AN ABSTRACT EXTENT FUNCTION[1]

BY

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It is the purpose of this note to define an abstract process, under which, as special cases, will fall the apparently diverse concepts of Lebesgue m-area for mappings from finitely triangulable spaces, the various Lebesgue-Williams areas for mappings from compact metric spaces, outer measures in an abstract set, metric outer measures, such as Hausdorff r-measure, in a metric space, the Daniell-Stone upper integral, the Burkill lower integral for interval functions, and perhaps others.

The form of our definition of the extent function $M(u)$ was suggested by the definitions of m-area given by R. F. Williams [2]. Its substance can be regarded as an extension of the ideas of Lebesgue [6], and of Fréchet [1], who was, apparently, the first to notice that, in the classical case of surface area, Lebesgue's definition may be viewed as a process for extending a semi-continuous function. See also M. H. Stone [4] in connection with what we call measuring systems.

Although our abstract process does not, in general, provide semi-continuous extensions in Fréchet's sense [1], the extent function $M(u)$ which arises is always semi-continuous. It will be clear that while our present definition leads to properties of one-sided lower-semi continuity, the definition may be modified so that, in general, functions exhibiting any of four types of semi-continuity will arise.

Measuring systems and the definition of $M(u)$

A function σ on $U \times U$ to R, where U is a set and R is the set of non-negative real numbers, will be called an *écart* for U and the pair (U, σ) will be called an *écarted space* if σ satisfies the following two conditions:

$$(1) \qquad \sigma(u, u) = 0 \qquad \text{for all } u \in U,$$

$$(2) \qquad \sigma(u, v) \leq \sigma(u, w) + \sigma(w, v) \qquad \text{for all } u, v, w \in U.$$

If U is a set, then a quintuple $\mathfrak{M} = [\sigma, A, q, d, v]$, where σ is an écart for U, A is a set, q is a function on A to U, d is a function on A to R and v is a function on A to R will be called a *measuring system* for U.

For a given measuring system $\mathfrak{M} = [\sigma, A, q, d, v]$ for U, we define, for each $u \in U$, the following subset $R_{\mathfrak{M}}(u)$ of R:

$R_{\mathfrak{M}}(u) = \{r \in R \mid$ for every $\varepsilon > 0$, there exists an $a \in A$ such that $\sigma(u, q(a)) < \varepsilon$, $d(a) < \varepsilon$ and $v(a) < r + \varepsilon\}$.

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For each $r \in R$, we define the following subset $U_{\mathfrak{M}}(r)$ of U:

$U_{\mathfrak{M}}(r) = \{u \in U \mid$ for every $\varepsilon > 0$, there exists an $a \in A$ such that $\sigma(u, q(a)) < \varepsilon, d(a) < \varepsilon$ and $v(a) < r + \varepsilon\}$.

The function $M(u)$ is defined as follows:

$$M(u) = \inf R_{\mathfrak{M}}(u), \quad \text{if} \quad R_{\mathfrak{M}}(u) \neq \emptyset.$$
$$= +\infty, \quad \text{if} \quad R_{\mathfrak{M}}(u) = \emptyset.$$

It is clear that $u \in U_{\mathfrak{M}}(r)$ if and only if $r \in R_{\mathfrak{M}}(u)$ and that $M(u) = +\infty$ if and only if $R_{\mathfrak{M}}(u) = \emptyset$.

Properties of the function $M(u)$

We prove below the following theorems with no further hypotheses on the nature of the elements of the measuring system $\mathfrak{M}$. In a few of these, the concept of accessibility is required. An element $u \in U$ will be said to be *accessible to the measuring system* $\mathfrak{M} = [\sigma, A, q, d, v]$ provided that for every $\varepsilon > 0$, there exists an $a \in A$ such that $\sigma(u, q(a)) < \varepsilon$ and $d(a) < \varepsilon$.

THEOREM 1. *If $R_{\mathfrak{M}}(u) \neq \emptyset$, then $M(u) \in R_{\mathfrak{M}}(u)$.*

If $R_{\mathfrak{M}}(u) \neq \emptyset$, then $M(u) = \inf R_{\mathfrak{M}}(u) < +\infty$, and for each $\varepsilon > 0$, there exists an $r \in R_{\mathfrak{M}}(u)$ such that $r < M(u) + \varepsilon/2$. Hence, there exists an $a \in A$ such that $\sigma(u, q(a)) < \varepsilon, d(a) < \varepsilon$ and $v(a) < r + \varepsilon/2 < M(u) + \varepsilon$.

THEOREM 2. *For each $r \in R$, $U_{\mathfrak{M}}(r) = \{u \in U \mid M(u) \leq r\}$.*

If $u \in U_{\mathfrak{M}}(r)$, then $r \in R_{\mathfrak{M}}(u)$ and $M(u) = \inf R_{\mathfrak{M}}(u) \leq r$. Conversely, if $u \in U$ is such that $M(u) \leq r$, then $r \in R_{\mathfrak{M}}(u)$ and $u \in U_{\mathfrak{M}}(r)$.

THEOREM 3. *For each $r \in R$, $U_{\mathfrak{M}}(r)$ is closed in the upper topology for (U, σ), where an upper ε-neighborhood of $u_0 \in U$ is a set of the form*

$$\{u \in U \mid \sigma(u_0, u) < \varepsilon\}.$$

Suppose u_0 is in the closure of $U_{\mathfrak{M}}(r)$. For each $\varepsilon > 0$, there exists a $u \in U_{\mathfrak{M}}(r)$ such that $\sigma(u_0, u) < \varepsilon/2$. Since $u \in U_{\mathfrak{M}}(r)$, there exists an $a \in A$ such that $\sigma(u, q(a)) < \varepsilon/2, d(a) < \varepsilon/2$ and $v(a) < r + \varepsilon/2$. Since $\sigma(u_0, q(a)) \leq \sigma(u_0, u) + \sigma(u, q(a)) < \varepsilon$, it follows that $u_0 \in U_{\mathfrak{M}}(r)$.

THEOREM 4. *The function $M(u)$ is lower-semi continuous with respect to the upper topology for U at each point $u_0 \in U$.*

If $u_0 \in U$ and $M(u_0) = +\infty$, then $R_{\mathfrak{M}}(u_0) = \emptyset$ and for each $r \in R$, $u_0 \notin U_{\mathfrak{M}}(r)$. Since $U_{\mathfrak{M}}(r)$ is closed, there exists a $\delta > 0$, such that if $u \in U$ and $\sigma(u_0, u) < \delta$, then $u \notin U_{\mathfrak{M}}(r)$. From Theorem 2, it follows that $r < M(u)$. If $M(u_0) < +\infty$, then for each $\varepsilon > 0$, $M(u_0) - \varepsilon \notin R_{\mathfrak{M}}(u_0)$. Since $u_0 \notin U_{\mathfrak{M}}(M(u_0) - \varepsilon)$, and since $U_{\mathfrak{M}}(M(u_0) - \varepsilon)$ is closed, there is a $\delta > 0$ such that if $u \in U$ and $\sigma(u_0, u) < \delta$, then $M(u_0) - \varepsilon < M(u)$.

THEOREM 5. *If* $\{a_n\}$ *is any sequence in* A *such that* $\sigma(u, q(a_n)) \to 0$ *and* $d(a_n) \to 0$, *then* $M(u) \leq \lim \inf v(a_n)$.

Let $\{a_n\}$ be such a sequence. If $M(u) < +\infty$, then for each $\varepsilon > 0$, $M(u) - \varepsilon \notin R_{\mathfrak{M}}(u)$. There exists a $\delta > 0$, then, such that if $a \in A$ has the properties $\sigma(u, q(a)) < \delta$ and $d(a) < \delta$, then

$$M(u) - \varepsilon < M(u) - \varepsilon + \delta \leq v(a).$$

We conclude that $M(u) - \varepsilon \leq \lim \inf v(a_n)$ for each $\varepsilon > 0$ and that $M(u) \leq \lim \inf v(a_n)$.

If $M(u) = +\infty$, then $R_{\mathfrak{M}}(u) = \emptyset$ and for each $r \in R$, there exists a $\delta > 0$ such that if $a \in A$ has the properties $\sigma(u, q(a)) < \delta$ and $d(a) < \delta$, then $r < v(a)$. We conclude that $r < \lim \inf v(a_n)$ for each $r \in R$ and that $\lim \inf v(a_n) = +\infty$.

THEOREM 6. *If* u *is accessible to* $\mathfrak{M}$, *then there exists a sequence* $\{a_n\}$ *in* A *such that* $\sigma(u, q(a_n)) \to 0$, $d(a_n) \to 0$ *and* $\lim v(a_n) = M(u)$.

If $M(u) = +\infty$, any sequence satisfying the hypotheses will do, as the second part of the proof of Theorem 5 shows. If $M(u) < +\infty$, then since $M(u) \in R_{\mathfrak{M}}(u)$, there is a sequence $\{a_n\}$ in A with the property that for each positive integer n, $\sigma(u, q(a_n)) < 1/n$, $d(a_n) < 1/n$ and $v(a_n) < M(u) + 1/n$. It follows that $\lim \inf v(a_n) \leq \lim \sup v(a_n) \leq M(u)$ and, by Theorem 5, that $\lim v(a_n) = M(u)$.

THEOREM 7. *For each* $u \in U$, $M(u) = \inf_{[\phi]} \lim \inf v(a_n)$, *where* $[\phi]$ *is the collection of all sequences* $\phi = \{a_n\}$ *in* A *such that* $\sigma(u, q(a_n)) \to 0$ *and* $d(a_n) \to 0$.

If u is accessible to $\mathfrak{M}$, this theorem is a consequence of Theorems 5 and 6. If u is inaccessible to $\mathfrak{M}$, then necessarily $M(u) = +\infty$ and under the convention regarding the infimum taken over the empty set, the theorem holds in this case also.

THEOREM 8. *Let*

$$A_{\mathfrak{M}}(u, \varepsilon) = \{a \in A \mid \sigma(u, q(a)) < \varepsilon \text{ and } d(a) < \varepsilon\}$$

and let

$$V_{\mathfrak{M}}(u, \varepsilon) = \inf \{v(a) \mid a \in A_{\mathfrak{M}}(u, \varepsilon)\};$$

then

$$M(u) = \sup_\varepsilon V_{\mathfrak{M}}(u, \varepsilon) = \lim_{\varepsilon \to 0} V_{\mathfrak{M}}(u, \varepsilon).$$

Let $N(u) = \sup_\varepsilon V_{\mathfrak{M}}(u, \varepsilon)$. It is clear that $V_{\mathfrak{M}}(u, \varepsilon)$ is a decreasing function of ε. If $M(u) < +\infty$, then for each $\varepsilon > 0$, there exists an $a \in A_{\mathfrak{M}}(u, \varepsilon)$ such that $v(a) < M(u) + \varepsilon$. It follows that, for every $\varepsilon > 0$,

$$V_{\mathfrak{M}}(u, \varepsilon) < M(u) + \varepsilon$$

and that

$$N(u) = \lim_{\varepsilon \to 0} V_{\mathfrak{M}}(u, \varepsilon) \leq M(u).$$

This relation also holds if $M(u) = +\infty$. If $N(u) < +\infty$, then for each $\varepsilon > 0$, we have $V_{\mathfrak{M}}(u, \varepsilon) \leq N(u)$. Consequently, $V_{\mathfrak{M}}(u, \varepsilon) < +\infty$ for each $\varepsilon > 0$. From the definition of $V_{\mathfrak{M}}(u, \varepsilon)$, it follows that there exists an $a \in A_{\mathfrak{M}}(u, \varepsilon)$ such that $v(a) < V_{\mathfrak{M}}(u, \varepsilon) + \varepsilon$. Hence, for each $\varepsilon > 0$, there exists an $a \in A$ such that $\sigma(u, q(a)) < \varepsilon$, $d(a) < \varepsilon$ and $v(a) < N(u) + \varepsilon$, which shows that $M(u) \leq N(u)$.

THEOREM 9. *If $\mathfrak{M} = [\sigma, A, q, d, v]$ and $\mathfrak{M}' = [\sigma', A', q', d', v']$ are two measuring systems for U, then in order that $M'(u) \leq M(u)$ at a point $u \in U$ for which $M(u) < +\infty$, it is necessary and sufficient that for every $\varepsilon > 0$, there exist a $\tau > 0$ such that whenever $a \in A$ has the properties $\sigma(u, q(a)) < \tau$ and $d(a) < \tau$, then there exists an $a' \in A'$ with the properties $\sigma'(u, q'(a')) < \varepsilon$, $d'(a') < \varepsilon$ and $v'(a') < v(a) + \varepsilon$.*

To show the necessity of this condition, we observe that since

$$M'(u) - \varepsilon/2 \notin R_{\mathfrak{M}}(u)$$

for any given $\varepsilon > 0$, there exists a $\tau > 0$ such that if $a \in A$ has the properties $\sigma(u, q(a)) < \tau$ and $d(a) < \tau$, then $M'(u) - \varepsilon/2 < v(a)$. Since $M'(u) \in R_{\mathfrak{M}'}(u)$, there exists an $a' \in A'$ for which $\sigma'(u, q'(a')) < \varepsilon/2$, $d'(a') < \varepsilon/2$ and $v'(a') < M'(u) + \varepsilon/2 < v(a) + \varepsilon$.

As for the sufficiency, since $M(u) < +\infty$, then u is accessible to $\mathfrak{M}$ and by Theorem 6 there exists a sequence $\{a_n\}$ in A such that $\sigma(u, q(a_n)) \to 0$, $d(a_n) \to 0$ and $M(u) = \lim v(a_n)$. For each n so large that $1/n < \tau$, there exists an $a'_n \in A'$ such that $\sigma'(u, q'(a'_n)) < 1/n$, $d'(a'_n) < 1/n$ and $v'(a'_n) < v(a_n) + 1/n$. Hence, $\sigma'(u, q'(a'_n)) \to 0$, $d'(a'_n) \to 0$ and $\liminf v'(a'_n) \leq M(u)$. From Theorem 5, we conclude that $M'(u) \leq M(u)$.

Examples

1. *Lebesgue m-area for mappings from a finitely triangulable space X into a Euclidean space E_n.* Let U be the set of all continuous mappings $f : X \to E_n$. Let $\sigma(f, g) = \| f - g \|$, the uniform metric for U. Let A be the collection of all triples $a = (t, K, h)$, where K is a geometric m-complex, $t : X \to K$ is a homeomorphism onto, and $h : K \to E_n$ is a quasi-linear mapping. Let $q(a) = ht$, let $d(a) = 0$, and let $v(a) = \sum a_m(h(s))$, where $a_m(h(s))$ is the elementary m-area of the image of the simplex $s \in K$ and the summation is over all m-simplexes of K. Then $M(f) = L_m(f)$. See [2].

2. *Williams-Lebesgue m-area for mappings from a compact metric space X of covering dimension $\leq m$ into E_n.*

I. Let U and σ be as in Example 1. Let A be the set of all triples $a = (\alpha, g, h)$, where α is a finite open cover for X of order $\leq m$, g is a canonical map of X into X_α, the nerve of α, and h is a simplicial map $h : X \to E_n$. Let $q(a) = hg$, let $d(a) = \text{mesh } \alpha$, where mesh $\alpha = \sup_{V \in \alpha} \text{diam } V$, and let $v(a)$ be as in Example 1 except that the summation is over all m-simplexes $s \in X_\alpha$. Them $M(f) = L_m^p(f)$. See [2].

II. Let U, σ, A, q, d be as in I of this example but let $v(a) = e_m^*(\alpha, g, h)$, where $e_m^*(\alpha, g, h)$ is the elementary area defined in [2]. Then $M(f) = L_m^*(f)$.

3. *Metric outer measures from arbitrary non-negative set functions.* Let (X, ρ) be a metric space. Let U be the collection of all subsets of X. Let $\sigma(E, F) = 0$ if $E \subset F$ and let $\sigma(E, F) = 1$ otherwise. Let $\mathcal{C}$ be a class of subsets of U such that the empty set belongs to $\mathcal{C}$. Let τ be a non-negative set function defined on $\mathcal{C}$ such that $\tau(\emptyset) = 0$. Let A be the collection of all sequences with values in $\mathcal{C}$. If $a = \{C_n\}$, $C_n \in \mathcal{C}$, let $q(a) = \bigcup_{n=1}^{\infty} C_n$, let $d(a) = \sup_n \operatorname{diam} C_n$, and let $v(a) = \sum_{n=1}^{\infty} \tau(C_n)$. From Theorem 8, it can be seen that $M(E)$ is the result of a standard construction leading to metric outer measures. See [3, p. 105].

4. *Outer measures from arbitrary non-negative set functions in an abstract set* X. Let U, σ, A, q, v be as in Example 3 for an abstract set equipped with the trivial metric $\rho(x, y) = 0$. Then $M(E)$ is an outer measure.

5. *The Daniell-Stone upper integral.* Let U be the collection of all extended real-valued functions on a set X. Let $E(e)$ be a positive linear functional defined on a subspace U_0 of U. Let $\sigma(f, g) = 0$ if $|f| \leq |g|$ and let $\sigma(f, g) = 1$ otherwise. Let A be the collection of all sequences whose values are non-negative functions in U_0. Let $q(a) = \sum_{n=1}^{\infty} e_n$, let $d(a) = 0$, and let $v(a) = \sum_{n=1}^{\infty} E(e_n)$. Then $M(f) = N(f)$, the Daniell-Stone upper integral. See [4].

6. *The Burkill lower integral for non-negative interval functions.* Let $U = I$ be an interval in the line. Let S be the collection of all subintervals J of I. Let $F(J)$ be a non-negative interval function defined on S. Let A be the set of all decompositions of I into a finite number of non-overlapping elements of S. Let $\sigma(I, I) = 0$, let $q(a) = I$ for all $a \in A$, let $d(a) = \text{mesh } a$, where mesh $a = \sup_{J \in a} |J|$, and let $v(a) = \sum_{J \in a} F(J)$. Then $M(I) =$ the Burkill lower integral of $F(J)$ over I. See [5, p. 165].

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