combination of the $Y_k$'s with coefficients in $A^*(K, G)$. Combining these two expressions we may write $Y_j \otimes Y_i$ as a linear combination of elements of $D_Q(K, G)$.

Using the relation $M_i^\tau = Y_i \cdot (M_h^\tau)\chi$ we may then obtain any product $M_j^\tau \otimes M_i^\sigma$, showing that the relations above are sufficient.

## 4 Characters

4.1. LEMMA 12. *If* $|Q| = 1$, $D_Q(K, G)$ *is semisimple.*

*Proof.* When $|Q| = 1$, $D_Q(K, G)$ has as basis $U_1, \cdots, U_s$ since $U_\tau = M_1^\tau$. Let $\psi_1, \cdots, \psi_s$ be the $s$ distinct characters of $A^*(K, G)$ (see Green [4 Theorem 1]). Define the map $\Psi_\sigma : D_Q(K, G) \to \boldsymbol{C}$ by

$$\Psi_\sigma(U_\tau) = \psi_\sigma\{(U_\tau)\chi\} \qquad (\tau, \sigma = 1, \cdots, s).$$

Since $\chi$ is an $A^*(K, G)$-isomorphism, $\Psi_\sigma$ is a character of $D_Q(K, G)$ proving the lemma.

We now assume $|Q| > 1$. Since $A(K, Q)$ can be expressed as an extension of $A(K, \bar{Q})$ with extension relations as in Green [3, 2.8c, d, e; 2.9c, d], $D_Q(K, Q)$ can be expressed as an extension of $D_{\bar{Q}}(K, \bar{Q})$ and so a character $\varphi$ of $D_Q(K, Q)$ will be a character on restriction to $D_{\bar{Q}}(K, \bar{Q})$ and will be consistent with the extension relations, i.e., if $z_r = \varphi(V_r)$ $(1 \leq r < h;$ $(p, r) = 1)$ and $y = z_{q+1} - z_{q-1}$ then

$$(4.11) \qquad \begin{aligned} z_r y &= z_{r+q} - z_{q-r}, & 1 \leq r < q, \\ z_r y &= z_{r+q} - z_{r-q}, & q < r < (p-1)q, \\ z_r y &= z_{r-q} - z_{2h-(r+q)}, & (p-1)q < r < h. \end{aligned}$$

Let $\psi$ be a character of $A^*(K, G)$ and $\varphi$ a character of $D_Q(K, Q)$. Define the mapping $\eta : D_Q(K, G) \to \boldsymbol{C}$ by

$$(4.12) \qquad \eta(M_i^\tau) = \{\psi(R)\}^{(i-1)/2} \psi\{M_1^\tau \chi\} \varphi(V_i).$$

Then $\eta(Y_1) = \psi\{(Y_1)\chi\} \neq 0$ since $(Y_1)\chi$ has an inverse in $A^*(K, G)$. Also $\psi(\mathcal{R}) \neq 0$. For if $\psi(\mathcal{R}) = 0$, since $M_1^\tau = M_1^{\tau'} \cdot \mathcal{R}$ for some $\tau'$, $1 \leq \tau, \tau' \leq s$, then $\psi(M_1^\tau \chi) = \psi(M_1^{\tau'} \chi)\psi(\mathcal{R}) = 0$ whence $\psi\{(Y_1)\chi\} = 0$.

LEMMA 13. *Let $\eta$ be defined as in* (4.12). *Then $\eta$ is a character of $D_Q(K, G)$.*

*Proof.* The proof is by induction on $|Q|$. By Lemma 12 the result holds for $|Q| = 1$. Assume it true for $|Q| < h$. Let $|Q| = h$ and define $\eta$ as in (4.12) where $\psi$ is a character of $A^*(K, G)$ and $\varphi$ a character of $D_Q(K, Q)$. Since $A^*(K, \bar{G}) \cong A^*(K, G)$ and $\varphi$ restricts to a character on $D_{\bar{Q}}(K, \bar{Q})$, $\eta$ restricted to $D_{\bar{Q}}(K, \bar{G})$ by the hypothesis is a character of $D_{\bar{Q}}(K, \bar{G})$. It will thus be a character of $D_Q(K, G)$ if it is consistent with the extension relations formulated in Lemma 11.

$\eta$ is trivially consistent with the relation $M_1^\tau = Y_i \cdot (M_h^\tau)\chi$ $(i = 1, \cdots, h; (i, p) = 1)$. Write $C_i = \eta(Y_i)$ and $\nu = \{\psi(R)\}^{1/2}$. Then $\eta$ is consistent with (3.11) if and only if, with $(r, p) = 1$,

$$C_r(C_{q+1} - \nu^2 C_{q-1}) = C_1(C_{q+r} - \nu^2 C_{q-r}), \qquad 1 < r < q,$$
$$C_r(C_{q+1} - \nu^2 C_{q-1}) = C_1(C_{q+r} - \nu^{2q} C_{r-q}), \qquad q < r < (p-1)q,$$
$$C_r(C_{q+1} - \nu^2 C_{q-1}) = C_1(\nu^{2q} C_{r-q} - \nu^{2r_1} C_{h-r_1}), \qquad (p-1)q < r < h.$$

Since $C_r = \nu^{r-1}\psi\{(Y_1)\chi\}Z_i$ it is trivial, using (4.11), to verify that these equations hold.

THEOREM 1. *$D_Q(K, G)$ is semisimple.*

*Proof.* Since $D_Q(K, Q)$ is semisimple it has $(p-1)q$ distinct characters $\varphi_j$, $1 \leq j \leq (p-1)q$. Similarly $A^*(K, G)$ has $s$ distinct characters $\psi_\sigma$ $(1 \leq \sigma \leq s)$. Thus there exist $s(p-1)q$ characters $\eta_{j\sigma}$ defined as in (4.12). These are distinct. For if $\eta_{j\sigma} = \eta_{k\rho}$ then $\eta_{j\sigma}(M_1^\tau) = \eta_{k\rho}(M_1^\tau)$ $(1 \leq \tau \leq s)$, i.e., $\psi_\sigma(M_1^\tau \chi) = \psi_\rho(M_1^\tau \chi)$ for all $\tau$ since $M_h^\tau \chi$ is the sum of terms of the form $M_1^\tau \chi$. By Lemma 4, $\psi_\sigma = \psi_\rho$. Thus $\varphi_j(V_i) = \varphi_k(V_i)$ $(1 \leq i < h, (i, p) = 1)$ whence $i = j$.

Since the dimension of $D_Q(K, G)$ is $s(p-1)q$, $D_Q(K, G)$ is semisimple.

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