MULTIPLICITY OF SOLUTIONS IN FRAME MAPPINGS

BY

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The Kakutani Theorem, the Dyson theorem and their extensions as well as various forms of the Knaster conjecture have generally been considered from the viewpoint of sufficient conditions for existence of one solution. The present note considers the problem of the multiplicity of such solutions.

Throughout this paper k is invariably an odd prime, k-tuple is understood to mean orthogonal k-tuple and all omology [4] is over the coefficient field J_k, the integers mod k. The space of ordered k-tuples in S^{n-1} is denoted by W and is a Stiefel manifold, viz.

$$W = V_{n,k} = SO(n)/SO(n-k) \approx O(n)/O(n-k). \tag{1.1}$$

However W counts two different orderings of a k-tuple as two distinct k-tuples. If P_k is the permutation group on k letters then the distinct unordered k-tuples constitute the orbit space W/P_k. Write w for an arbitrary point of S^{n-1} and $\bar{w} = (w^1, \cdots, w^k)$ for an ordered k-tuple. Similarly write x for an arbitrary point of R^l and $\bar{x} = (x^1, \cdots, x^k)$ for a point of the k-fold topological product $R^l \times \cdots \times R^l = R^{lk}$. The mapping $F : W \to R^{lk}$ is defined by $F(\bar{w}) = \bar{u} = (f(w^1), \cdots, f(w^k))$. Write P_k again, for the permutation group giving the reorderings of $x^1, \cdots, x^k$. The mapping F is P_k equivariant, that is to say $gF = Fg$, for all $g \in P_k$. We say F is *free* equivariant on a set X_0 if the group acts freely both on X_0 and on $F(X_0)$.

Our methods require the computation of certain indices and a knowledge of the cohomology rings of certain orbit spaces. Little is known about these when P_k is the group. We therefore restrict our attention to the cyclic subgroup C of order k (isomorphic to J_k) throughout this paper. Results of Borel (or Bott) are then available for practical computation. Specifically for some $g \in C$,

$$g : (w^1, \cdots, w^k) = w^2, \cdots, w^k, w^1.$$
$$g : (x^1, \cdots, x^k) = x^2, \cdots, x^k, x^1.$$

Let $\Delta = \{\bar{x} \mid x^1 = \cdots = x^k\}$ be the diagonal of R^{lk} and write

$$\begin{aligned} D &= F^{-1}\Delta \\ A &= W - D = F^{-1}(R^{lk} - \Delta). \end{aligned} \tag{1.2}$$

Then A and D are invariant sets under C. Moreover F is free equivariant on A with respect to C but not on D. (If we used P_k it would be more natural to introduce δ the subspace of R^{lk} for which two coordinates (or more) of

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$\bar{x}$ are equal instead of Δ. Thus here F would be free equivariant on $F^{-1}(R^{lk} - \delta)$ and not on $F^{-1}(\delta)$. In brief P_k is linked to problems of a common image for two or more points of a k-tuple. This is another reason for introducing C.) We shall refer to the elements of W/C as *cyclic classes.* If the space K is invariant under C we write K' for the orbit space K/C. Accordingly F induces F' on $W' = W/C$ to $Y' = R^{lk}/C$.

The index properties central for our investigation are summarized below for the simplicial case [4]. Suppose C acts freely on a space X. If $I(0)$ is the 0-dimensional unit cocycle on $X' = X/C$ then the Smith homomorphism $s(m)$, [4, p. 329], takes $H^n(X')$ into $H^{n+m}(X')$ and yields

$$(1.3)\qquad \begin{aligned} I(2j) &= s(2j)I(0) = (I(2))^j \\ I(2j+1) &= s(2j+1)I(0) = I(1)I(2)^j \end{aligned}$$

where powers are in the sense of cup products. The index $\nu(X)$ [4, 135.4] is the largest integer m for which $s(m)I(0) \neq 0$. The critical property is that under free equivariant maps the *index cannot decrease.* We denote the universal space for "sufficiently high n" for a group L by $E(L)$ and the classifying space $E(L)/L$ by B_L [1]. Write

$$B_L^n = H^n(B_L).$$

The inclusion homomorphism $L \to M$ is denoted by $\rho(L, M)$; thus

$$(1.4)\qquad B_M^* \xrightarrow{\rho^*(L,M)} B_L^*.$$

We require generalization of the index ν to the case of an A which is open in W and is invariant under C. Such a generalization is essentially covered by [10]. Alternatively the cofinal primitive open coverings can be shown to exist by arguments in say [9]. The desideratum is a cohomology theory for manifolds, admitting exactness and Poincare duality for the pairs X', U', U open. Cech, Alexander omology groups with closed supports will satisfy these conditions. In fact this is a specialization of results in [8]. (I am indebted to Raymond for some comments on his results.) Indeed in the special case of the locally Euclidean manifold and the coefficient field J_k we have [8, Eq. 6.2]

$$(1.5)\qquad \begin{array}{ccccccc} \to H^m(X', U') & \longrightarrow & H^m(X') & \longrightarrow & H^m(U') & \xrightarrow{d} & \\ \downarrow\approx & & \downarrow\approx & & \downarrow\approx & & \\ \to H_{N-m}(X' - U') & \to & H_{N-m}(X') & \to & H_{N-m}(X', X' - U') & \to & \end{array}$$

with exact rows and commutativity in the squares. Here $N = \dim X'$ [7]. Since scalar or cap products pair $H_{N-m}(X - U)$ and $\dot{H}^{N-m}(X - U)$ orthogonally to J_k (where $\dot{H}$ refers to cohomology with compact supports),

$$(1.6)\qquad \dim (X' - U') = M \Rightarrow H_{M+i}(X' - U') = 0 \text{ for } i > 0.$$

To bring out the underlying ideas we take up the generalized Kakutani problem separately. Here, $k = n = 2m + 1$ is an odd prime. The cohomology ring $H^*(SO(n)/C)$ is essentially covered by Lemmas 10.1, 10.3 and 7.4 of [2]. More precisely we start with the principal bundle

$$Q = [SO(n) \times E(C)]_C, \quad SO(n), \quad B_C, \quad p \tag{2.1}$$

where the bracket notation indicates cosets with respect to C. A Vietoris-Begle theorem argument yields

$$H^*(SO(n)/C) \approx H^*([SO(n) \times E]_C). \tag{2.2}$$

Since there is no k torsion $k \neq 2$, [1],

$$\begin{aligned} E_2^{pq} &= H^p(B_{J_k}, H^q(SO(n))) \\ &= B_{J_k}^p \otimes H^q(SO(n)). \end{aligned} \tag{2.3}$$

Let T be the torus of maximal rank in $SO(n)$ [1]. Let G be the subgroup of elements of order k in T. The following facts are known [2].

$$\begin{aligned} H^*SO(n) &\approx \Lambda(u_3, u_7, \cdots, u_{4m-1}) \\ B_C^* &\approx \Lambda(a) \otimes J_k(b) \qquad \dim a = 1, \quad \dim b = 2 \\ B_G^* &\approx \Lambda(a_1, \cdots, a_m) \otimes J_k(b_1, \cdots, b_m) \\ B_T^* &\approx J_k(t_1, \cdots, t_m) \qquad \dim t_i = 2, \end{aligned} \tag{2.4}$$

where $J_k(\quad)$ and $\Lambda(\quad)$ refer to the polynomial and to the exterior algebra respectively. Here $\{u_{4i-1}\}$ constitute universally transgressive generators (forming a basis for the module of primitive elements) of $H^*(SO(n), J_k)$ [1]. Moreover

$$\begin{aligned} &\text{(a)} \quad \rho^*(T, SO(2m+1))B^*_{SO(2m+1)} = J_k(p_0, p_4, \cdots, p_{4i}, \cdots, p_{4m}) \\ &\text{(b)} \quad \rho^*(T, SO(2m)) = J_k(\sigma_0, \cdots, \sigma_{m-1}, \sqrt{\sigma_m}) \end{aligned} \tag{2.5}$$

where p_{4i}, the Pontriagin class of dimension $4i$ reduced mod k, is explicitly

$$p_{4i} = \sigma_i = \sum t_{j_1}^2 \cdots t_{j_i}^2 \tag{2.6}$$

i.e. the elementary symmetric function in $\{t_j^2 \mid 1 \leq j \leq m\}$, ($\sqrt{\sigma_m}$ is usually written W_{2n} and is the product $t_1 \cdots t_m$ [3]). Thus

$$\rho^*(T, SO(2m+1))B^*_{SO(2m+1)} = J_k\left(\textstyle\prod_{i=1}^{i=m} (1 + t_i^2)\right). \tag{2.7}$$

The passage from B_T^* to B_G^* is a monomorphism which replaces t_i by b_i and that from B_G^* to B_C^* takes b_1 to ib. Hence writing $P\,|_{4i}$ for the term of degree $4i$ in the polynomial P, and $v_{4i} \in B^{4i}_{SO(m)}$ for the image by transgression of u_{4i-1},

$$\begin{aligned} \rho^*(C, SO(n))v_{4i} &= \textstyle\prod_{i=1}^{i=m} (1 + (ib)^2)\,|_{4i} \\ &= \sum \Delta_i(1^2, \cdots, m^2)b^{2i}\,|_{4i} \end{aligned} \tag{2.8}$$

where Δ_i is the i^{th} symmetric function in the arguments $1^2, \cdots, m^2$ (that is to say Δ_i consists of the sums of products of i different squares chosen from $1^2, \cdots, m^2$).

LEMMA 1.
$$\Delta_0 = 1$$
$$\Delta_i = 0 \qquad i < m \bmod 2m + 1.$$

The proof is elementary.

Write A for Δ_m. Then by Lemma 1,

$$\prod_{i=1}^{i=m} (1 + (ib)^2) = 1 + Ab^{2m}. \tag{2.9}$$

Since $SO(n)$ is k torsion free, for $k \neq 2$ [1], it is easy to see that $SO(n)$ and B_C^* satisfy the conditions for [1, Proposition 2.21] so that the transgression relation takes the form

$$d_{4i}\, p^2{}_{4i}(1 \otimes u_{4i-1}) = p^2{}_{4i}(\rho^*(C, SO(n))v_{4i} \otimes 1) \tag{2.10}$$

where $p^2{}_{4i}$ is defined in [4, p. 431]. The right hand side vanishes according to (2.8) unless $i = m$, in which case

$$\rho^*(C, SO(n))v_{4m} = Ab^{2m} \tag{2.11}$$

so

$$d_{4m}(p^2{}_{4m}(1 \otimes u_{4m-1})) = p^2{}_{4m}(Ab^{2m} \otimes 1). \tag{2.12}$$

Since $d_{4m+j} = 0$ for $j > 0$ it follows that $E_\infty = E_{4m+1}$ and hence s in [2, Proposition 10.3] is $2m$. In short

$$E_\infty = \Lambda(a) \otimes J_k(b)/\mathcal{I}(b^{2m}) \otimes \Lambda(u_3, \cdots, u_{4m-5}) \tag{2.13}$$

where $\mathcal{I}(b^{2m})$ is the ideal generated by b^{2m}. By Theorems 7.4 and 7.5 of [2], E_∞ can be identified with $H^*(SO(n)/C)$. Accordingly ab^{2m-1} corresponds to the highest nonzero element, $I(4m - 1) = I(2n - 3)$.

To indicate the space, $I(m, Y)$ is written for $I(m)$ of (1.3). Moreover as remarked earlier (since A is metric) Theorem 5.3, page 382 of [4], can be extended to assert that the inclusion map implies $I(i, X) \to I(i, A)$.

THEOREM 2. *If f maps S^{n-1} into R, n an odd prime, then the ordered n-tuples $w^1, \cdots, w^n$ with $f(w^1) = \cdots = f(w^n)$ constitute a set D considered imbedded in $SO(n)$. Identification into cyclic n-tuple classes yields the orbit space D' with*

$$H_{N-j}(D') \neq 0, \qquad n - 1 \leq j \leq 2n - 3, \quad \dim SO(n) = N.$$

It is known [4], [10] that[2] $\nu(S^{2N+1}) = 2N + 1$. Moreover $\nu(R^n - \Delta) = \nu(S^{n-2})$.

[2] A simple calculation in the spirit of this paper is of interest. The cohomology ring of the lens space S^{2N+1}/J_k is

$$H(S^{2N+1}/J_k) = \Lambda(a) \otimes J_k(b)/I(b^{N+1})$$

with $\dim a = 1$, $\dim b = 2$ and $I(b^{N+1})$ the ideal generated by b^{N+1}. Then $I(1)$ and $I(2)$ can be identified with a and b and plainly the highest nonvanishing product is ab^N. This has dimension $2N + 1$.

By the italicized critical property of the index

$$n - 2 = \nu(R^n - \Delta) \geq \nu(A).$$

Thus $I(i, A) = 0, n - 1 \leq i$. On the other hand $I(i, SO(n)) \neq 0, i \leq 2n - 3$. Hence by exactness in (1.5) with $X' = SO(n)/C$, there is an antecedent of $I(i, SO(n))$ in $H^i(X', A')$ for $n - 1 \leq i \leq 2n - 3$. Transferring this data to the lower row in (1.5) shows for instance that there is a nontrivial homology class in $H_{N-i}(X' - A')$ for $i = n - 1$. Moreover by commutativity in the squares of (1.5) a representative cycle for this homology class does not bound in X'.

COROLLARY 3. *Under the hypotheses above*

$$\dim D \geq \tfrac{1}{2}(n - 1)(n - 2).^3$$

Since the orbits consist of k points, by (1.6)

$$N = \dim \frac{SO(n)}{C} = \dim SO(n) = \frac{n(n-1)}{2}.$$

Hence by virtue of Theorem 2 and (1.6)

$$\begin{aligned} N - n + 1 &= \tfrac{1}{2}(n - 1)(n - 2) \\ &= \dim SO(n - 1). \end{aligned}$$

In particular the sharper form of the original Kakutani theorem is that for $n = 3$ there is a nonbounding cycle on D/J_3. Cf. Corollary 5.

COROLLARY 4. *For some $x_0 \in R$, there is a subset D_0 of D consisting of n-tuples for which $x_0 = f(w^1) = \cdots = f(w^n)$ where* $\dim D_0 \geq \frac{1}{2}n(n - 3)$.

Consider the map $F \mid D : D \to R$. Since D is closed in $SO(n)$ and therefore compact $F \mid D$ is a closed mapping. By [7, p. 91] for some x_0

$$\dim F^{-1}(x_0) \geq \dim D - \dim R = \tfrac{1}{2}n(n - 3).$$

COROLLARY 5. *Let K be a convex body in R^n. The dimension of the set of circumscribing cubes is at least $\frac{1}{2}(n - 1)(n - 2)$. There is a set of dimension $\geq \frac{1}{2}n(n - 3)$ of such cubes of the same edge length.*

Every point w on S^{n-1} determines a direction. Define $f(w)$ as the distance between the two support planes orthogonal to this direction. Then Corollaries 3 and 4 apply. We give a little more detail. Thus to every $\bar{w}$ corresponds a circumscribing orthogonal parallelopipedon $L(\bar{w})$ of side lengths

$$(f(w'), \cdots, f(w^n)).$$

Since $f(w) = f(-w)$, $L(\bar{w})$ is unaffected by replacing w^i by $-w^i$ or by action of the permutation group P_r. If L is the set of such parallelopipedons there

[3] A paper of Cairns [6] is concerned with the multiplicity question for the Kakutani Theorem and asserts the first conclusion of our Corollary 3 for all n. The argument proposed is an application of [7, p. 91] but seems inadequate as it stands.

is a 1-1 correspondence

$$L \leftrightarrow \frac{O(n)}{(J_2)^n \times P_n}.$$

Assign to L the topology of the right hand space. We have

$$L \xleftarrow{q} O(n) \xrightarrow{i} SO(n) \xrightarrow{\eta} SO(n)/C$$

where q, i and η are the obvious projections and since the antecedents of points under these projections are finite point sets,

$$\dim D' = \dim D = \dim (i^{-1}D) = \dim (qi^{-1}D = L).$$

It seems quite likely that for K a 3-dimensional cube there is no continuum of circumscribing cubes of constant edge length though by the first part of Corollary 5 there is one of continuously varying nonconstant edge length. This would indicate that the lower dimensional bounds for the two situations described in Corollary 5 are really different and in fact are best possible at least for $n = 3$. Similar comments hold for Corollaries 3 and 4.

The more general case where $k = 2e + 1 \neq n = 2m + 1$, k a prime, proceeds similarly. The orbit space $W' = V_{n,k}/C$ can also be represented topologically by the left coset space

$$W' = SO(n)/(SO(n-k) \times C). \tag{3.1}$$

This representation is suggested formally by

$$SO(n)/SO(n-k)/\frac{SO(n-k) \times C}{SO(n-k)}.$$

We give the justification.

The group $SO(n) = \{s\}$ acts on W by

$$s\bar{w} = s(w^1, \cdots, w^k) = (sw^1, \cdots, sw^k) \quad \text{and} \quad gs\bar{w} = sg\bar{w}.$$

The action on $W' = \{[\bar{w}]_C\}$ is again denoted somewhat ambiguously by s and is given by

$$s[\bar{w}]_C = [s\bar{w}]_C.$$

Let the initial n-tuple be

$$e_1 = (1, 0, \cdots), \cdots, e_n = (0, \cdots, 1)$$

and let the initial k-tuple be $\bar{w}_0 = (e_1, \cdots, e_k)$. Then

$$s_0[\bar{w}_0]_C = [\bar{w}_0]_C$$

implies and is implied by

$$s_0 \sim (\mathbf{M}^i_{\mathbf{r}(n-k)}) \qquad 0 \leq i \leq k-1$$

where M^i is the i^{th} power of the obvious cyclic permutation matrix [5, Eq. 4.01] and $\mathbf{r}$ is the matrix representative of some $r \in SO(n-k)$. Thus

$$s_0 \in C \times SO(n-k) \quad \text{and} \quad ss_0[w_0]_C = [sw_0]_C.$$

Denote $SO(n - k) \times C$ by S. To the point $[w] = [sw_0]_C$ of the orbit space W' make correspond the coset $[s]_S$ composed of $\bigcup_{s_0 \in S} ss_0$. The correspondence is evidently 1-1 and justifies (3.1).

The parallel to (2.1), (2.2) and (2.3) is the principal bundle,

$$Q = [SO(n) \times E_S]_S , B_S , SO(n)$$

whence, using the Vietoris-Begle theorem,

$$H^* (SO(n)/S) \approx H^*([SO(n) \times E_S]_S).$$

Furthermore from [2, Theorem 7.4]

$$(3.2) \qquad \begin{aligned} E_2^{pq} &= H^p(B_S) \otimes Hq(SO(n)) \approx E_2^{p0} \otimes E_2^{0q} \\ E_\infty &= H(SO(n)/S) \end{aligned}$$

We denote the maximal torus for $SO(n - k)$ by T' and use $\{s_i\}$ in place of $\{t_i\}$ for the arguments. Thus with $r = \frac{1}{2}(n - k)$

$$B_{T'}^* = J_k(s_1 , \cdots , s_r).$$

The Kunneth Theorem applied to $B_{J_k \times T'}^*$ enables us to represent $\rho^*(J_k \times T', G)$ by

$$(3.3) \qquad \begin{aligned} b_i &\to ib & i &\leq e \\ b_i &\to s_{i-e} & e < i &\leq \tfrac{1}{2}(n + k - 1). \end{aligned}$$

The compositions for $\rho^*(S, SO(n))$ yield

$$(3.4) \qquad \begin{aligned} g_{4i} &= \rho^*(S, SO(n))v_{4i} \\ &= \rho^*(S, J_k \times T')\rho^*(J_k \times T', G^m)\rho^*(G^m, T)\rho^*(T, SO(n))v_{4i} . \end{aligned}$$

Since $\rho^*(S, J_k \times T')$ is a monomorphism we may omit it.

The Kunneth Theorem decomposition of B_S^* indicates g_{4i} is obtained as a sum of terms of the form $\alpha\beta$, where α relates to $B_{SO(n-k)}^*$ and β to $B_{J_k}^*$ and $\dim \alpha + \dim \beta = 4i$. A convenient representation is given by

$$g_{4i} = \rho^*(S, SO(n))v_{4i} = \sum \sigma_i(s_1^2 , \cdots , s_r^2) \prod_{i=1}^{i=k'} (1 + (ib)^2)|_{4i} .$$

By Lemma 1 this is

$$(3.5a) \qquad \begin{aligned} g_{4i} &= (\sum (\sigma_i)(1 + Ab^{k-1}) \,|_{4i} \\ &= \sigma_i + \sigma_{i-e}\, Ab^{k-1} & i \leq r. \end{aligned}$$

Since $r + e = m$

$$(3.5b) \qquad g_{4m} = \sigma_r\, Ab^{k-1}.$$

Here A is a nonvanishing constant, and v_{4i} is again the image by transgression of u_{4i-1}.

The parallel to (2.10) is then

$$
(3.6)\qquad \begin{aligned} d_{4i}\, p^2_{4i}(1 \otimes u_{4i-1}) &= p^2_{4i}((\rho^*(S, SO(n)v_{4i}) \otimes 1 \\ &= p^2_{4i}(g_{4i} \otimes 1). \end{aligned}
$$

That is to say $p^2_{4i}(g_{4i} \otimes 1)$ is a boundary for $i \leq m$. The ideal of these bounding terms is denoted by $\mathcal{J}$. Since $\{p_{4i} \mid i \leq r\}$ is a collection of algebraically independent terms, (3.5a) guarantees $\{g_{4i} \mid i \leq r\}$ is also. Since $(n - k)$ is even (2.5b) applies so

$$
\begin{aligned} E_{4i+1} &= \frac{B^*_S}{\mathcal{J}} \otimes \Lambda(u_{4r+3}, \cdots, u_{4m-1}) \\ &= \frac{J_k(\sqrt{\sigma_r})}{\mathcal{J}(\sigma_r)} \otimes \Lambda(a) \otimes J_k(b) \otimes \Lambda(u_{4r+3}) \end{aligned}
$$

where $\mathcal{J}(\sigma_r)$ is the ideal in $J_k(\sqrt{\sigma_r})$ generated by σ_r. Let q be the smallest integer such that $r < qe$. Then

$$g_{4qe} = \sigma_{(q-1)e} Ab^{k-1}.$$

Now

$$\sigma_{je} = -\sigma_{(j-1)e} Ab^{k-1} \mod \mathcal{J}.$$

Hence

$$g_{4qe} = \pm A^q b^{q(k-1)}.$$

Note also that $(q + 1)e > r + e = m$. Since u_{4m-1} is the term of maximum degree in the exterior product arising from $B^*_{SO(n)}$ in E_2, no boundary terms enter for $r > qe$ (cf. 3.6). Moreover $d_s = 0$ for $4r + 1 \leq s < 4qe$. Accordingly in view of (3.2)

$$
\begin{aligned} H^*(SO(n)/S) &= E_\infty = E_{4qe+1} \\ &= \frac{J_k(\sqrt{\sigma_r})}{\mathcal{J}(\sigma_r)} \otimes \frac{(a) \otimes J_k(b)}{\mathcal{J}(b^{qk-1})} \otimes \Lambda(u_{4qe+3}, \cdots, u_{4m-1}). \end{aligned}
$$

In short

$$\nu(V_{n,k}) = \dim (ab^{2qe-1}) = 4qe - 1 \qquad [5, \text{Eq. } 4.07].$$

THEOREM 6. $H_{N-j}(D/J_k) \neq 0 \quad (k - 1)l \leq j \leq 2(k - 1)\left[\dfrac{n-1}{k-1}\right] - 1,$

where [ ] *is the integer part*

$$\dim D/J_k = \dim D \geq \tfrac{1}{2}k(2n - k - 1) - (k - 1)l.$$

The proof is similar to that of Theorem 2. Remark first that $\nu(A) \leq (k - 1)l - 1$. Hence the element $I(j, X')$ of $H^j(X')$ maps into 0 in $H^j(A')$ for j restricted as above. Then appeal to exactness in (1.5) when $X' = V_{n,k}/C$ yields the first assertion of the theorem. Next note

$$
\begin{aligned} N = \dim (V_{n,k}/C) &= \dim (V_{n,k}) \\ &= \dim SO(n) - \dim SO(n - k) = \tfrac{1}{2}k(2n - k - 1). \end{aligned}
$$

COROLLARY 7. *For some point x in R^l the dimension of the cyclic classes mapping into $\bar{x}$ is at least*

$$N - kl = k\left(\frac{2n - k - 1}{2} - l\right).$$

Again since mappings of compact spaces are necessarily closed and since $F \mid D$ maps the compact subset D into Δ, then for some element $\bar{x}$ of Δ

$$\dim F^{-1}(\bar{x}) \geq \dim D - \dim \Delta = k\left(\frac{2n - k - 1}{2} - l\right).$$

The conclusions in this paper admit extension to general values of k. Thus the decompositions in (2.5) and in (3.9) depend on results valid for cohomology over J_p when p is a prime dividing k [2, Sections 10 and 11] though for $p = 2$ some complications enter due to torsion. What is needed for the methods in this paper is an extension of the index. For instance if $\boldsymbol{K}$, C is a couple in the sense of [4, p. 332] where C is cyclic of nonprime order with composition series $\{C_i\}$ and composition factors $\{\bar{C}_i\}$ an individual index can be associated with each $\boldsymbol{K}/C_i$, $\bar{C}_i$.

Added in proof. The text conjecture following the proof of Corollary 5 has been validated in joint work with C. W. Mendel in the form that up to obvious symmetries there is at most a single circumscribing cube K with assigned edge length for each cube k.

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