COMMUTATOR CLOSED GROUPS

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Introduction

The commutator operation is a useful tool of group theory. However, only very little is known about mappings that carry every element g of G into the commutator $x \circ g$ for fixed x. If the group has the property that a certain power of each of such mappings maps every element onto 1, then G is an engel group, the aspects of which have been investigated by Zorn [1], Levi [1], Baer [1], Gruenberg [1], [2] and the author [1]. We will investigate here another class of groups, namely those groups, where the set of the mappings indicated above is a semigroup with respect to the usual product operation. Actually, corresponding to the different ways of commutator bracketing, we have two definitions:

G is right commutator closed, if for any two elements a, b of the group G there is an element $c \in G$ such that $a \circ (b \circ g) = c \circ g$ holds for all $g \in G$.

G is left commutator closed, if for any two elements a, b of the group G there is an element $c \in G$ such that $(g \circ b) \circ a = g \circ c$ holds for all $g \in G$.

These two classes of groups contain for instance the two-engel groups considered by Levi [1], for in these groups we have the identities

$$x \circ (y \circ z) = (y \circ x) \circ z; \qquad (z \circ y) \circ x = z \circ (x \circ y)$$

which give us directly the element c needed. Unlike the two-engel groups, however, the finite right commutator closed groups may be nilpotent of any class wanted, as Example 1 in Section 5 shows. It will be shown, that right and left commutator closed groups are metabelian (Theorem 1.4). Right commutator closed groups are nilpotent whenever $G/C(G')$ is finitely generated (Theorem 2.1). This may be considered as best possible because there are infinitely generated right commutator closed groups that are neither residually nilpotent nor locally nilpotent (and therefore not Z-A-groups either); see the examples in Section 5. Right commutator closed groups are left commutator closed (Lemma 1.3); see Section 3 for left commutator closed groups that are not right commutator closed.

Not every subgroup of a right (left) commutator closed group is itself right (left) commutator closed; a counterexample is Example 2 in Section 5. So we define the corresponding local properties as follows:

The group G is locally right (left) commutator closed, if any finitely generated

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subgroup of G is contained in a finitely generated right (left) commutator closed subgroup of G.

Please note that right commutator closed groups need not be locally right commutator closed, as shown by Example 2 in Section 5. To avoid this inconvenience seems to be impossible since we deal with a closure property.

In this paper we see that the consequence arising from conditions that are only different with respect to "right" and "left" may in fact be different. In an earlier paper [2] the author could prove that the statement "$a^{(n+1)} \circ g = 1$ for all $g \in G$" is a consequence of "$g^{(n)} \circ a = 1$ for all $g \in G$". There was no indication, however, whether in the same way $g \circ^{(n+1)} a = 1$ for all $g \in G$ if $a \circ^{(n)} g = 1$ for all $g \in G$, and this paper does not add hope for this—to the knowledge of the author, undecided—conjecture.

Definitions and notations

$x \circ y = x^{(1)} \circ y = x \circ^{(1)} y = x^{-1}y^{-1}xy.$

$x^{(n)} \circ y = x \circ (x^{(n-1)} \circ y)$; $x \circ^{(n)} y = (x \circ^{(n-1)} y) \circ y$ for $n > 1$.

$x \circ B$ is the subgroup generated by all $x \circ b$ with $b \in B$.

$A \circ B$ is the subgroup generated by all $a \circ B$ with $a \in A$.

$Z(G)$ = center of G.

$C(A)$ = centralizer of A.

$N(A)$ = normalizer of A.

The series

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_k \supseteq \cdots,$$

where $G_n = G_{n-1} \circ G$ for $n > 1$, is called the *lower central series*, while the series

$$1 \subseteq Z(G) \subseteq Z_2(G) \subseteq \cdots \subseteq Z_k(G) \subseteq \cdots,$$

where $Z_n(G)/Z_{n-1}(G) = Z(G/Z_{n-1}(G))$ for $n > 1$ and not a limit ordinal, $Z_n(G) = \bigcup_{k<n} Z_k(G)$ if n is a limit ordinal, is called the *upper central series*.

The group G is *nilpotent*, if the lower central series ends after a finite number of steps with 1. This is equivalent to the fact that G is a finite member of the upper central series. G is *nilpotent of* (finite) *class* c, if $G_{c+1} = 1$; this is equivalent to $Z_c(G) = G$.

G is a *Z-A-group*, if G is some (possibly transfinite) term of the upper central series.

G is *residually nilpotent* if 1 is the intersection of all normal subgroups of G with nilpotent quotient group (and this is equivalent to $\bigcap_{i=1}^{\infty} G_i = 1$).

1. The general case

THEOREM 1.1. *The following properties of the group G are equivalent:*

(i) *G is right commutator closed.*

(ii) *To every pair x, y of elements in G there exists an element $z \in G$ such that*

$$(x \circ g)^{-1}(y \circ g) = z \circ g$$

for every $g \in G$.

(iii) *To every finite set of elements* $x_1, x_2, \cdots, x_n$ *in* G *there exists an element* $z \in G$ *such that*

$$(x_1 \circ (x_2 \circ (\cdots \circ (x_n \circ g) \cdots))) = z \circ g$$

for every $g \in G$.

(iv) *To every finite set of elements* $x_1, x_2, \cdots, x_n$ *in* G *and to every sequence* $m_1, m_2, \cdots, m_n$ *of integers there exists an element* $z \in G$ *such that*

$$(x_1 \circ g)^{m1}(x_2 \circ g)^{m2} \cdot \cdots (x_n \circ g)^{m_n} = z \circ g$$

for every $g \in G$.

Proof. Assume G is right commutator closed. Then to every pair a, b of elements in G there exists an element c in G such that $(a \circ (b \circ g)) = c \circ g$ for every $g \in G$. Hence

$$(a^{-1}g^{-1}a)(a^{-1}b^{-1}gba)(b^{-1}g^{-1}b)g = (c^{-1}g^{-1}c)g,$$

$$g^{-1}(b^{-1}gb)(ab^{-1}g^{-1}ba^{-1}) = ac^{-1}g^{-1}ca^{-1},$$

$$(b \circ g)^{-1}(ba^{-1} \circ g) = ca^{-1} \circ g.$$

So for $x = b$ and $y = ba^{-1}$ we have found the element z. But for every pair x, y there exists a pair a, b satisfying these two equations; and the element z needed for (ii) can be found. Hence (ii) is a consequence of (i). By (ii) there is an operation $\oplus$ of all elements in $G/Z(G)$ defined as follows:

$$(x \oplus y) \circ g = (x \circ g)(y \circ g).$$

For if we let $y = 1$ in (ii) we obtain an element $\bar{x}$ such that

$$(x \circ g)^{-1} = \bar{x} \circ g \qquad \text{for all } g \in G.$$

Hence

$$(x \circ g)(y \circ g) = (\bar{x} \circ g)^{-1}(y \circ g) = z \circ g$$

for all $g \in G$ and a certain $z \in G$. The element z exists by (ii) and is defined mod $Z(G)$. This yields $x \oplus y = z$. By (ii) (letting $y = 1$) every element has an inverse with respect to this operation $\oplus$, and clearly the operation is associative. Then equation (ii) says that $G/Z(G)$ is a group with respect to operation $\oplus$, and (iv) follows. Furthermore we deduce

(iv′) The set of all $x \circ g$ coincides with $x \circ G$.

If we use the equation

$$x_1 \circ (x_2 \circ g) = (x_1 \circ g)(x_2 x_1 \circ g)^{-1}(x_2 \circ g)$$

$n - 1$ times, we are able to give the commutator

$$x_1 \circ (x_2 \circ (\cdots \circ (x_n \circ g) \cdots))$$

the form of a product of simple commutators $a \circ g$ with a independent of g. Therefore (iii) follows from (iv). But (i) is just a special case of (iii). This proves the equivalence of all four properties.

LEMMA 1.2. *If the group G is right commutator closed, then $x \in G \circ x$ if and only if $x = 1$.*

Proof. By Theorem 1.1(iv′) and (iv), x is contained in $G \circ x$ if and only if $x = w \circ x$ for a certain $w \in G$. But this yields $w^{-1}x^{-1}w = 1$, proving $x = 1$.

LEMMA 1.3. *Right commutator closed groups are left commutator closed.*

Proof. By Theorem 1.1(iv) there exists for every element $x \in G$ another element $\bar{x}$, such that

$$x \circ g = (\bar{x} \circ g)^{-1} = g \circ \bar{x} \qquad \text{for all } g \in G.$$

As G is right commutator closed, there is an element $z \in G$ such that

$$\bar{y} \circ (\bar{x} \circ g) = z \circ g \qquad \text{for all } g \in G.$$

Therefore

$$(g \circ x) \circ y = (\bar{x} \circ g) \circ y = \bar{y} \circ (\bar{x} \circ g) = z \circ g = g \circ \bar{z} \quad \text{for all } g \in G,$$

which proves the lemma.

Remark. See Theorem 3.3 for a partial converse.

THEOREM 1.4. *Left commutator closed groups and right commutator closed groups are metabelian.*

Proof. By Lemma 1.3 it suffices to show that left commutator closed groups are metabelian.

Now assume that for any two elements a, b of G there exists an element c such that $(g \circ a) \circ b = g \circ c$ for all $g \in G$. Letting $g = a$ we see that c and a commute. Furthermore

$$aga^{-1} \circ a = a(g \circ a)a^{-1}$$

and

$$aga^{-1} \circ c = a(g \circ c)a^{-1}.$$

Therefore

$$\begin{aligned} a(g \circ a)^{-1}a^{-1}b^{-1}a(g \circ a)a^{-1}b &= a(g \circ a)a^{-1} \circ b \\ &= (aga^{-1} \circ a) \circ b \\ &= aga^{-1} \circ c \\ &= a(g \circ c)a^{-1} \\ &= a((g \circ a) \circ b)a^{-1} \\ &= a(g \circ a)^{-1}b^{-1}(g \circ a)ba^{-1}. \end{aligned}$$

Consequently

$$b^{-1}a(g \circ a)a^{-1}b = ab^{-1}(g \circ a)ba^{-1},$$

$$(1) \qquad (a \circ b^{-1}) \circ (g \circ a) = 1 \qquad \text{for all } a, b, g \in G.$$

Now

$$g^{-1}(f \circ a)g \circ b = g^{-1}(f \circ a)^{-1}gb^{-1}g^{-1}(f \circ a)gb$$
$$= (g \circ a)(fg \circ a)^{-1}(fgb \circ a)(gb \circ a)^{-1}$$

by $x^{-1}(y \circ z)x = (yx \circ z)(x \circ z)^{-1}$.

This implies by (1)

$$\begin{aligned}(g \circ a)^{-1}(g^{-1}(f \circ a)g \circ b)(g \circ a) &= (g \circ a)^{-1}(g \circ a)(fg \circ a)^{-1}(fgb \circ a)(gb \circ a)^{-1}(g \circ a)\\ &= (g \circ a)(fg \circ a)^{-1}(fgb \circ a)(gb \circ a)^{-1}\\ &= g^{-1}(f \circ a)g \circ b.\end{aligned}$$

By $yx \circ z = x^{-1}(y \circ z)x(x \circ z)$ and the preceding argument we obtain

$$\begin{aligned}((fg \circ a) \circ b) &= (g^{-1}(f \circ a)g)(g \circ a)) \circ b\\ &\qquad (\text{letting } y = f,\ x = g \text{ and } z = a)\\ &= (g \circ a)^{-1}((g^{-1}(f \circ a)g) \circ b)(g \circ a)((g \circ a) \circ b)\\ &\qquad (\text{letting } y = g^{-1}(f \circ a)g,\ x = g \circ a \text{ and } z = b)\\ &= ((g^{-1}(f \circ a)g) \circ b)((g \circ a) \circ b).\end{aligned}$$

But

$$fg \circ c = g^{-1}(f \circ c)g(g \circ c) = g^{-1}((f \circ a) \circ b)g((g \circ a) \circ b),$$

and this implies

$$g^{-1}((f \circ a) \circ b)g = (g^{-1}(f \circ a)g) \circ b,$$
$$g^{-1}b^{-1}(f \circ a)bg = b^{-1}g^{-1}(f \circ a)gb;$$

$$(b^{-1} \circ g^{-1}) \circ (f \circ a) = 1 \qquad \text{for all } a, b, f, g \in G. \tag{2}$$

Therefore the group G is metabelian.

LEMMA 1.5. *If the group G is right (left) commutator closed, then the normalizer of the subgroup U of G is right (left) commutator closed, whenever the minimum condition is satisfied for subgroups isomorphic to U in G.*

Proof. Let N be the normalizer of U in G. The element x is contained in N if and only if all the commutators $x \circ u$ $(u \circ x)$ are contained in U, where u is in U. So if x and y are contained in N, then $x \circ u$ and $y \circ (x \circ u)$ $(u \circ x$ and $(u \circ x) \circ y)$ are contained in U for all u in U. As G is right (left) commutator closed, there is an element $z \in G$ satisfying $y \circ (x \circ g) = z \circ g$ $((g \circ x) \circ y = g \circ z)$ for all $g \in G$. Since $z \circ U = U \circ z \subseteq U$, we have $z^{-1}Uz \subseteq U$. But the descending chain of isomorphic subgroups $z^{-k}Uz^{k}$ with positive integers k is only finite by the minimum condition, so for certain k we have $z^{-k}Uz^{k} = z^{-k-1}Uz^{k+1}$; proving $U = z^{-1}Uz$. Thus z is contained in N, proving the lemma.

We will denote by the endomorphism ring induced by G in (the abelian normal subgroup) A the ring spanned by the automorphisms of A induced by G. It will cause no confusion to denote by the element $g \in G$ at the same time the mapping of a onto $a^g = g^{-1}ag$ for all $a \in A$; and if u and v are endomorphisms, then $a^{u+v} = a^u a^v$ for all $a \in A$ as usual.

LEMMA 1.6. *If G is right commutator closed, then in the endomorphism ring induced by G in G' the set of elements $1 - g$ with $g \in G$ is an ideal J.*

Proof. By Theorem 1.4 we know that G' is abelian; so we may form the commutative endomorphism ring P of G' spanned by all the automorphisms induced by G in G'. The set J of all $1 - g$, where $g \in G$, is an additive subgroup of P by Theorem 1.1(ii). To show further that J is an ideal it suffices to show $gJ \subseteq J$ for all $g \in G$. However, if $1 - h$ is an arbitrary element of J, then $g(1 - h) = g - gh = (1 - gh) - (1 - g)$ belongs to J for all $g \in G$. So J is an ideal of P.

2. Groups G with finitely generated $G/C(G')$

In this section we want to prove

THEOREM 2.1. *If G is a right commutator closed group such that the quotient group $G/C(G')$ is finitely generated, then G is nilpotent.*

Proof. G is metabelian by Theorem 1.4. We want to show that for every x in G there is an integer n dependant only on x such that $x^{(n)} \circ y = 1$ for all $y \in G'$. Then the Theorem follows by Gruenberg [2, Theorem 3.1, p. 449].

Assume now the existence of an element $x \in G$, which does not satisfy an equation of the form $x^{(n)} \circ y = 1$ for all $y \in G'$. We consider the endomorphism ring X of G' sapnned by 1 and x. Clearly X is a quotient ring of the ring of polynomials in x with integral coefficients. We are especially interested in the polynomials in $1 - x$ with constant term 1, for in the endomorphism ring P induced by G in G' these correspond to automorphisms induced by G in G'. Let the group of automorphisms induced by G in G', which is essentially $G/C(G')$, be generated by k elements. Take $k + 1$ irreducible polynomials in $1 - x$ with constant term 1, for instance $P_1, \cdots, P_{k+1}$. They generate a subgroup of $G/C(G')$, which has a basis of k elements and is abelian. Hence, by suitable choice of the subscripts, we have

$$\prod_{i \in I_1} P_i^{n_i} = \prod_{i \in I_2} P_i^{n_i} \tag{1}$$

where $0 \leq n_i$, not all $n_i = 0$.

Now the ring of polynomials with integral coefficients is a unique factorization domain, so the equation above, which is true in P and in X, cannot be true in the ring of polynomials. As all $P_i \equiv 1 \bmod (1 - x)$, the difference of the right and the left side of (1) is divisible by $1 - x$, and we deduce from (1) an equation of the form

$$(1 - x)^k(s + (1 - x)f(1 - x)) = 0, \tag{2}$$

where $k > 0$ and $s \neq 0$ are integers, $f(1 - x)$ is a polynomial in $1 - x$. If $s = \pm 1$, then $s + (1 - x)f(1 - x)$ is a unit of P, and (2) yields in P (and therefore in X) $(1 - x)^k = 0$, contrary to the hypothesis on x. So s is not equal to 1, 0, and -1. By Lemma 1.6 we know of the existence of an element z in G such that $(1 - x)f(1 - x) = 1 - z$; furthermore there are z_i such that $i(1 - x)f(1 - x) = 1 - z_i$. By formula (2) we have

$$(3) \qquad (1 - x)^k z_i = (1 - x)^k(1 + si).$$

Now it is well known that in the set of integers $1 + si$ there are infinitely many that are pairwise relatively prime. Take $k + 1$ such integers $1 + si$; then the corresponding elements z_i will satisfy an equation of the form

$$(4) \qquad \prod_{i \in I_1} z_i^{n_i} = \prod_{i \in I_2} z_j^{n_j},$$

as the basis of $G/C(G')$ has only k elements and $z_i \in G$. We deduce from (3)

$$(5) \qquad (1 - x)^k \prod_{i \in I_1} (1 + si)^{n_i} = (1 - x)^k \prod_{j \in I_2} (1 + sj)^{n_j}.$$

As the ring of integers is a unique factorization domain, we obtain from (5)

$$(6) \qquad (1 - x)^k \cdot m = 0,$$

where $m \neq 0$. Assume p divides m and $m = pm'$. Then $m'(1 - x)^k X$ is a quotient ring of the polynomial ring in $(1 - x)$ modulo p. Now we take $k + 1$ polynomials $Q_1, \cdots, Q_{k+1}$ in $(1 - x)$ with the following properties: they are irreducible mod p and have constant term 1; and they are different mod p. As $G/C(G')$ is generated by k elements and the Q_i correspond to automorphisms, there is an equation of the form

$$\prod_{i \in I_1} Q_i^{n_i} = \prod_{j \in I_2} Q_j^{n_j},$$

and because the ring of polynomials mod p is a unique factorization domain, the coefficients of the terms $(1 - x)^t$ of both the sides of the equation can not be congruent mod p for all t. By multiplication with $(1 - x)^k m'$ and by use of (6) we find a new equation of the form

$$(8) \qquad (1 - x)^{k+k_1} m'(t + (1 - x)f_1(1 - x)) = 0,$$

where $t \not\equiv 0 \bmod p$ is a constant. By multiplication of (8) with a suitable constant we are able to get $t \equiv 1 \bmod p$, therefore we may assume without loss of generality $t = 1$. But then the right-hand bracket in (8) is a unit of P, and (8) yields

$$(9) \qquad (1 - x)^{k+k_1} m' = 0.$$

By an obvious inductive argument on the number of prime factors of m we are able to show the existence of an integer k_i such that

$$(10) \qquad (1 - x)^{k_i} = 0,$$

a contradiction proving the nonexistence of our element x. Thus Theorem 2.1 has been proved.

Remark. Obviously the case of the finitely generated group G is contained in Theorem 2.1. But conversely not every group satisfying the conditions of Theorem 2.1 is finitely generated, as may be seen by the following example: Take the direct product A of infinitely many cyclic groups of order p^3, p odd. Define $x^{p^2} = 1$ and $x^{-1}ax = a^{1+p}$ for all $a \in A$. Then $\{x, A\}$ is right commutator closed (see Section 5), $\{x^p, A\}$ is the centralizer of the commutator group of $\{x, A\}$, and the quotient group is cyclic, while $\{x, A\}$ is not finitely generated.

3. Left commutator closed groups

If the reader had not been warned by the remarks of the introduction and following Lemma 1.3, he might be tempted to assume that the properties of being right commutator closed and of being left commutator closed are equivalent. While by Lemma 1.3 all right commutator closed groups are left commutator closed, the converse is not true. A counterexample is the noncommutative group of order 6. We shall see that this group is in a sense typical of the situation.

LEMMA 3.1. *If G is a left commutator closed torsion group, then the following statements are true:*

(I) *G^2 is locally nilpotent.*

(II) *If x is an element of a locally nilpotent normal subgroup of G but no right engel element, then the order of x is divisible by* 3.

Proof. It is well known that an element x is contained in a locally nilpotent normal subgroup of the soluble group G, if x is a left engel element (see Gruenberg [1, Theorem 4, p. 160]). Therefore it suffices for the proof of (I) to show that x^2 is a left engel element for every $x \in G$.

If x is not a left engel element there exists $y = y(x)$ such that $y \circ^{(k)} x \neq 1$ for all positive integers k. The group G is an extension of a locally finite group by another locally finite group (for abelian torsion groups are locally finite); therefore G is locally finite by Schmidt's Theorem (see Kurosh [1, p. 153]). Hence $H = \{x, y\}$ is finite. In the endomorphism ring P induced by G in G' the elements x and y generate a subring a quotient ring of which is the endomorphism ring X induced by H in H'. Our assumption that x was not a left engel element in H has its expression in X as

$$(1) \qquad (x - 1)^k \neq 0 \qquad \text{for all positive } k.$$

Now H and X is finite, so there exist powers of $x - 1$ that equal each other. Then we have for some $r \geq 0$, $s > 0$

$$(2) \qquad (x - 1)^r = (x - 1)^{r+s}.$$

Without loss of generality we may assume that s is minimal satisfying equation (2). If $s = 2n$, we deduce from (2)

$$(x - 1)^r(1 - (x - 1)^n)(1 + (x - 1)^n = 0.$$

Now G is left commutator closed; therefore in P there is an element z of the automorphism group of G' induced by G such that $(x - 1)^n = z - 1$. This shows $(1 + (x - 1)^n)^k = 1$ for a certain integer k, because G is a torsion group. This relation remains true in X, and we would find

$$(x - 1)^r(1 - (x - 1)^n) = 0,$$

contrary to the minimality of $s = 2n$. Thus s is odd. As x is not a left engel element in H, the inverse x^{-1} is not a left engel element in H either; and there are integers $u \geq 0$ and $v > 0$ such that

$$(3) \qquad (x^{-1} - 1)^u = (x^{-1} - 1)^{u+v}.$$

Without loss of generality we may assume that v is minimal; under this assumption v is odd. Then by iterated use of (2) and (3) and by multiplication with $(x - 1)^u$ and $(x^{-1} - 1)^r$ respectively we obtain

$$(4) \qquad (x - 1)^{u+r} = (x - 1)^{u+r+sv}$$

$$(5) \qquad (x^{-1} - 1)^{u+r} = (x^{-1} - 1)^{u+r+sv}.$$

If we multiply (5) by $(-x)^{u+r+sv}$ we find

$$(6) \qquad (x - 1)^{u+r}(-x)^{sv} = (x - 1)^{u+r+sv}.$$

The comparison with (4) yields

$$(7) \qquad (x - 1)^{u+r} = -x^{sv}(x - 1)^{u+r}.$$

If x^2 were not a left engel element, then we would obtain for x^2 the same types of equation as we had for x, and an equivalent in x^2 for (7) would be

$$(8) \qquad (x^2 - 1)^t = -(x^2 - 1)^t(x^2)^w,$$

where the integer w may be assumed without loss of generality to be odd. By iterated use of (8) we deduce $(x^2 - 1)^t = -(x^2 - 1)^t x^{2wsv}$, observing that sv is odd. On the other hand (7) leads us to $(x - 1)^{u+r} = x^{2wsv}(x - 1)^{u+r}$. Therefore we have

$$\begin{aligned}(x^2 - 1)^{t+u+r} &= (x^2 - 1)^t(x + 1)^{u+r}(x - 1)^{u+r} \\ &= (x^2 - 1)^t(x + 1)^{u+r}(x - 1)^{u+r}x^{2wsv} \\ &= (x^2 - 1)^{t+u+r}x^{2wsv}\end{aligned}$$

by (7), while by (8) we obtain

$$\begin{aligned}(x^2 - 1)^{t+u+r} &= (x^2 - 1)^{u+r}(x^2 - 1)^t \\ &= -(x^2 - 1)^{u+r}(x^2 - 1)^t x^{2wsv}.\end{aligned}$$

This is impossible unless

$$(9) \qquad 2(x^2 - 1)^{t+u+r} = 0.$$

By use of (2) we obtain

$$\begin{aligned}(x^2 - 1)^{t+u+r}(1 + (x - 1)^s) &= (x^2 - 1)^{t+u+r}(1 - (x - 1)^s)\\ &= (x^2 - 1)^{t+u}(x + 1)^r(x - 1)^r(1 - (x - 1)^s)\\ &= 0,\end{aligned}$$

and $1 + (x - 1)^s$ is a unit of P, which proves

$$(x^2 - 1)^{t+u+r} = 0, \tag{10}$$

contrary to the assumption that x^2 was not a left engel element. This proves statement (I) of the theorem. To prove (II), we consider the element $z \in G$ (which exists by definition) such that $(x - 1)^s = z - 1$. We obtain by (2)

$$2(x - 1)^r = (x - 1)^r z. \tag{11}$$

By (I), z^2 is a left engel element, so there is an integer k such that

$$0 = (x - 1)^r(z^2 - 1)^k = (x - 1)^r 3^k. \tag{12}$$

If y is contained in a locally nilpotent normal subgroup, this proves (II).

COROLLARY. *In a finite left commutator closed group all the elements of order prime to* 6 *are contained in the hypercenter.*

Finally we prove the equivalence of being left and right commutator closed for nilpotent groups.

THEOREM 3.2. *If G is nilpotent and left commutator closed, then G is right commutator closed.*

Proof. By induction on the nilpotency class of G. For nilpotent groups of class at most 2 there is nothing to prove. Now assume the class of G is $c > 2$, and the theorem has already been proved for all nilpotent groups of class at most $c - 1$. Then $G/Z(G)$, which is of class $c - 1$, is left commutator closed and therefore right commutator closed by induction hypothesis By Theorem 1.1(ii) we are able to find an element x' for each $x \in G$ such that

$$g \circ x' \equiv x \circ g \mod Z(G) \qquad \text{for all } g \in G. \tag{1}$$

Therefore

$$\begin{aligned} y \circ (x \circ g) &= ((x \circ g) \circ y)^{-1} = ((g \circ x') \circ y)^{-1}\\ &= (g \circ z)^{-1} \qquad = z \circ g \qquad \text{for all } g \in G,\end{aligned} \tag{2}$$

where z exists because G is left commutator closed; and this proves the theorem.

4. Local properties

THEOREM 4.1. *If the group G is locally right commutator closed, then G has the following properties:*

(i) *G is metabelian.*
(ii) *G is locally nilpotent.*

(iii) *For every element* $x \neq 1$ *of* G *we have* $x \notin G \circ x$
(iv) G *is locally left commutator closed.*

Proof. The subgroup generated by the arbitrary four elements w, x, y, z is contained in a finitely generated right commutator closed subgroup, which is metabelian by Theorem 1.4. Hence

$$(w \circ x) \circ (y \circ z) = 1,$$

which proves (i).

Every finitely generated subgroup of G is contained in a finitely generated right commutator closed subgroup, which is nilpotent by Theorem 2.2. Hence (ii) is true.

Now assume x were contained in $G \circ x$; then there would be elements $g_1, g_2, \cdots, g_n$ and integers $m_1, m_2, \cdots, m_n$ such that

$$x = (g_1 \circ x)^{m_1}(g_2 \circ x)^{m_2} \cdots (g_n \circ x)^{m_n}.$$

But the group $M = \{x, g_1, g_2, \cdots, g_n\}$ is contained in a right commutator closed subgroup K, and x is not contained in $K \circ x$ by Lemma 1.2, which contradicts the assumption and proves (iii).

Every finite set of elements is contained in a finitely generated right commutator closed group, which is left commutator closed by Lemma 1.3; hence (iv) is true.

Remark. As there are finitely generated left commutator closed groups that are not right commutator closed, the properties of being locally left commutator closed and locally right commutator closed are not equivalent.

THEOREM 4.2. *If* G *is a locally left commutator closed torsion group, then* G *has the following properties:*
(i) G *is metabelian.*
(ii) G^2 *is locally nilpotent.*
(iii) *The elements of* G' *of order prime to* 3 *are right engel elements.*

Proof. The subgroup generated by the arbitrary four elements w, x, y, z is contained in a left commutator closed group, which is metabelian by Theorem 1.4; hence $(w \circ x) \circ (y \circ z) = 1$, which proves (i). On the other hand every square is a left engel element in this subgroup, so $y \circ^{(n)} (x^2) = 1$ for almost all integers n, and (ii) follows.

Now if z has an order prime to 3 and is contained in G', then it is contained in M' for some finitely generated subgroup M of G. Let x be an arbitrary element of G. Then $\{x, M\}$ is contained in a finitely generated left commutator closed subgroup U of G. Obviously $z \in U'$, and, by Lemma 3.2, $z \circ^{(n)} x = 1$ for almost all integers n. This shows the validity of (iii).

5. Examples

In order to simplify the arguments we begin with a general construction principle.

LEMMA 5.1. *If A is an abelian normal subgroup of $G = AB$ and B is abelian, then the following statements are true*:

(R) *If for every pair $b_1, b_2 \in B$ there exists an element $b_3 \in B$ such that $b_1 \circ (b_2 \circ a) = b_3 \circ a$ for all $a \in A$, then G is right commutator closed.*

(L) *If for every pair $b_1, b_2 \in B$ there exists an element $b_3 \in B$ such that $(a \circ b_2) \circ b_1 = a \circ b_3$ for all $a \in A$, then G is left commutator closed.*

Proof. First we collect identities we shall need further on. We have

$$A' = B' = 1, \tag{1}$$

and as $G/A \cong B/B \cap A$ is abelian, we observe

$$G' \subseteq A. \tag{2}$$

Every element of G can be written as a product ba where $b \in B$ and $a \in A$. Then (1) and (2) yield

$$ba \circ g = a^{-1}(b \circ g)a(a \circ g) = (b \circ g)(a \circ g) \text{ for all } a \in A, b \in B \text{ and } g \in G. \tag{3}$$

Now we want to show that G is right (left) commutator closed if and only if for every pair $x, y \in G$ there is an element $z \in G$ such that

$$x \circ (y \circ w) = z \circ w \quad ((w \circ y) \circ x = w \circ z)$$

for all $w \in A$ and all $w \in B$. The necessity of this condition is clear. However for the right case we have

$$\begin{aligned} x \circ (y \circ ba) &= x \circ ((y \circ a)(y \circ b)) && \text{by (3)} \\ &= (x \circ (y \circ a))(x \circ (y \circ b)) && \text{by (1)} \\ &= (z \circ a)(z \circ b) \\ &= z \circ ba && \text{by (3),} \end{aligned}$$

and the left case follows in the same manner:

$$\begin{aligned} (ba \circ y) \circ x &= ((b \circ y)(a \circ y)) \circ x \\ &= ((b \circ y) \circ x)((a \circ y) \circ x) \\ &= (b \circ z)(a \circ z) \\ &= ba \circ z. \end{aligned}$$

Now if $g_1 = b_1 a_1$, $g_2 = b_2 a_2$, $g_3 = b_3 a_3$ with $a_1, a_2, a_3 \in A$ and b_1, b_2, b_3 as in the lemma, then

$$g_1 \circ (g_2 \circ x) = g_3 \circ x \quad ((x \circ g_2) \circ g_1 = x \circ g_3)$$

if $x \in A$, as can be seen by (1)–(3). We want to choose a_3 such that the formula is true for any $x = b$ in B also. By (1)–(3) this is true for the right commutator closed case, if

$$a_3 \circ b = g_3 \circ b = g_1 \circ (g_2 \circ b) = g_1 \circ (a_2 \circ b) = b_1 \circ (a_2 \circ b) = (b_1 \circ a_2) \circ b;$$

similarly it is true for the left commutator closed case, if

$$b \circ a_3 = b \circ g_3 = (b \circ g_2) \circ g_1 = (b \circ a_2) \circ g_1 = (b \circ a_2) \circ b_1 = b \circ (a_2 \circ b_1).$$

Therefore with $\qquad a_3 = b_1 \circ a_2 \quad (a_3 = a_2 \circ b_1)$

we have constructed $g_3 = b_3 a_3$ needed to prove the lemma.

This lemma enables us to construct new right or left commutator closed groups out of known ones; for if N is a normal subgroup of the right or left commutator closed group G and $N \subseteq G'$, then the splitting extension of N by the group of automorphisms induced by G in N (the holomorph of N in G) is right or left commutator closed.

Example 1. Take a cyclic group A of order p^n and all the automorphisms of this group mapping each element x onto the *k-th* power x^k, where $k \equiv 1 \bmod p$. It can be seen easily that these automorphisms form a group B of order p^{n-1}, which, together with the cyclic group A, satisfies the conditions of Lemma 5.1(R). Therefore $G = AB$ is right commutator closed. We observe $G_n \neq 1$ and $G_{n+1} = 1$, which shows that there is no bound on the nilpotency class for right commutator closed groups.

Example 2. Denote by Q the ring of rational functions in one variable x, where the denominator polynomials have the constant term 1 and the coefficients of the polynomials are taken either from the integers or from a field of order p. Then multiplication with a polynomial with constant term 1 is an automorphism of Q. Denote the additive group of Q by $\tilde{Q}$, and call U the group of all multiplications (and divisions) by polynomials with constant term 1. By Lemma 5.1(R) the group $G = \tilde{Q}U$ is right commutator closed. The intersection of the terms of the descending central series is trivial, for in G_n we have all quotients of $\tilde{Q}$ where the numerator polynomial has rx^n as leading term ($G_n = x^nQ$). G has trivial centre, and the group $\{q, u\}$ generated by the elements $q \in \tilde{Q}$ and $u \in U$ is nilpotent only if one of the elements equals 1. Thus G is not locally nilpotent.

If u corresponds to the multiplication of the quotients with $1 + x$, and if $q \in \tilde{Q}$ corresponds to the constant 1, then the group $\{q, u\}$ is its own normalizer and is not commutator closed.

Example 3. Denote by P the additive group of all formal power series $y = \sum_{i=d}^{-\infty} a_i x^i$ in one variable x; the coefficients a_i may be chosen as in Example 2. All the multiplications of the power series by those power series of the form

$$t = 1 + \sum_{i=-1}^{-\infty} b_i x^i$$

are automorphisms of P. Call the group of all these automorphisms T. We form equivalence classes in P by calling two elements in P equivalent, if all the coefficients a_i for $i = 0, 1, 2, \cdots$ are equal. The additive group P^* of these equivalence classes admits the automorphisms in T. Hence by Lemma 5.1(R) the group $F = P^*T$ is right commutator closed, and all elements in

P^* with leading coefficient ax^n are contained in $Z_{n+1}(F)$; therefore F is a Z-A-group. Furthermore we have $F_2 = F_3$, which proves F is not residually nilpotent.

If we take the direct product of the groups G and F of Example 2 and 3, we have a group that is neither residually nilpotent nor a Z-A-group.

Literature

R. Baer

1. *Engelsche Elemente Noetherscher Gruppen*, Math. Ann., vol. 133 (1957), pp. 256–270.

K. W. Gruenberg

1. *The engel elements of a soluble group*, Illinois J. Math., vol. 3 (1959), pp. 151–168.
2. *The upper central series in soluble groups*, Illinois J. Math., vol. 5 (1961), pp. 436–466.

H. Heineken

1. *Engelsche Elemente der Länge drei*, Illinois J. Math., vol. 5 (1961), pp. 681–707.
2. *Eine Bemerkung über engelsche Elemente*, Arch. Math., vol. XI (1960), p. 321.

A. G. Kurosh

1. *The theory of groups*, second English edition, New York, Chelsea, 1960, translated from the Russian.

F. W. Levi

1. *Groups in which the commutator operation satisfies certain algebraic conditions*, J. Indian Math. Soc., vol. VI (1942), pp. 88–96.

M. Zorn

1. *Nilpotency of finite groups*, Bull. Amer. Math. Soc., vol. 42 (1936), pp. 485–486.

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