ON TOPOLOGIES OF FINITE W^*-ALGEBRAS

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1. Let M be a W^*-algebra (namely, a C^*-algebra with a dual structure as a Banach space [4], [7]).

We may consider the following five typical topologies on M: (1) the norm topology u as a Banach space; (2) the Mackey topology τ defined by uniform convergences on every relatively $\sigma(M_*, M)$-compact convex set of M_*, where M_* is the associated space of M (namely, M is the dual of M_*); (3) the topology $\mathfrak{s}^*$ defined by a family of semi-norms $\{\alpha_\varphi, \alpha_\varphi^* \mid \text{all positive } \varphi \in M_*\}$, where $\alpha_\varphi(x) = \varphi(x^*x)^{1/2}$ and $\alpha_\varphi^*(x) = \varphi(xx^*)^{1/2}$ for $x \in M$; (4) the topology $\mathfrak{s}$ defined by a family of semi-norms $\{\alpha_\varphi \mid \text{all positive } \varphi \in M_*\}$; (5) the weak*-topology σ (namely, $\sigma(M, M_*)$).

We can easily see that $u \prec \tau$ if M is infinite-dimensional. By introducing the τ-topology into W^*-algebras, the author [7] simplified the proof that two topologies $\mathfrak{s}$ and σ have the same dual M_* as a set—that is, we showed $\tau \precsim \mathfrak{s}^* \precsim \mathfrak{s} \precsim \sigma$, so that by the theorem of Mackey these four topologies have the same dual M_* as a set.

Considering this fact, the extremal property of the τ-topology must be a powerful tool in the theory of W^*-algebras.

On the other hand, for the $\mathfrak{s}^*$-, $\mathfrak{s}$- and σ-topologies, we have nice concrete representations—in fact, the $\mathfrak{s}^*$ (resp. $\mathfrak{s}$ and σ) coincides with the strong*-operator topology—namely, the operator topology is defined by a family of semi-norms $\{\| x\xi \|, \| x^*\xi \| \mid \xi \in \mathfrak{H}\}$ (resp. the strong operator topology and the weak operator topology) on bounded spheres, when M is faithfully represented as a weakly closed*-algebra on a hilbert space $\mathfrak{H}$. Therefore, it is also important to have an analogous representation for the τ-topology. In this note, we shall show a concrete representation of the τ-topology of finite W^*-algebras as follows: the τ-topology of finite W^*-algebras is equivalent to the $\mathfrak{s}$-topology on bounded spheres. As a corollary of this result, we shall show that every σ-continuous linear mapping of a finite W^*-algebra into another W^*-algebra is $\mathfrak{s}$-continuous on bounded spheres. For non-finite W^*-algebra, we have no solution; clearly τ is $\prec$ $\mathfrak{s}$ on bounded spheres for non-finite ones, because $\mathfrak{s}^*$ is $\prec$ $\mathfrak{s}$ on bounded spheres (cf. [5], [7]).

Our conjecture is as follows: can we conclude that the τ-topology is equivalent to the $\mathfrak{s}^*$-topology on bounded spheres for all W^*-algebras?

2. Let M be a finite W^*-algebra, M_* the associated space of M.

LEMMA 1. *Let (f_i) be a countable family of elements in M_*; then there is a*

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central projection z of M such that M_z is countably decomposable and

$$f_i(M(1-z)) = 0$$

for all i, where 1 is the identity of M.

Proof. Let $f_i = L_{v_i}|f_i|$ be the polar decomposition of f_i (cf. [6], [7]). Put $\varphi = \sum_{i=1}^{\infty} |f_i|/2^i|f_i|(1)$; then φ is a normal positive functional on M. Let $s(\varphi)$ be the support of φ; then $s(\varphi)Ms(\varphi)$ is countably decomposable; let z be the central envelope of $s(\varphi)$; then M_z is also countably decomposable, for M is finite. Then $f_i(M(1-z)) = |f_i|(v_i M(1-z)) = 0$ for all i, since the support $s(|f_i|)$ of $|f_i|$ is contained in $s(\varphi)$. This completes the proof.

LEMMA 2. *Suppose that a sequence (f_i) converges weakly to f_0 in M_*; then there is a normal finite trace t on M as follows: for arbitrary sequence (a_n) such that*

$$\| a_n \| \leq 1 \quad \textit{and} \quad t(a_n^* a_n) \to 0 \ (n \to \infty),$$

we have $\lim_{n\to\infty} f_i(a_n) = 0$, uniformly for $i = 1, 2, 3, \cdots$.

Proof. By Lemma 1, there is a central projection z such that M_z is countably decomposable and $f_i(M(1-z)) = 0$ for $i = 1, 2, \cdots$ and so

$$f_0(M(1-z)) = 0.$$

Let t be a normal finite trace on M which is faithful on M_z.

We shall define a norm $\| \cdot \|_2$ on M_z as follows: $\| a \|_2 = t(a^*a)^{1/2}$ for $a \in M_z$.

Let S be the unit sphere of M; then S_z is the unit sphere of M_z. We define a metric $d(x, y)$ on S_z such that $d(x, y) = \| x - y \|_2$; then this metric defines a topology on S_z equivalent to the s-topology; hence S_z is a complete metric space under the metric $d(\ ,\ )$. The family $\{f_i \mid i = 0, 1, 2, \cdots\}$ can be considered as continuous functions on the metric space S_z and

$$\lim_{i\to\infty} f_i(a) = f_0(a)$$

for all $a \in S_z$.

Put $H_i = \{a \mid |f_j(a) - f_0(a)| \leqq \varepsilon \text{ for } j \geqq i;\ a \in S_z\}$ for arbitrary positive $\varepsilon > 0$; then $S_z = \bigcup_{i=1}^{\infty} H_i$; by the theorem of Baire, there is a set H_{j_0}, which is of the second category; since H_{j_0} is closed, it contains an open set; hence there is an element a_0 of S_z, a positive number $\delta(\varepsilon)$ such that

$$d(a, a_0) < \delta(\varepsilon) \ (a \in S_z) \quad \text{implies} \quad |f_j(a) - f_0(a)| \leqq \varepsilon \qquad \text{for } j \geqq j_0.$$

Now suppose that a sequence of self-adjoint elements (b_n) in S_z satisfies $\lim_n t(b_n^2) = 0$. By the theorem of Segal [8, Cor. 13.1], there is a subsequence (b_{n_p}) of (b_n) which converges metrically nearly everywhere to 0—namely, for every positive $\varepsilon' > 0$ there exists a sequence $P_{n_p}(\varepsilon')$ of projections in M_z such that $P_{n_p}(\varepsilon') \uparrow z$ as $n_p \uparrow \infty$ and $\| b_{n_p} P_{r_p}(\varepsilon') \| < \varepsilon'$ $(p = 1, 2, \cdots)$.

Therefore

$$\begin{aligned}|(f_j-f_0)(b_{n_p})| &= |(f_j-f_0)(P_{n_p}(\varepsilon')b_{n_p}P_{n_p}(\varepsilon'))| \\ &\quad + |(f_j-f_0)((z-P_{n_p}(\varepsilon'))b_{n_p}P_{n_p}(\varepsilon'))| \\ &\quad + |(f_j-f_0)(P_{n_p}(\varepsilon')b_{n_p}(z-P_{n_p}(\varepsilon'))| \\ &\quad + |(f_j-f_0)((z-P_{n_p}(\varepsilon'))b_{n_p}(z-P_{n_p}(\varepsilon')))| \\ &\leqq 6\sup_{0\leqq i<\infty}\|f_i\|\varepsilon' \\ &\quad + |(f_j-f_0)((z-P_{n_p}(\varepsilon'))b_{n_p}(z-P_{n_p}(\varepsilon'))|.\end{aligned}$$

Now put

$$x_{n_p} = P_{n_p}(\varepsilon')a_0P_{n_p}(\varepsilon') + (z-P_{n_p}(\varepsilon'))b_{n_p}(z-P_{n_p}(\varepsilon'));$$

then $x_{n_p}\in S_z$ and

$$\begin{aligned}d(x_{n_p}, a_0) &= \|(z-P_{n_p}(\varepsilon'))a_0P_{n_p}(\varepsilon') + P_{n_p}(\varepsilon')a_0(z-P_{n_p}(\varepsilon')) \\ &\quad + (z-P_{n_p}(\varepsilon'))a_0(z-P_{n_p}(\varepsilon')) \\ &\quad - (z-P_{n_p}(\varepsilon'))b_{n_p}(z-P_{n_p}(\varepsilon'))\|_2 \\ &\leqq 4\|z-P_{n_p}(\varepsilon')\|_2 .\end{aligned}$$

Take p_0 such that $\|z-P_{n_p}(\varepsilon')\|_2 < \delta(\varepsilon)/4$ for $p\geqq p_0$; then

$$\begin{aligned}|(f_j-f_0)(x_{n_p})| &= |(f_j-f_0)(P_{n_p}(\varepsilon')a_0P_{n_p}(\varepsilon')) \\ &\quad + (f_j-f_0)((z-P_{n_p}(\varepsilon'))b_{n_p}(z-P_{n_p}(\varepsilon')))| \\ &\leqq \varepsilon\end{aligned}$$

for $p \geq p_0$ and $j\geqq j_0$. Moreover,

$$d(P_{n_p}(\varepsilon')a_0P_{n_p}(\varepsilon'), a_0) \leqq 3\|z-P_{n_p}(\varepsilon')\|_2 < \delta(\varepsilon)$$

for $p\geqq p_0$ and $\|P_{n_p}(\varepsilon')a_0P_{n_p}(\varepsilon')\|\leqq 1$, so that

$$|(f_j-f_0)(P_{n_p}(\varepsilon')a_0P_{n_p}(\varepsilon'))| \leqq \varepsilon$$

for $p\geqq p_0$ and $j\geqq j_0$; therefore

$$|(f_j-f_0)((z-P_{n_p}(\varepsilon'))b_{n_p}(z-P_{n_p}(\varepsilon')))| \leqq 2\varepsilon$$

for $p\geqq p_0$ and $j\geqq j_0$.

Hence

$$|(f_j-f_0)(b_{n_p})| \leqq 6\sup_{0\leqq i<\infty}\|f_i\|\varepsilon' + 2\varepsilon$$

for $p\geqq p_0$ and $j\geqq j_0$.

Put $\varepsilon' = \varepsilon/6\sup_{0\leqq i<\infty}\|f_i\|$; then $|(f_j-f_0)(b_{n_p})|\leqq 3\varepsilon$ for $p\geqq p_0$ and $j\geqq j_0$.

Since $b_{n_p}\to 0$ $(p\to\infty)$ in the s-topology, there is a positive integer p_1 such that $|(f_j-f_0)(b_{n_p})|\leqq 3\varepsilon$ for $p\geqq p_1$ and $1\leqq j\leqq j_0$; hence

$$|(f_j-f_0)(b_{n_p})| \leqq 3\varepsilon$$

for $p \geqq \max(p_0, p_1)$ and $j = 1, 2, 3, \cdots$—namely, $\lim_{p\to\infty} f_j(b_{n_p}) = 0$, uniformly for $j = 1, 2, 3, \cdots$.

Now we want to show that $\lim_n f_j(b_n) = 0$, uniformly for $j = 1, 2, 3, \cdots$. Suppose that this is false; then there is a positive number ε, a subsequence (b_{n_q}) of (b_n) and a subsequence (f_{j_q}) of (f_j) such that $|f_{j_q}(b_{n_q})| > \varepsilon$ for $q = 1, 2, 3, \cdots$.

On the other hand, by the above discussions, we can choose a subsequence $(b_{n_{q_r}})$ of (b_{n_q}) such that $\lim_r f_j(b_{n_{q_r}}) = 0$, uniformly for $j = 1, 2, \cdots$. This is a contradiction; hence we have $\lim_n f_j(b_n) = 0$, uniformly for $j = 1, 2, 3, \cdots$.

Now let (a_n) be a sequence of elements in S such that $t(a_n^* a_n) \to 0$. Put $a_n = h_n + ik_n$ (h_n, k_n self-adjoint); then $t(a_n^* a_n) = t(h_n^2) + t(k_n^2)$; hence

$$t((h_n z)^2) = t(h_n^2) \to 0 \quad \text{and} \quad t((k_n z)^2) = t(k_n^2) \to 0,$$

so that $\lim_n f_j(h_n z) = \lim_n f_j(h_n) = 0$, uniformly for $j = 1, 2, \cdots$ and $\lim_n f_j(k_n z) = \lim_n f_j(k_n) = 0$, uniformly for $j = 1, 2, \cdots$ and so

$$\lim_n f_j(a_n) = 0,$$

uniformly for $j = 1, 2, 3, \cdots$. This completes the proof.

LEMMA 3. *Let K be a relatively $\sigma(M_*, M)$-compact set in M_*; then there is a normal finite trace t on M as follows: for arbitrary directed set (a_α) such that $\| a_\alpha \| \leqq 1$ and $\lim_\alpha t(a_\alpha^* a_\alpha) = 0$, we have $\lim_\alpha f(a_\alpha) = 0$, uniformly for $f \in K$.*

Proof. We shall show that for any positive ε, there is a positive $\delta(\varepsilon)$ and a finite set $\{f_1, f_2, \cdots, f_p\} \subseteqq K$ such that if $|f_i|(a^*a) < \delta(\varepsilon)$ for $i = 1, 2, \cdots, p$ and $a \in S$, then $|f(a)| < \varepsilon$ for $f \in K$. Suppose that this is false for some ε. Let $f_1 \in K$ be arbitrary; then there is an $a_1 \in S$ and an $f_2 \in K$ such that $|f_1|(a_1^* a_1) < 2^{-1}$ and $|f_2(a_1)| \geq \varepsilon$. By induction, construct sequences $\{f_i\} \subset K$ and $\{a_i\} \subset S$ such that $|f_i|(a_j^* a_j) < 2^{-j}$ for $1 \leqq i \leqq j < \infty$, and $|f_{j+1}(a_j)| \geqq \varepsilon$ for $1 \leqq j < \infty$.

Since K is relatively $\sigma(M_*, M)$-compact, there is a subsequence (f_{i_p}) of (f_i) which converges weakly in M_*. Put $\varphi = \sum_{p=1}^{\infty} |f_{i_p}|/2^p$, and let $s(\varphi)$ be the support of φ.

$$\begin{aligned}\varphi(a_j^* a_j) &= \textstyle\sum_{p=1}^{\infty} |f_{i_p}|(a_j^* a_j)/2^p \\ &\leqq \textstyle\sum_{\{p:i_p \leqq j\}} |f_{i_p}|(a_j^* a_j)/2^p + \sum_{\{p:i_p > j\}} \|f_{i_p}\|/2^p \\ &\leqq \frac{\frac{1}{2} - 1/2^{p(j)+1}}{1 - \frac{1}{2}} \cdot 2^{-j} + \frac{1/2^{p(j)+1}}{1 - \frac{1}{2}} \cdot \sup_{f \in K} \|f\|\end{aligned}$$

where $p(j)$ is the greatest p such that $i_p \leqq j$. Therefore $\lim_{j\to\infty} \varphi(a_j^* a_j) = 0$ and so the sequence $\{a_j s(\varphi)\}$ is s-convergent to 0 (cf. [2, Prop. 4, Chap. 1, §4]).

Let z be the central envelope of $s(\varphi)$, t a normal finite trace on M which is faithful on Mz; then

$$t((a_j\, s(\varphi))^*(a_j\, s(\varphi))) \to 0 \qquad (j \to \infty),$$

so that by Lemma 2, $\lim_{j\to\infty} f_{i_p}(a_j\, s(\varphi)) = 0$, uniformly for $p = 1, 2, 3, \cdots$.

Since $s(|f_{i_p}|) \leqq s(\varphi)$, $f_{i_p}(a_j\, s(\varphi)) = f_{i_p}(a_j)$; hence $\lim_{j\to\infty} f_{i_p}(a_j) = 0$, uniformly for $p = 1, 2, \cdots$. This contradicts $|f_{j+1}(a_j)| \geqq \varepsilon$ for all j.

Therefore there is a sequence of elements (g_n) in K such that $|g_n|(a^*a) = 0$ for all n implies $f(a) = 0$ for all $f \in K$. Put

$$\psi = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|g_n|}{|g_n|(1)}$$

and let z_0 be the central envelope of the support $s(\psi)$ of ψ; then

$$|g_n|(M(1 - z_0)) = 0$$

for all n, so that $f(M(1 - z_0)) = 0$ for all $f \in K$. Let t be a normal finite trace on M which is faithful on Mz_0. Let (a_α) be a directed set of elements in S such that $t(a_\alpha^* a_\alpha) \to 0$; then $\lim_\alpha f(a_\alpha) = 0$ uniformly for $f \in K$—in fact, suppose that this is false; then there is a subsequence (a_{α_n}) of (a_α), a subsequence (f_n) of K, a positive number ε such that $t(a_{\alpha_n}^* a_{\alpha_n}) \to 0$ and

$$|f_n(a_{\alpha_n})| \geqq \varepsilon.$$

This contradicts Lemma 2. This completes the proof.

Remark 1. The proof of Lemma 3 is a modification of the discussions of Bartle-Dunford-Schwartz [1].

Now we shall show the following.

THEOREM. *Let M be a finite W^*-algebra; then the τ-topology is equivalent to the s-topology on bounded spheres.*

Proof. Suppose that a directed set of elements (x_α) in S is s-convergent to 0. Let K be arbitrary relatively $\sigma(M_*, M)$-compact set in M_*; then by Lemma 3 there is a finite normal trace t such that $\lim_{t(a_\alpha^* a_\alpha)\to 0, \|a_\alpha\| \leqq 1} f(a_\alpha) = 0$, uniformly for $f \in K$; since $t(x_\alpha^* x_\alpha) \to 0$, we have $\lim_\alpha f(x_\alpha) = 0$, uniformly for $f \in K$, so that $\{a_\alpha\}$ is τ-convergent to 0. This completes the proof.

COROLLARY 1. *Let ρ be a σ-continuous linear mapping of a finite W^*-algebra into another W^*-algebra; then ρ is s-continuous on bounded spheres.*

Proof. By the general theory of locally convex spaces, ρ is τ-continuous. By the above theorem, the τ-topology coincides with the s-topology on bounded spheres of finite W^*-algebras, so that ρ is s-continuous on bounded spheres.

Remark 2. This corollary can not be extended to general W^*-algebras—in fact, let M be a W^*-algebra; then we can construct a W^*-algebra N such that there is an anti-$*$-isomorphism ρ of M onto N (cf. [3], [7]); ρ is always σ-bicontinuous (cf. [7]); however, if M is not finite, ρ is always s-discontinuous on bounded spheres (cf. [5], [7]).

COROLLARY 2. *Let Φ be a normal positive mapping of a finite W^*-algebra into another W^*-algebra; then it is $\mathfrak{s}$-continuous on bounded spheres.*

Proof. Since a normal positive mapping is σ-continuous, by Corollary 1 it is $\mathfrak{s}$-continuous on bounded spheres.

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