LEBESGUE MEASURE AND THE n-AREA OF HOMEOMORPHISMS

BY

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It is known [1], that the Lebesgue 2-area for homeomorphisms f from planar admissible sets M into E_2 agrees with the Lebesgue 2-measure μ_2 of the interior $f(M)^0$ of the image of M under f.

In this note, we show that a similar relation holds for homeomorphisms from compact metric spaces into E_n, Lebesgue n-measure, and the Lebesgue-type n-area $L_n^p(f)$ introduced by R. F. Williams [2].

Theorem. *If X is a compact metric space and $f : X \to E_n$ is a homeomorphism, then $L_n^p(f) = \mu_n[f(X)^0]$.*

The proof will be given by means of several lemmas. The terminology used concerning $L_n^p(f)$ is, for the most part, that of Williams's paper [2]. We denote composition of mappings by $f \circ g$ or by fg. The Euclidean norm of a point p in E_n is denoted by $|p|$ and the uniform norm of a mapping $f : X \to E_n$ is denoted by $\| f \|$.

Lemma 1. *If Y is a compact subset of E_n and $i_Y : Y \to E_n$ is the inclusion mapping, then if K is any compact subset of Y^0, there exists an $\varepsilon > 0$ such that if $q : Y \to E_n$ and $\| i_Y - q \| < \varepsilon$, then $K \subset q(Y)$.*

This may be proved by an extension of the argument given in [3] to apply first to a suitable open n-cell about each point of K and then to K itself by the standard argument based on the compactness of K.

Lemma 2. *If Y and i_Y are as in Lemma 1, then $\mu_n[Y^0] \leq L_n^p(i_Y)$.*

If $Y^0 = \emptyset$, this is clearly true. If $Y^0 \neq \emptyset$, then for any given $\varepsilon > 0$, there exists a compact set $K \subset Y^0$ such that $\mu_n[Y^0] - \varepsilon < \mu_n[K]$. Let $\delta > 0$ be such that if $q : Y \to E_n$ and $\| i_Y - q \| < \delta$, then $K \subset q(Y)$. We may assume that $\delta \leq \varepsilon$. Let (α, g, h) be any n-canonical map triple,

$$\begin{tikzcd} Y \arrow[rr, "i_Y"] \arrow[dr, "g"'] & & E_n \\ & {|Y_\alpha|} \arrow[ur, "h"'] & \end{tikzcd}$$

such that mesh $\alpha < \delta$ and $\| i_Y - hg \| < \delta$. Since $hg(Y) \subset h(|Y_\alpha|)$, then $\mu_n[hg(Y)] \leq \mu_n[h(|Y_\alpha|)]$. Since $\mu_n[h(|Y_\alpha|)] = \sum \mu_n[\sigma']$, where the summation extends over all n-simplexes σ' of $h(|Y_\alpha|)$, and since

$$\sum \mu_n[\sigma'] \leq \sum \mu_n[h(\sigma)] = e_n(h),$$

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where the latter summation extends over all n-simplexes σ of Y_α, then

$$\mu_n[Y^0] - \varepsilon < \mu_n[K] \leq \mu_n[hg(Y)] \leq e_n(h).$$

LEMMA 3. *If Y and i_Y are as in Lemma 1, then $L_n^p(i_Y) \leq \mu_n[Y^0]$.*

Let $\varepsilon > 0$ be given and let T be a subdivision of some closed n-simplex in E_n that is large enough to contain Y, into a finite number of closed n-simplexes $\bar{\sigma}_n$, each of diameter $< \varepsilon/2$. Let $T(Y)$ be the collection of all closed n-simplexes of T which meet Y. Let $|T(Y)|$ be the union of all closed n-simplexes of $T(Y)$.

We define a mapping $g : Y \to |T(Y)|$ by means of the following partial mappings $g_{\bar{\sigma}_n}$, $\bar{\sigma}_n \in T(Y)$, defined on the sets $S_{\bar{\sigma}_n} = Y \cap \bar{\sigma}_n$, $\bar{\sigma}_n \in T(Y)$.

(a) If $\bar{\sigma}_n \in T(Y)$ and $\sigma_n \subset Y$, let $g_{\bar{\sigma}_n} = i_Y \mid S_{\bar{\sigma}_n}$.

(b) If $\bar{\sigma}_n \in T(Y)$ and $\sigma_n \cap E_n - Y \neq \emptyset$, there exists a point $p \in \sigma_n$ and a retraction $r_p : \bar{\sigma}_n - p \to \partial\bar{\sigma}_n$, where $\partial\bar{\sigma}_n$ denotes the point set boundary of $\bar{\sigma}_n$. Let $g_{\bar{\sigma}_n} = r_p \circ i_Y \mid S_{\bar{\sigma}_n}$.

The mapping $g : Y \to |T(Y)|$ is continuous and leaves fixed each point of Y that lies on the boundary of a closed n-simplex of $T(Y)$.

Let T' be the collection of all open simplexes $\sigma_k \in T(Y)$, $k \leq n$, that meet $g(Y)$, together with all their faces. Let St (v) denote the open star (relative to T') at the vertex v of T'. The collection of inverse images of these stars under g, $\alpha = \{g^{-1}[\text{St } (v)] \mid v \text{ is a vertex of } T'\}$, is an open cover for Y (in the subspace topology of Y), T' is a geometric realization of the nerve Y_α of α, and g is canonical with respect to α. If we let h be the inclusion mapping of $|T'|$ into E_n, then h is clearly simplicial and (α, g, h) is an n-canonical map triple.

Since each open n-simplex σ_n of T', if any, is contained in Y^0, then

$$e_n(h) = \sum \mu_n[h(\sigma_n)] \leq \mu_n[Y^0],$$

where the summation extends over all n-simplexes σ_n of T'. Also, mesh $\alpha < \varepsilon$ and $\| i_Y - hg \| < \varepsilon/2$.

LEMMA 4. *If X and Y are compact metric spaces of dimension $\leq m$ and are homeomorphic under the homeomorphism $\phi : X \to Y$, then for any mapping $f : Y \to E_n$, $L_m^p(f, Y) = L_m^p(f \circ \phi, \phi^{-1}(Y))$.*

We show first that $L_m^p(f \circ \phi, \phi^{-1}(Y)) \leq L_m^p(f, Y)$. Let $\varepsilon > 0$ be given. By the uniform continuity of ϕ^{-1}, there exists a $\delta > 0$, which we may assume is no larger than ε, such that if A is any subset of Y of diameter $< \delta$, then $\phi^{-1}(A)$ has diameter $< \varepsilon$. By the definition of $L_m^p(f, Y)$, there exists an m-canonical map triple (α, g, h) such that mesh $\alpha < \delta$, $\| f - hg \| < \delta$ and $e_m(h) < L_m^p(f, Y) + \delta$. Let $\alpha' = \{\phi^{-1}(A) \mid A \in \alpha\}$, let $g' = g \circ \phi$, and let $h' = h$. Since $X_{\alpha'}$ is isomorphic to Y_α, then $g' : X \to |Y_\alpha|$ is canonical with respect to α', mesh $\alpha' < \varepsilon$, and $|f(\phi(x)) - h(g(\phi(x)))| < \varepsilon$ for all

$x \in \phi^{-1}(Y)$. Since $e_m(h') = e_m(h) < L_m^p(f, Y) + \varepsilon$, it follows that

$$L_m^p(f \circ \phi, \phi^{-1}(Y)) \leq L_m^p(f, Y).$$

If we replace Y by $\phi^{-1}(Y)$, f by $f \circ \phi$, and X by Y, then the hypotheses of the lemma are satisfied with ϕ^{-1} in place of ϕ. We conclude that

$$L_m^p(f, Y) = L_m^p(f \circ \phi \circ \phi^{-1}, \phi(\phi^{-1}(Y))) \leq L_m^p(f \circ \phi, \phi^{-1}(Y)).$$

The theorem is now proved, for if X is a compact metric space and $f : X \to E_n$ is a homeomorphism, then $\dim X \leq n$ and

$$X \xrightarrow{f} f(X) = Y \xrightarrow{i_Y} E_n .$$

By Lemmas 2 and 3, $L_n^p(i_Y , Y) = \mu_n[Y^0]$. By Lemma 4,

$$L_n^p(i_Y , Y) = L_n^p(i_Y \circ f, f^{-1}(Y)) = L_n^p(f, X).$$

References

1. L. Cesari, *Surface area*, Princeton University Press, 1956, p. 123.
2. R. F. Williams, *Lebesgue area of maps from Hausdorff spaces*, Acta Math., vol. 102 (1959), pp. 33–46.
3. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1948, p. 75.

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