

SKEW PRODUCT TRANSFORMATIONS AND THE ALGEBRAS GENERATED BY $\exp(p(n))$

BY
F. J. HAHN¹

Introduction

The circle of ideas examined in this paper separates itself naturally into two types of problems discussed in Sections 1 and 2. In Section 1 we examine the ergodic and topological properties of skew product dynamical systems (X, T) . (See Section 1 for definitions.) Ergodic properties of such transformations on the torus were first studied by Anzai [1]. The question of strict ergodicity of such transformations on the torus was studied by Furstenberg [4]. Such a system (X, T) is an example of an affine transformation of a compact abelian group. The ergodic properties of such transformations were studied by the author in [6]. Furstenberg's technique for demonstrating strict ergodicity depends on the fact that $C(X)$ is separable and thus X is metric. The author [6] used these techniques to get another proof of the famous theorem of Weyl [9] on the distribution of the numbers $p(n)$, $n = 1, 2, 3, \dots$, where p is a polynomial with at least one irrational coefficient which is not the constant term. (See Theorem 1.) In Section 1 we use Weyl's theorem and a theorem of Oxtoby [7] (this is our Theorem 2) to study the strict ergodicity of (X, T) where X is not restricted to being metric. In Section 1 it is shown that minimality, ergodicity, and strict ergodicity are all equivalent and are implied by the non-existence of elements of finite order in the character group of a group G used to define the space X .

In Section 2 we examine the algebras of sequences of the form $\exp(p(n))$, $[\exp \theta = e^{2\pi i \theta}]$ where $p(n)$ is a polynomial of degree less than or equal to some preassigned number $m = 1, 2, 3, \dots, \infty$ and with coefficients from some fixed subgroup Λ of the additive group of the reals. We relate the maximal ideal spaces of these algebras to skew product dynamical systems. Finally in some concluding remarks we point out the relationship between these algebras and the subject of [2].

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1. Ergodicity and minimality of skew product dynamical systems

By a dynamical system we mean the pair (X, T) where X is a compact Hausdorff space and T a homeomorphism of X . We wish to define carefully a skew product dynamical system. We remark first that a group G is called a monothetic group with generator γ_0 if G is a topological group and the set of

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elements $\{n\gamma_0 : n \text{ an integer}\}$ is dense in G . We see easily that G must be abelian.

DEFINITION. (X, T) is a skew product dynamical system if $X = \prod_{i=1}^m G_i$ where $m = 1, 2, 3, \dots, \infty$ and each $G_i = G$. Moreover G is a compact monothetic group with generator γ_0 . T is the homomorphism of X given by

$$T(x_1, x_2, x_3, \dots, x_j, \dots) = (x_1 + \gamma_0, x_2 + x_1, x_3 + x_2, \dots, x_j + x_{j-1}, \dots).$$

If we let S be defined by

$$S(x_1, x_2, x_3, \dots, x_j, \dots) = (x_1, x_2 + x_1, x_3 + x_2, \dots, x_j + x_{j-1}, \dots)$$

then we see that $T(x) = \gamma + S(x)$ where $\gamma = (\gamma_0, 0, 0, 0, \dots)$. This allows us to see that if μ is normalized Haar measure on X then μ is invariant under T . This is true since S is an automorphism of X and thus preserves μ . Consequently S followed by a translation preserves μ .

In order to prove the main theorems in this section we will make use of the following theorems due to Weyl and to Oxtoby. By $C(X)$ we shall always mean the Banach algebra of continuous complex-valued functions on X where $\|f\| = \sup \{|f(x)| : x \in X\}$.

THEOREM 1. (Weyl) Let $p(z)$ be a polynomial of degree m :

$$p(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0, \quad a_i \text{ real numbers.}$$

Suppose a_l is irrational for some $l, m \geq l \geq 1$. Under these conditions

$$\sum_{n=0}^N \exp(p(n)) = o(N)$$

uniformly in a_0, a_1, \dots, a_{l-1} .

Proof. See [9, p. 326], or see [6] and observe that uniformity is implied by the strict ergodicity of the transformation.

THEOREM 2. (Oxtoby) Let (X, φ) be a minimal dynamical system. If for each $f \in C(X)$

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(\varphi^n(x))$$

exists uniformly in x then (X, φ) is strictly ergodic. By this we mean there is a unique φ -invariant positive measure μ for which $\mu(X) = 1$.

Proof. The proof is simple enough to outline here. Let

$$f^*(x) = \lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(\varphi^n(x)).$$

Because of the uniformity of convergence we see $f^* \in C(X)$. We also see $f^*(\varphi(x)) = f^*(x)$. The minimality of (X, φ) implies f^* is constant. Let μ be defined by $\int f d\mu = f^*$. If μ is not unique there must be another ergodic measure ν and it follows from the ergodic theorem that

$$f^* = \int f d\nu$$

and thus $\nu = \mu$.

In order to prove the next theorem we shall need more detailed knowledge of how T acts on X . If

$$T^n(x) = (y_1^n, y_2^n, \dots, y_j^n, \dots)$$

then it is not hard to show by induction that

$$\begin{aligned}
 y_1^n &= p_1(n)\gamma_0 + x_1 \\
 y_2^n &= p_2(n)\gamma_0 + p_1(n)x_1 + x_2 \\
 \vdots & \\
 y_j^n &= p_j(n)\gamma_0 + p_{j-1}(n)x_1 + p_{j-2}(n)x_2 + \dots + x_j \\
 \vdots &
 \end{aligned}
 \tag{1.1}$$

where $p_j(n) = (1/j!)(n(n-1)(n-2)\dots(n-j+1))$ for $j = 1, 2, 3, \dots$ and $n = 0, 1, 2, \dots$. We observe that degree of $p_j(n)$ is j and that the coefficients are integers.

For the next few formulas we assume $X = \prod_{i=1}^m G_i = G$, and $m < \infty$. If $f \in \hat{X}$ then

$$f(x) = f_1(x_1) \cdot f_2(x_2) \cdot \dots \cdot f_m(x_m) \quad \text{where } f_i \in \hat{G}_i. \tag{1.2}$$

We wish to examine the values $f(T^n x)$. To do this we establish the following conventions. Let

$$\begin{aligned}
 f_i(x_j) &= \exp \theta_i(x_j) \\
 f_i(\gamma_0) &= \exp \alpha_i
 \end{aligned}
 \tag{1.3}$$

for $i = 1, 2, \dots, m, j = 1, 2, \dots, m$ and $0 \leq \theta_i(x_j) < 1, 0 \leq \alpha_i < 1$. We see that $f(T^n x) = f(y_1^n, \dots, y_m^n) = f_1(y_1^n) f_2(y_2^n) \dots f_m(y_m^n)$. We have

$$\begin{aligned}
 f_j(y_j^n) &= f_j(p_j(n)\gamma_0 + \sum_{i=1}^j p_{j-i}(n)x_i) \\
 &= \exp(p_j(n)\alpha_j + \sum_{i=1}^j p_{j-i}(n)\theta_j(x_i)).
 \end{aligned}
 \tag{1.4}$$

Consequently we obtain

$$\begin{aligned}
 f(T^n x) &= \exp(\sum_{j=1}^m p_j(n)\alpha_j + \sum_{j=1}^m \sum_{i=1}^j p_{j-i}(n)\theta_j(x_i)) \\
 &= \exp q(n)
 \end{aligned}
 \tag{1.5}$$

where we let

$$\begin{aligned}
 a_m n^m + a_{m-1} n^{m-1} + \dots + a_1 n + a_0 &= q(n) \\
 &= \sum_{j=1}^m p_j(n)\alpha_j \\
 &\quad + \sum_{j=1}^m \sum_{i=1}^j p_{j-i}(n)\theta_j(x_i).
 \end{aligned}$$

We now have the necessary language for Theorem 3.

THEOREM 3. *If $f \in C(X)$ then*

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = f^*(x)$$

exists uniformly in x .

Proof. We first observe that it suffices to prove the theorem in the case where X is the product of G taken only finitely often. To see this we remark that since the characters of X are dense in $C(X)$ it is sufficient to prove the theorem for f a character. If f is a character it depends on only finitely many coordinates. That is f is of the form

$$f(x) = f_1(x_1) \cdot f_2(x_2) \cdots f_m(x_m)$$

where each f_i is a character of G . So if the theorem is proven for X having only finitely many factors then it holds in the infinite case also. We thus assume that $X = \prod_{i=1}^m G_i$, $G_i = G$, and $m < \infty$. We carry out the proof by examining three cases. The first and third case we reduce to Theorem 1 and the second case we solve with the aid of the ergodic theorem.

Case I. f_m is not of finite order. Since the multiples of γ_0 are dense in G it follows that $\exp \alpha_m$ is not of finite order and thus α_m is irrational. Examining 1.5 we see that only $p_m(n)$ contributes a term of degree m and thus in $q(n)$ we have $a_m = \alpha_m/m!$ which is irrational. From Theorem 1 we have

$$\sum_{n=0}^{N-1} f(T^n x) = \sum_{n=0}^{N-1} \exp(q(n)) = o(N)$$

uniformly in $a_0, a_1, a_2, \dots, a_{m-1}$. However these a_i are functions of $\theta_j(x_j)$ and thus

$$\sum_{n=0}^{N-1} f(T^n x) = o(N)$$

uniformly in x . This completes Case I.

Case II. f_i is of finite order for $i = 1, 2, \dots, m$. Our aim is to obtain a decomposition $\mathfrak{F} = \{F_0, F_1, F_2, \dots, F_\lambda\}$ of X . The sets F_i are to be pairwise disjoint and each should have positive Haar measure. Moreover for each $i = 1, 2, 3, \dots, \lambda$ we shall have $f(T^n x) = k_i(n)$ if $x \in F_i$. If this is done I claim this case is also solved. We reason in this manner. It follows from the ergodic theorem that

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

exists for almost all x . In particular this limit must exist for at least one x in each F_i . But for each x in F_i we have

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \lim_N \frac{1}{N} \sum_{n=0}^{N-1} k_i(n).$$

This limit must exist for every x in each F_i and consequently exists for each

x in G . Since there are only finitely many F_i we see that

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

exists uniformly in x . Thus we see that in order to complete Case II we must only obtain the decomposition \mathfrak{F} . We now obtain this decomposition.

For each f_i we let $\mathfrak{C}_i = \{H_0^i, H_1^i, \dots, H_{\nu(i)}^i\}$ be the decomposition of G given by the cosets determined by $H_0^i = \text{kernel } f_i$. We observe that each coset has positive Haar measure in G . Let $\mathfrak{C} = \{H_0, H_1, \dots, H_\nu\}$ be the decomposition of G which is the coarsest refinement of all the \mathfrak{C}_i . That is

$$\mathfrak{C} = \mathfrak{C}_1 \wedge \mathfrak{C}_2 \wedge \dots \wedge \mathfrak{C}_m.$$

We observe again that each set of \mathfrak{C} has positive Haar measure in G . Furthermore each f_i is constant on each set of \mathfrak{C} . Finally we let

$$\mathfrak{F} = \mathfrak{C} \times \mathfrak{C} \times \dots \times \mathfrak{C} \text{ (} m \text{ times),}$$

that is, $\mathfrak{F} = \{F_0, F_1, \dots, F_\lambda\}$ where the F_i are gotten by taking all possible subsets of X of the form $H_{i_1} \times H_{i_2} \times \dots \times H_{i_m}$, where $H_{i_k} \in \mathfrak{C}$. Since Haar measure on X is the product measure of the Haar measure on G it follows that each member of \mathfrak{F} has positive Haar measure in X . Since each $f_i, i = 1, 2, \dots, m$, is constant on the sets of \mathfrak{C} we see that $f(T^n x) = k_i(n)$ for each $x \in F_i, i = 1, 2, \dots, \lambda$. Thus Case II has been disposed of.

Case III. f_k is not of finite order for some $k, 1 \leq k < m$ but

$$f_{k+1}, f_{k+2}, \dots, f_m$$

are all of finite order. We prove this case by combining the techniques of Cases I and II. Let $\mathfrak{C}_{k+1}, \mathfrak{C}_{k+2}, \dots, \mathfrak{C}_m$ be defined as in Case II. Let

$$\mathfrak{C} = \mathfrak{C}_{k+1} \wedge \mathfrak{C}_{k+2} \wedge \dots \wedge \mathfrak{C}_m$$

and let $\mathfrak{F} = \mathfrak{C} \times \mathfrak{C} \times \dots \times \mathfrak{C}$ (m times), $\mathfrak{F} = \{F_0, F_1, \dots, F_\lambda\}$. We see that each $f_{k+1}, f_{k+2}, \dots, f_m$ is constant on the sets of \mathfrak{C} . For the time being fix some set F_ν and let $x \in F_\nu$. According to 1.5 we have

$$f(T^n x) = \exp(q(n))$$

where

$$\begin{aligned} q(n) &= a_m n^m + a_{m-1} n^{m-1} + \dots + a_1 n + a_0 \\ &= \sum_{j=1}^m p_j(n) \alpha_j + \sum_{j=1}^m \sum_{i=1}^j p_{j-i}(n) \theta_j(x_i). \end{aligned}$$

We see from 1.3 that $\theta_{k+1}, \theta_{k+2}, \dots, \theta_m$ are constant on the sets of \mathfrak{C} . Recalling that the degree of $p_{j-i}(n)$ is $j - i$ we see that the coefficients a_k, a_{k+1}, \dots, a_m depend only on $\theta_{k+1}, \theta_{k+2}, \dots, \theta_k$ and on no θ_i with $i \leq k$. Thus if we allow x to vary in F_ν the numbers a_k, a_{k+1}, \dots, a_m are constant. We now examine a_k and show that it is irrational. We must have

$$a_k = r_1 \alpha_k + r_2$$

where r_1 is rational and r_2 is a rational combination of $\theta_{k+1}, \theta_{k+2}, \dots, \theta_m, \alpha_{k+1}, \dots, \alpha_m$. Since $f_{k+1}, f_{k+2}, \dots, f_m$ are of finite order it follows that r_2 is rational. Since f_k is not of finite order it follows that α_k is irrational and thus a_k is irrational. Theorem 1 states that

$$\sum_{n=0}^{N-1} \exp(q(n)) = o(N)$$

uniformly in a_0, a_1, \dots, a_{k-1} . Thus we see that

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

exists uniformly for $x \in F_\nu$. Since there are only finitely many F_ν in \mathfrak{F} we see that

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

exists uniformly for $x \in X$. This completes Case III and thus the theorem is proved.

If (Z, T) is a dynamical system we let $\mathbf{O}(z) = \{T^n z : n \text{ an integer}\}$ be called the orbit of z for any $z \in Z$. By $\text{Cl } \mathbf{O}(z)$ we mean the closure of $\mathbf{O}(z)$. We also define the product dynamical system $(Z \times Z, T \times T)$ by the formula $T \times T(z_1, z_2) = (Tz_1, Tz_2)$. We let $\Delta \subset Z \times Z$ be the set

$$\Delta = \{(z, z) : z \in Z\}.$$

DEFINITION. A dynamical system (Z, T) is distal if for each

$$(z_1, z_2) \in Z \times Z$$

we have either $\text{Cl } \mathbf{O}(z_1, z_2) \cap \Delta = \emptyset$ or $\text{Cl } \mathbf{O}(z_1, z_2) \subset \Delta$. The orbit of (z_1, z_2) is taken in the product dynamical system $(Z \times Z, T \times T)$.

LEMMA 4. A skew product dynamical system is distal.

Proof. Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be elements of X . Suppose $\text{Cl } \mathbf{O}(x, y) \cap \Delta \neq \emptyset$; then we wish to show $\text{Cl } \mathbf{O}(x, y) \subset \Delta$. Since X is compact this implies that there is a $z \in X$ and a subset of integers

$$\{n(\lambda) : \lambda \in \Lambda\},$$

where Λ is a directed set, such that

$$\lim_\lambda T^{n(\lambda)} x = \lim_\lambda T^{n(\lambda)} y = z.$$

Since $T(x) = (x_1 + \gamma_0, x_2 + x_1, \dots, x_j + x_{j-1}, \dots)$ we see that $x_1 = y_1$. Suppose it has been shown that $x_1 = y_1, x_2 = y_2, \dots, x_{j-1} = y_{j-1}$. Since the j^{th} coordinate of $T^n(x)$ is of the form x_j plus a polynomial in x_1, x_2, \dots, x_{j-1} and similarly for the j^{th} coordinate of $T^n(y)$ we see that $x_j = y_j$. Thus by induction it follows that $x = y$ and consequently $\mathbf{O}(x, y) \subset \Delta$. Since Δ is closed we have $\text{Cl } \mathbf{O}(x, y) \subset \Delta$.

THEOREM 5. *Let (X, T) be a skew product dynamical system. If the character group of G has no elements of finite order then (X, T, μ) is ergodic. In any case the following statements are equivalent:*

- (a) (X, T, μ) is ergodic.
- (b) (X, T) is minimal.
- (c) (X, T) is strictly ergodic.

Proof. We first show that if \hat{G} has no elements of finite order then (X, T, μ) is ergodic. To do this we assume that the preceding statement is false; that is we simultaneously assume that (X, T, μ) is not ergodic and that \hat{G} has no elements of finite order. We will arrive at a contradiction.

If (X, T, μ) is not ergodic then it is certainly not strongly mixing. It follows from [6, Corollary 3] that there is a positive integer p and a character $\eta \in \hat{X}$ such that $\eta \neq 1$ and $\hat{S}^p \eta = \eta (\hat{S}^p \eta(x) = \eta(S^p x))$. We let

$$\eta = (\eta_1, \eta_2, \dots, \eta_m),$$

that is $\eta(x) = \eta_1(x_1)\eta_2(x_2) \cdots \eta_m(x_m)$, where $\eta_i \in \hat{G}$ for $i = 1, 2, \dots, m$. It is an easy computation to see that

$$1.6 \quad \hat{S} \eta = (\eta_1 \eta_2, \eta_2 \eta_3, \dots, \eta_{m-1} \eta_m, \eta_m).$$

We observe that $\hat{S}^p(\eta) = \eta$ implies $\eta_{m-1} \eta_m^p = \eta_{m-1}$. Since \hat{G} has no elements of finite order we conclude $\eta_m \equiv 1$. We now proceed downward by induction. Suppose we have shown that $\eta_j \equiv \eta_{j+1} \equiv \dots \equiv \eta_m \equiv 1$ for some $j, 2 < j \leq m$. Then $\hat{S}^p(\eta) = \eta$ implies $\eta_{j-2} \eta_{j-1}^p = \eta_{j-2}$ and we again conclude $\eta_{j-1} \equiv 1$. So far we have shown by induction

$$\eta_2 \equiv \eta_3 \equiv \dots \equiv \eta_m \equiv 1.$$

We will now show that $\eta_1 \equiv 1$. It follows from [6, Theorem 4] that for at least one η for which $\hat{S}^p \eta = \eta$ we have $\hat{T}^p \eta = \eta$. Thus we have

$$\eta = \hat{T}^p \eta = \eta_1^p(\gamma_0) \eta.$$

We conclude $\eta_1^p(\gamma_0) = 1$. Since the multiples of γ_0 are dense in G we conclude that $\eta_1^p \equiv 1$ and the fact that G has no elements of finite order allows us to assert $\eta_1 \equiv 1$. Consequently $\eta \equiv 1$ which contradicts the assumption $\eta \neq 1$.

We now go on to prove the equivalence of (a), (b), and (c).

(a) implies (b). Since (X, T, μ) is ergodic it follows that

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = f^*(x) = \int f d\mu$$

for almost all $x \in X$. By Theorem 3 the first equality holds uniformly in x for each $f \in C(X)$. Thus $f \in C(X)$ implies $f^*(x)$ is continuous and since $f^*(x)$ is constant almost everywhere it is constant. Thus

$$\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f d\mu$$

for each $x \in X$ and $f \in C(X)$. This implies that for each $x \in X$ the sequence $x, Tx, T^2x, \dots, T^n x, \dots$ is dense in X which implies (X, T) minimal.

(b) implies (c). We need merely apply Theorems 2 and 3.

(c) implies (a). It is known that if there is a unique invariant measure then it is an ergodic measure (see [7]).

Because of the nature of Theorem 5 some remarks and examples are in order. We remark first that every skew product dynamical system has quasi-discrete spectrum. Thus we may use the results of L. M. Abramov, *Metric automorphisms with quasi-discrete spectrum*, *Izvestia of Acad. of Sci., U. S. S. R.*, 1962, pp. 513-530, to obtain the first part of Theorem 5 for X metric.

Example 1. \hat{G} may have elements of finite order and (X, T, μ) may be ergodic. Let $G = Z_2$ and $X = Z_2 \times Z_2$. In this case (X, T) is transitive and thus in particular it is ergodic.

Example 2. If \hat{G} has an element of order p then (X, T^{2p}, μ) is not ergodic. Let $\eta_1^p \equiv 1, \eta_1 \not\equiv 1$, for some $\eta_1 \in \hat{G}$ and define $\eta = (\eta_1, 1, 1, \dots, 1)$. We now have

$$\begin{aligned} \eta(T^{2p}x) &= \eta(\gamma_0 + S\gamma_0 + S^2\gamma_0 + \dots + S^{2p-1}\gamma_0)\eta(x) \\ &= \eta_1^{p(2p-1)}(\gamma_0)\eta(x) = \eta(x). \end{aligned}$$

Thus T^{2p} has a non-constant eigenfunction of eigenvalue one and is not ergodic.

Example 3. If $X = G$ then (X, T, μ) is always ergodic and has discrete spectrum. This is the well known case of translation by a fixed element γ_0 .

Example 4. There exist cases where \hat{G} has an element of finite order and (X, T, μ) is not ergodic. Let \hat{G} be the discrete group of the complex numbers of modulus one. We see that G is compact. Let $X = G \times G$ and let γ_0 be that element of G defined by $\xi(\gamma_0) = \xi$ for each $\xi \in \hat{G}$. The multiples of γ_0 are dense in G . Let $\xi_1 = e^{i\pi/2}$ and $\xi_2 = e^{i\pi}$, ξ_1 and $\xi_2 \in \hat{G}$. In $L_2(X)$ let $f = (\xi_1, \xi_2) + i(\xi_1 \xi_2, \xi_2)$. We observe that f is not a constant and we compute

$$\begin{aligned} Tf(x_1, x_2) &= f(x_1 + \gamma_0, x_2 + x_1) \\ &= (\xi_1, \xi_2)(x_1 + \gamma_0, x_2 + x_1) + i(\xi_1 \xi_2, \xi_2)(x_1 + \gamma_0, x_2 + x_1) \\ &= \xi_1(\gamma_0)(\xi_1 \xi_2, \xi_2)(x_1, x_2) + i\xi_1(\gamma_0)\xi_2(\gamma_0)\xi_2^2(x_1)(\xi_1, \xi_2)(x_1 x_2) \\ &= i(\xi_1 \xi_2, \xi_2)(x_1 x_2) + (\xi_1 \xi_2)(x_1 x_2) \\ &= f(x_1, x_2). \end{aligned}$$

Thus T has a non-constant eigenfunction with eigenvalue one so (X, T, μ) is not ergodic.

Example 5. Because of Theorem 5 we wish to indicate that the class of

compact monothetic groups whose character group has no elements of finite order is a rich class. First consider all groups G_p , p a prime, whose character group \hat{G}_p consists of the discrete additive group of all rationals r which can be written in the form $r = q/p^k$, q and k integers. These groups G_p are the p -adic solenoids. Since \hat{G}_p is a subset of the reals we see that it has no elements of finite order. In order to see that G_p is monothetic we must only show that \hat{G}_p is isomorphic to a subgroup of the discrete group \mathbf{C}_1 of complex numbers of modulus one. Let $f: \hat{G}_p \rightarrow \mathbf{C}_1$ be defined by $f(r) = e^{ir}$. We see easily that f is a homomorphism and we need only check to see kernel $f = \{0\}$. We observe $e^{ir} = 1$ if and only if $r = 2\pi n$ for some integer n . Since π is irrational we have $r = 0$.

Another class of interesting examples of the same type is obtained as follows. Let E be a subset of the reals with the following property: If $\xi_1, \xi_2, \dots, \xi_n$ are in E then the numbers $1, \xi_1, \xi_2, \dots, \xi_n$ are linearly independent over the rationals. Let \hat{G} be the discrete subgroup of the additive group of reals generated by E . We again see that \hat{G} has no elements of finite order since it is a subgroup of the reals. In order to see that G is monothetic we define the homomorphism $f: \hat{G} \rightarrow \mathbf{C}_1$ by $f(\xi) = \exp \xi$. We need only show kernel $f = \{0\}$. If $f(\xi) = 1$, where $\xi \in \hat{G}$, then $\xi = m$ an integer. Since $\xi \in \hat{G}$ we see that $\xi = q_1 \xi_1 + q_2 \xi_2 + \dots + q_n \xi_n$, q_i are integers and $\xi_i \in E$. Thus $0 = -m + q_1 \xi_1 + q_2 \xi_2 + \dots + q_n \xi_n$, but because of the independence of $1, \xi_1, \xi_2, \dots, \xi_n$ over the rationals we have $m = q_1 = q_2 = \dots = q_n = 0$ and thus $\xi = 0$.

The above class of groups G contains all the finite dimensional tori. It also yields a large class of compact monothetic groups which are not metrizable and thus will not yield to the treatment of [6]. To obtain this class we must only be sure that the set E is non-countably infinite. Such sets E exist and then $C(G)$ is not separable so G is not metrizable.

2. Algebras generated by $\exp(p(n))$ and their maximal ideal spaces

To begin with we let B be the Banach algebra of all bounded bisequences of complex numbers $\{z = z_n : n \text{ an integer}\}$ with

$$\|z\| = \sup \{|z_n| : n \text{ an integer}\}.$$

In this algebra we wish to examine certain closed subalgebras, their maximal ideal spaces, and relate some of the results of Section 1 with these maximal ideal spaces. We first describe the subalgebras which we consider.

Let Λ be any subgroup of the additive group of reals. Let

$$2.1 \quad \Gamma(\Lambda) = \{\exp \lambda : \lambda \in \Lambda\}.$$

For any positive integer m we let

$$2.2 \quad P_m(\Lambda) = \{p(n) = \lambda_m n^m + \lambda_{m-1} n^{m-1} + \dots + \lambda_1 n + \lambda_0 : \lambda_i \in \Lambda\}$$

and let

$$2.3 \quad P_\infty(\Lambda) = \bigcup_{m=1}^\infty P_m(\Lambda).$$

For each $m = 1, 2, 3, \dots, \infty$ we let

$$2.4 \quad Q_m(\Lambda) = \{\exp(p(n)) : p(n) \in P_m(\Lambda)\}.$$

Thus we see that for each m , $Q_m(\Lambda)$ consists of exponentials of polynomials in n with coefficients in Λ and degree no larger than m . $Q_m(\Lambda)$ is a subalgebra of B . The subalgebras of B which we wish to examine are the closed algebras generated by $Q_m(\Lambda)$. Since $Q_m(\Lambda)$ is a group under multiplication we see that these subalgebras are merely the closed linear spans of $Q_m(\Lambda)$. Thus we let

$$2.5 \quad \begin{aligned} L_m(\Lambda) &= \text{closed subalgebra generated by } Q_m(\Lambda) \\ &= \text{closed linear subspace generated by } Q_m(\Lambda). \end{aligned}$$

We let $\mathfrak{M}_m(\Lambda)$ be the maximal ideal space of $L_m(\Lambda)$ and let

$$\theta_m : L_m(\Lambda) \rightarrow C(\mathfrak{M}_m(\Lambda))$$

be the Gelfand mapping of the elements of $L_m(\Lambda)$ to the continuous functions on $\mathfrak{M}_m(\Lambda)$. Since $L_m(\Lambda)$ is a B^* algebra with identity it is known that θ_m is an isomorphic isometry onto $C(\mathfrak{M}_m(\Lambda))$. We also observe that the mapping $\varphi : B \rightarrow B$ defined by $(\varphi z)_n = z_{n+1}$ is an isomorphic isometry of B onto B and $L_m(\Lambda)$ is preserved by φ . If we let

$$2.5a \quad \eta_m = \theta_m \varphi \theta_m^{-1}$$

then we see $\eta_m : \mathfrak{M}_m(\Lambda) \rightarrow \mathfrak{M}_m(\Lambda)$ and

$$2.5b \quad \eta_m \theta_m = \theta_m \varphi.$$

We wish to describe both η_m and $\mathfrak{M}_m(\Lambda)$.

We do this by defining a skew-product dynamical system (X_m, T_m) . The X_m depends of course on Λ but we omit it from our notation. In this system we let Y_m be a particular orbit closure. We then show that there is an isomorphism ν of the dynamical systems (Y_m, T_m) and $(\mathfrak{M}_m(\Lambda), \eta_m)$. That is we show that there is a homeomorphism $\nu : Y_m \rightarrow \mathfrak{M}_m(\Lambda)$ such that $\eta_m \nu = \nu T_m$.

From now on we will consider the group Λ as fixed and for the sake of brevity we will omit Λ from our notation. We let Γ_d be the group Γ with the discrete topology. If $\hat{\Gamma}_d$ is the character group of Γ_d then $\hat{\Gamma}_d$ is compact. In $\hat{\Gamma}_d$ we use the additive notation, that is $x + y$ is defined by $(x + y)\gamma = x(\gamma) \cdot y(\gamma)$ for each $\gamma \in \Gamma_d$. We let

$$2.6 \quad X_m = \prod_{i=1}^m \hat{\Gamma}_d, \quad m = 0, 1, 2, \dots, \infty.$$

In order to define a skew product dynamical system we need to choose a generator ξ_0 of $\hat{\Gamma}_d$. We let ξ_0 be the element defined by $\xi_0(\gamma) = \gamma$ for each

$\gamma \in \Gamma_d$. The multiples of ξ_0 are dense in $\hat{\Gamma}_d$ for the following reason. Since

$$\hat{\Gamma}_d = \Gamma_d$$

we see that each element $\gamma \in \Gamma_d$ is a character of $\hat{\Gamma}_d$. These characters separate points. If $\xi_0(\gamma) = 1$ then $\gamma = 1$ by definition of ξ_0 . Thus the multiples of ξ_0 are dense in $\hat{\Gamma}_d$. We let (X_m, T_m) be given by

$$2.7 \quad T_m(x) = (x_1 + \xi_0, x_2 + x_1, \dots, x_m + x_{m-1}).$$

Furthermore we let $0 = (0, 0, 0 \dots) \in X_m$ and set

$$2.8 \quad Y_m = \text{Cl } \mathbf{O}(0).$$

THEOREM 6. *The dynamical system (Y_m, T_m) is distal, minimal and strictly ergodic.*

Proof. By Lemma 4, (X_m, T_m) is distal. It is easy to see that this property is inherited by (Y_m, T_m) . From Ellis' Theorem [3] it follows that (Y_m, T_m) is pointwise almost periodic. Since Y_m is an orbit closure (Y_m, T_m) must be minimal. The Tietze extension theorem says that any continuous complex-valued function on Y_m may be extended to X_m . If we now apply Theorem 3 and Theorem 2 we see that (Y_m, T_m) is strictly ergodic.

THEOREM 7. *There is an isomorphism $\nu : (Y_m, T_m) \rightarrow (\mathfrak{M}_m, \eta_m)$.*

Proof. We reduce the problem by examining the space \mathfrak{M}_m . This space \mathfrak{M}_m is completely determined by $C(\mathfrak{M}_m)$. We have already remarked that $\theta : L_m(\Lambda) \rightarrow C(\mathfrak{M}_m)$ is an isomorphic isometry onto. Thus \mathfrak{M}_m is completely determined by $L_m(\Lambda)$. Consider $\mathbf{O}(0) \subset Y_m$, the orbit of 0 under T_m . For each $f \in L_m(\Lambda)$ define the function $\tilde{f} : \mathbf{O}(0) \rightarrow \mathbf{C}$ as follows: $\tilde{f}(T^n(0)) = f(n)$. I claim that the theorem is proven if we can show that each such \tilde{f} has a continuous extension to all of Y_m . This is not difficult to see. For if it is possible to extend each such \tilde{f} then since $\mathbf{O}(0)$ is dense in Y_m we see that $L_m(\Lambda)$ and $C(Y_m)$ are isometrically isomorphic under the mapping

$$\zeta_m : f \rightarrow \tilde{f}.$$

We also see that

$$2.7a \quad \zeta_m \varphi = T_m \zeta_m.$$

Letting $\nu = \theta_m \zeta_m^{-1}$ we obtain a homeomorphism of Y_m and \mathfrak{M}_m . From 2.5a, 2.5b, and 2.7a we obtain

$$\nu T_m = \theta_m \zeta_m^{-1} T_m = \theta_m \varphi \zeta_m^{-1} = \eta_m \theta_m \zeta_m^{-1} = \eta_m \nu$$

which would complete our theorem. Consequently all that remains to be shown is that \tilde{f} can be extended to a continuous function on Y_m for each $f \in L_m(\Lambda)$. Since finite linear combinations of the elements of $Q_m(\Lambda)$ are dense in $L_m(\Lambda)$ it suffices to show that \tilde{f} can be extended to a continuous function on Y_m for each $f \in Q_m(\Lambda)$. The plan for doing this is the following:

given $f \in Q_m(\Lambda)$ then produce $\psi \in C(Y_m)$ such that $\psi(T^n(0)) = \hat{f}(T^n(0)) = f(n)$. We will actually find ψ as an element of \hat{X} .

If we let $f \in Q_m(\Lambda)$ then

$$\begin{aligned}
 2.9 \quad & f(n) = \exp(p(n)), \\
 & p(n) = a_s n^s + a_{s-1} n^{s-1} + \cdots + a_1 n + a_0
 \end{aligned}$$

where $s < m + 1$ and $a_i \in \Lambda, i = 1, 2, \dots, s$. We recall that

$$\Gamma_d = \{\exp \lambda : \lambda \in \Lambda\}$$

is the character group of $\hat{\Gamma}_d$. We denote the character $\exp \lambda$ by ψ_λ . From the definition of ξ_0 we have $\psi_\lambda(\xi_0) = \exp \lambda$. Let $\psi \in \hat{X}$ have the form

$$2.10 \quad \psi(x) = \psi_{\lambda_1}(x_1)\psi_{\lambda_2}(x_2) \cdots \psi_{\lambda_s}(x_s).$$

From 1.1 we obtain

$$2.11 \quad T^n(0) = (p_1(n)\xi_0, p_2(n)\xi_0, \dots, p_s(n)\xi_0, \dots).$$

2.10 and 2.11 yield

$$2.12 \quad \psi(x) = \exp(q(n))$$

where

$$\begin{aligned}
 q(n) &= \lambda_1 p_1(n) + \lambda_2 p_2(n) + \cdots + \lambda_s p_s(n) \\
 &= b_s n^s + b_{s-1} n^{s-1} + \cdots + b_1 n + b_0.
 \end{aligned}$$

In order to complete the theorem it suffices to show that we can choose $\lambda_1, \dots, \lambda_s \in \Lambda$ such that $p(n) = q(n)$ for then $\psi(T^n(0)) = f(n)$.

We do this by downward induction recalling that the degree of $p_j(n)$ is j . If we set $\lambda_s = a_s/s!$ we observe the following:

$$b_s = a_s.$$

$$2.13 \quad \lambda_s \text{ and } \lambda_s/s! \in \Lambda.$$

λ_s is a polynomial in $a_k, k \geq s$, with integral coefficients.

Suppose now that we have determined $\lambda_s, \lambda_{s-1}, \dots, \lambda_j$ so the conditions of 2.13 are satisfied. We wish to show that λ_{j-1} can be determined so the conditions of 2.13 are satisfied. The only polynomials p_i involving terms of degree greater than or equal to $j - 1$ are p_{j-1}, p_j, \dots, p_m . Thus we see that

$$b_{j-1} = \lambda_{j-1} \cdot \frac{1}{(j-1)!} + \tau$$

where τ is the sum of integral multiples of $\lambda_j/j!, \lambda_{j+1}/(j+1)!, \dots, \lambda_s/s!$. Thus we see that $\tau \in \Lambda$. If we let $\lambda_{j-1} = (j-1)!(a_{j-1} - \tau)$ then we see that $a_{j-1} = b_{j-1}, \lambda_{j-1}$ and $\lambda_{j-1}(j-1)! \in \Lambda$, and λ_{j-1} is an integral poly-

nomial in the a_k for $k \geq j - 1$. Thus our induction shows that $a_i = b_i$, $i = 0, 1, 2, \dots, s$ so $p(n) = q(n)$ and the theorem is complete.

THEOREM 8. *If Γ_a has no elements of finite order then $X_m = Y_m$.*

Proof. $X_m = Y_m$ if and only if (X_m, T) is minimal. We now need only use Theorem 5.

In conclusion I would like to point out certain relations between the results of this paper and those of [2]. In [2] the authors studied the algebra of sequences f_n which had the following properties: for each such f there is a distal dynamical system (X, T) and an $x \in X$ and an $\tilde{f} \in C(X)$ such that

$$f_n = \tilde{f}(T^n x).$$

This algebra is in general quite large. It is not true that all of its elements have mean values. The algebras $\{\exp(p(n))\}$ form smaller more manageable subalgebras of the larger algebra.

Added in proof. For references to [6] see also *Errata* to [6] in Amer. J. Math., vol. 86 (1964).

REFERENCES

1. H. ANZAI, *Ergodic skew transformations on the torus*, Osaka Math. J., vol. 3 (1951), pp. 83-99.
2. L. AUSLANDER AND F. HAHN, *Real functions coming from flows on compact spaces and concepts of almost periodicity*, Trans. Amer. Math. Soc., vol. 106 (1963), pp. 415-426.
3. R. ELLIS, *Distal transformation groups*, Pacific J. Math., vol. 8 (1958), pp. 401-405.
4. H. FURSTENBERG, *Strict ergodicity and transformations of the torus*, Amer. J. Math., vol. 83 (1961), pp. 573-601.
5. W. GOTTSCHALK AND G. HEDLUND, *Topological dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 36, 1955.
6. F. HAHN, *On affine transformations of compact abelian groups*, Amer. J. Math., vol. 85 (1963), pp. 428-446.
7. J. OXTOBY, *Ergodic sets*, Bull. Amer. Math. Soc., vol. 58 (1952), pp. 116-136.
8. C. RICKART, *Banach algebras*, Princeton, D. van Nostrand, 1960.
9. H. WEYL, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann., vol. 77 (1961), pp. 313-352.

YALE UNIVERSITY
NEW HAVEN, CONNECTICUT