JOINS OF SUBNORMAL SUBGROUPS

BY

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1. Introduction

(1.1) *Subnormal subgroups.* A subgroup H of a group G is said to be *subnormal* in G if there exist subgroups $H_0, H_1, \cdots, H_r$ such that

$$H = H_0 \leq H_1 \leq \cdots \leq H_r = G \tag{1}$$

where r is finite and, for $i = 0, 1, \cdots, r - 1$, H_i is normal in H_{i+1}. We shall use the familiar notation $H_i \lhd H_{i+1}$ to express this last fact; the relation of H to G is expressed by writing $H \lhd^r G$.

Let G be a group and let $\mathrm{sn}(G)$ denote the set of all subnormal subgroups of G. It is known that $\mathrm{sn}(G)$ is closed with respect to forming finite intersections of its members. In this paper we are concerned with the problem of deciding when a set of subnormal subgroups of G generate a subnormal subgroup and in particular, therefore, when $\mathrm{sn}(G)$ is closed with respect to forming finite joins of its members. Denote by $\mathfrak{S}$ the class of all groups in which the join of any pair (and hence of any finite number) of subnormal subgroups is subnormal. Then $\mathfrak{S}$ is the class of all groups G for which $\mathrm{sn}(G)$ is a lattice (with respect to the operations set intersection and group theoretical join).

In a well-known paper [7] Wielandt showed that $\mathfrak{S}$ contains the class of all groups which satisfy the maximal condition for subnormal subgroups. We are able to prove the following generalization of Wielandt's result.

Theorem 4.3(i). *If G is a group whose derived group G' satisfies the maximal condition for subnormal subgroups, then G belongs to $\mathfrak{S}$.*

Our main result on the class $\mathfrak{S}$ is the following.

Theorem 5.2. *An extension of a group in the class $\mathfrak{S}$ by a group which satisfies the maximal condition belongs to the class $\mathfrak{S}$.*

Another line of investigation is to impose conditions on a pair of subnormal subgroups which will make their join subnormal. A sample of our results in this direction is the following (Corollary 2 to Theorem 5.2).

If H and K are subnormal in G and their join J is an extension of an Abelian group by a group satisfying the maximal condition, then J is subnormal in G.

In §6 examples of groups which fail to belong to the class $\mathfrak{S}$ are given. Let $\mathfrak{S}^\infty$ be the class of all groups in which the join of any set of subnormal subgroups is always subnormal. In §8 it is shown that $\mathfrak{S}^\infty$ is a proper sub-

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class of $\mathfrak{S}$ and also that an analogue of Theorem 5.2 holds for the class $\mathfrak{S}^\infty$. This is

THEOREM 8.3. *An extension of a group in the class $\mathfrak{S}^\infty$ by a group which satisfies the maximal condition for subnormal subgroups belongs to the class $\mathfrak{S}^\infty$.*

(1.2) *Finiteness conditions.* A set of subgroups of a group is said to satisfy the *maximal condition* (Max) if and only if each ascending chain in the set has finite proper length (i.e., has finite length after deletion of any repetitions). If the set of all subgroups of a group G satisfies Max, we say that G satisfies Max; if merely the set $\mathrm{sn}(G)$ of all subnormal subgroups of G satisfies Max, we say that G satisfies Max-s, the *maximal condition for subnormal subgroups.* We recall two well-known results: (1) G satisfies Max if and only if each subgroup of G can be finitely generated; (2) each finitely generated nilpotent group satisfies Max, (see for example [5]).

(1.3) *Conjugates and commutators.* Let $x, y, z, \cdots$ be group elements. x^y denotes the conjugate $y^{-1}xy$ of x and $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy = x^{-1}x^y$. By $[x, y, z]$ we understand $[[x, y], z]$, and similarly for higher commutators. The following three identities are well known.

$$[xy, z] = [x, z]^y[y, z] \tag{2}$$

$$[x, yz] = [x, z][x, y]^z \tag{3}$$

$$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1. \tag{4}$$

If $(X_\lambda)_{\lambda\in\Lambda}$ is a set consisting of non-empty subsets and of elements of a group, $\{X_\lambda : \lambda \in \Lambda\}$ is the join or subgroup generated by the X_λ's. Let X, Y be non-empty subsets and let H be a subgroup of a group. X^H is defined to be the group generated by all the conjugates x^h, $x \in X$, $h \in H$. X^H is the *normal closure* of X in $\{X, H\}$, i.e., the smallest normal subgroup of $\{X, H\}$ which contains X. Clearly $\{X\}^H = X^H$ and $X^{\{X,H\}} = X^H$. By $[X, H]$ or $[H, X]$ we mean the group generated by all the commutators $[x, h]$, $x \in X$, $h \in H$.

It is clear from the definitions that

$$X^H = \{X, [X, H]\} \tag{5}$$

and

$$\{H, H^x\} = \{H, [H, x]\}, \tag{6}$$

(where of course $H^x = x^{-1}Hx$). By the identity (3),

$$[X, H]^H = [X, H]. \tag{7}$$

$[H, X, Y]$ is defined to be $[[H, X], Y]$ and similarly for higher commutator subgroups. We will occasionally employ the shorthand notation for higher commutator subgroups suggested by P. Hall in [2]. For example $[H, \underbrace{X, \cdots, X}_{l}]$ is written $\gamma^l HX^l$ and $[H, X, Y, X]$ is written $\gamma^3 HXYX$.

(1.4) *The standard series.* Let X be a non-empty subset of a group G. We define a descending chain of subgroups of G each member of which contains X as follows.

$$X^{G,0} = G$$
$$X^{G,\alpha+1} = X^{X^{G,\alpha}} \qquad \text{for each ordinal } \alpha,$$
$$X^{G,\mu} = \bigcap_{\beta<\mu} X^{G,\beta} \quad \text{for each limit ordinal } \mu.$$

Then $G = X^{G,0} \geq X^{G,1} \geq \cdots$ and $X^{G,\alpha+1} \lhd X^{G,\alpha}$ for all α; this is called the *standard series of* X in G. There exists a first ordinal α such that $X^{G,\alpha} = X^{G,\alpha+1}$. If α is finite and $X^{G,\alpha}$ happens to coincide with $\{X\}$, $\{X\}$ is subnormal in G. The importance of the standard series arises from the fact that the converse of this result is true. Let $\{X\}$ be subnormal in G and suppose that $\{X\} = H_0 \lhd H_1 \lhd \cdots \lhd H_r = G$, r being finite. $X^{G,0} = H_r$; let $i > 0$ and suppose that $X^{G,i-1} \leq H_{r-i+1}$; then

$$X^{G,i} = X^{X^{G,i-1}} \leq X^{H_{r-i+1}} \leq H_{r-i}^{H_{r-i+1}} = H_{r-i}\,.$$

Hence $X^{G,i} \leq H_{r-i}$, for $i = 0, 1, \cdots, r$ and so $\{X\} = X^{G,r}$. Thus $\{X\}$ is subnormal in G if and only if $\{X\} = X^{G,r}$, for some integer r.

The length of the standard series of a subnormal subgroup H of G is called its *index of subnormality* and is denoted by $S(G:H)$. Thus $S(G:H)$ is the least integer r for which $H = H^{G,r}$, and the length of any series from H to G cannot be less than r. This implies that if i is an integer such that $0 \leq i \leq S(G:H)$,

$$S(G:H^{G,i}) = i. \tag{8}$$

As H runs over sn(G), the function $S(G:H)$ assumes non-negative integral values. Obviously $S(G:H) = 0$ if and only if $H = G$ and $S(G:H) = 1$ if and only if H is a proper normal subgroup of G. By equation (8) either $S(G:H)$ takes all non-negative integral values or else it has a least upper bound d in which case $S(G:H)$ assumes each of the values $0, 1, \cdots, d$ and no others.

Let r be a non-negative integer and let X be a non-empty subset of a group G. Then

$$X^{G,r} = \{X, \gamma^r GX^r\}. \tag{9}$$

For $r = 0$ this is evident, if we interpret $\gamma^r GX^r$ as G. Let $r > 0$ and assume that $X^{G,r-1} = \{X, \gamma^{r-1}GX^{r-1}\}$. Then

$$X^{G,r} = X^{X^{G,r-1}} = X^{\gamma^{r-1}GX^{r-1}} = \{X, \gamma^r GX^r\}$$

by (5). One consequence of equation (9) is that in a nilpotent group of class c every subgroup is subnormal with index of subnormality at most equal to c. This is a well-known result.

(1.5) *Classes of groups.* By definition a *class of groups* $\mathfrak{X}$ has the properties (i) $G_1 \cong G \in \mathfrak{X}$ implies that $G_1 \in \mathfrak{X}$, and (ii) $\mathfrak{X}$ contains a unit group. A group in the class $\mathfrak{X}$ is called an *$\mathfrak{X}$-group*. If $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes of groups, $\mathfrak{X}\mathfrak{Y}$ is defined to be the class of all groups which are extensions of $\mathfrak{X}$-groups by $\mathfrak{Y}$-groups. The following alphabet of classes of groups is used:

$\mathfrak{A}$ = Abelian groups,	$\mathfrak{F}$ = finite groups,
$\mathfrak{N}$ = nilpotent groups,	$\mathfrak{G}$ = finitely generated groups,
$\mathfrak{N}_c$ = nilpotent groups of class $\leq c$,	$\mathfrak{M}$ = groups satisfying Max,
	$\mathfrak{M}_s$ = groups satisfying Max-s.

The term *closure operation* is used in the sense of Hall ([3]). We use only the closure operations s, s_n, Q; if $\mathfrak{X}$ is any class of groups then by

$$s\mathfrak{X}, \qquad s_n\,\mathfrak{X}, \qquad Q\mathfrak{X},$$

we mean respectively the class of groups embeddable in an $\mathfrak{X}$-group, the class of groups subnormally embeddable in an $\mathfrak{X}$-group, the class of quotient groups of an $\mathfrak{X}$-group. $\mathfrak{X}$ is said to be (for example) *s-closed* if and only if $\mathfrak{X} = s\mathfrak{X}$, that is, a subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.

2. Elementary results

The following facts about subnormal subgroups are fundamental and are, for the most part, immediate consequences of the definitions. Let H, K, L be subgroups of a group G.

(i) If H is subnormal in K and K is subnormal in L, then H is subnormal in L and

$$S(L:H) \leq S(L:K) + S(K:H).$$

(ii) If H is subnormal in K, then $H \cap L$ is subnormal in $K \cap L$ and

$$S(K \cap L : H \cap L) \leq S(K:H).$$

In particular if $H \leq L \leq K$, H is subnormal in L and

$$S(L:H) \leq S(K:H).$$

(iii) Let α be a homomorphism of G into some group. Then if H is subnormal in K, H^α is subnormal in K^α and

$$S(K^\alpha:H^\alpha) \leq S(K:H).$$

Suppose that H contains the kernel of α. Then H is subnormal in K if and only if H^α is subnormal in K^α, and,

$$S(K^\alpha:H^\alpha) = S(K:H)$$

if either side exists.

Intersections of subnormal subgroups.

LEMMA 2.1. *Let H_λ, K_λ, $(\lambda \in \Lambda)$, be subgroups of a group and let*

$H = \bigcap_{\lambda \in \Lambda} H_\lambda$, $K = \bigcap_{\lambda \in \Lambda} K_\lambda$. *If H_λ is subnormal in K_λ and $S(K_\lambda:H_\lambda) \leq d$ for all $\lambda \in \Lambda$ and some integer $d \geq 0$, then H is subnormal in K and $S(K:H) \leq d$.*

Proof. By hypothesis $\gamma^d K_\lambda H_\lambda^d \leq H_\lambda$ for each $\lambda \in \Lambda$. Therefore

$$\gamma^d KH^d \leq \gamma^d K_\lambda H_\lambda^d \leq H_\lambda$$

and so

$$\gamma^d KH^d \leq H.$$

In particular if $H_1, \cdots, H_n$ is a set of subnormal subgroups of a group G, $H = \bigcap_{i=1}^n H_i$ is subnormal in G and $S(G:H) \leq \max_{i=1,2,\cdots,n} S(G:H_i)$. Thus for any group G the set $\mathrm{sn}(G)$ is closed with respect to forming intersections of finite numbers of its members—this is a well-known fact. However it is easy to see that the intersection of even a countably infinite descending chain of subnormal subgroups need not be subnormal.

For example let G be the group with two generators a and b subject only to the defining relations

$$bab = a^{-1} \quad \text{and} \quad b^2 = 1.$$

G is of infinite dihedral type (and could also have been defined as a free product of two groups each of order 2). Let $H(m, n) = \{a^m b, a^{2^n}\}$, where the integers m and n satisfy $0 \leq m < 2^n$. A simple calculation shows that $H(m, n)$ is subnormal in G and its index of subnormality is equal to n. (Indeed apart from the subgroups of the infinite cyclic group $\{a\}$, which are obviously all normal in G, the $H(m, n)$'s are the only subnormal subgroups of G.) Define

$$H(m) = \bigcap_{2^n > m} H(m, n).$$

Then $H(m) = \{a^m b\}$, which is easily seen to coincide with its normalizer in G and, a fortiori, is not subnormal in G.

The join problem. We now consider the main topic of this paper, the problem of deciding when the join of two subnormal subgroups of a group is subnormal. The starting point for an attack on the "join problem" is the next lemma.

LEMMA 2.2. *Let H and K be subnormal subgroups of a group G and let $J = \{H, K\}$. Then if H is normal in J, J is subnormal in G and*

$$S(G:J) \leq S(G:H)S(G:K). \tag{10}$$

Proof. Let $H = H_r < H_{r-1} < \cdots < H_1 < H_0 = G$ be the standard series of H in G, where $r = S(G:H)$. We prove first that each H_i is normalized by K. If $i = 0$ this is clear, so let $i > 0$ and assume that $H_{i-1}^K = H_{i-1}$. Then

$$H_i^K = (H^{H_{i-1}})^K = (H^K)^{H_{i-1}} = H^{H_{i-1}} = H_i,$$

since $H \lhd J$. Hence $H_i^K = H_i$ and $H_i \lhd H_{i-1} K, (i > 0)$. Now K is subnormal in G (and hence in $H_{i-1}K$), so that H_iK is subnormal in $H_{i-1}K$ and also $S(H_{i-1}K:H_iK) \leq S(H_{i-1}K:K) \leq S(G:K)$, (using the properties (ii) and (iii) at the beginning of this section). Hence $J = HK$ is subnormal in G and $S(G:J) \leq r\cdot S(G:K) = S(G:H)S(G:K)$.

As a corollary of Lemma 2.2 we have a very useful criterion for two subnormal subgroups to have subnormal join.

LEMMA 2.3. *Let H and K be two subnormal subgroups of a group G and let $J = \{H, K\}$. Then the following statements are equivalent.*

(i) *J is subnormal in G.*
(ii) *H^K is subnormal in G.*
(iii) *$[H, K]$ is subnormal in G.*

Proof. Since $[H, K] \lhd H^K \lhd J$, (i) implies (ii) and (ii) implies (iii). Suppose that $[H, K]$ is subnormal in G; then since

$$[H, K] \lhd H^K = \{H, [H, K]\},$$

H^K is subnormal in G, by Lemma 2.2. Also $H^K \lhd J = \{H^K, K\}$, so J is subnormal in G, again by Lemma 2.2. Hence (iii) implies (i).

COROLLARY 1. $\mathfrak{N}\mathfrak{A} \leq \mathfrak{S}$: *all nilpotent-by-Abelian groups are in $\mathfrak{S}$.*

COROLLARY 2. *If in Lemma 2.3 one of the conditions is fulfilled, then*

$$S(G:[H, K]) - 1 \leq S(G:J) \leq \{(S(G:H) - 1)S(G:[H, K]) + 1\}S(G:K).$$

COROLLARY 3. *If H and K are subnormal in G, $J = \{H, K\}$ and $S(G:H) \leq 2$, then J is subnormal in G and*

$$S(G:J) \leq 2S(G:K).$$

Proofs. (i) If $G \in \mathfrak{N}\mathfrak{A}$, G' is nilpotent and so $[H, K]$ is subnormal in G' and hence in G.

(ii) Since $[H, K] \lhd J$, $S(G:[H, K]) \leq S(G:J) + 1$. The other part of the inequality is obtained by working through the proof of the last part of Lemma 2.3 and applying the inequality (10) twice.

(iii) Since $S(G:H) \leq 2$, $H \lhd H_1 \lhd G$, where $H_1 = H^G$. For any $x \in K$, $H^x \lhd H_1$, so $H^K \lhd H_1$. Hence H^K is subnormal in G with $S(G:H^K) \leq 2$. The required result follows from Lemmas 2.2 and 2.3.

LEMMA 2.4. *Let H and K be subnormal in G and let $J = \{H, K\}$. Then if $J = HKH$, J is subnormal in G and*

$$S(G:J) \leq rs(s + 1) \cdots (s + r - 1)$$

where $r = S(G:H)$ and $s = S(G:K)$.

Proof. Let $H = H_t < H_{t-1} < \cdots < H_1 < H_0 = J$ be the standard series of H in J, where $t = S(J:H)$. Let $K_i = H_i \cap K$. If x is any element of H_i, we can write $x = h_1kh_2$, $(h_1, h_2 \in H, k \in K)$. Since $H \leq H_i$,

$$k \in H_i \cap K = K_i$$

and so $H_i = HK_i H$. Also $H_{i+1} \lhd H_i$, so that

$$H_i = HK_i H \leq \{H_{i+1}, K_i\} = H_{i+1} K_i \leq H_i .$$

Hence

$$H_i = H_{i+1} K_i .$$

We will prove by induction on $t - i$ that H_i is subnormal in G. $H_t = H$, so we can assume that $i < t$ and that H_{i+1} is subnormal in G. Now H_i is subnormal in J with index of subnormality equal to i. From this it follows that K_i is subnormal in K and $S(K:K_i) \leq i$. Hence K_i subnormal in G and

$$S(G:K_i) \leq S(G:K) + i = s + i.$$

But we now have the following situation: H_{i+1} and K_i are subnormal in G, $H_i = H_{i+1} K_i$ and $H_{i+1} \lhd H_i$. By Lemma 2.2 H_i is subnormal in G and

$$S(G:H_i) \leq S(G:H_{i+1}) \cdot (s + i) \quad (i = 0, 1, \cdots, t - 1). \tag{11}$$

Since $H_0 = J$, J is subnormal in G. Also $t = S(J:H) \leq r$; therefore by (11)

$$S(G:J) \leq rs(s + 1) \cdots (s + r - 1).$$

COROLLARY. *The join of a pair of permutable subnormal subgroups is subnormal.*

This (probably well-known) result is an immediate consequence of Lemma 2.4.

3. Some lemmas on commutator subgroups

In the proofs of several of our theorems rather frequent use is made of some simple results on commutator subgroups and normal closures. The most important of these is Lemma 3.4, which may be of independent interest.

LEMMA 3.1. *Let H be a subgroup and let x be an element of a group. If $(h_\lambda)_{\lambda \in \Lambda}$ is a set of generators for H, then*

$$[H, x] = \{[h_\lambda, x]^H : \lambda \in \Lambda\}.$$

Proof. Since $[H, x]^H = [H, x]$, the right side is contained in the left. Hence it is sufficient to show that each commutator $[h, x]$, $(h \in H)$, is contained in $\{[h_\lambda, x]^H : \lambda \in \Lambda\} = R$. Let $h = h_{\lambda(1)}^{\varepsilon(1)} \cdots h_{\lambda(n)}^{\varepsilon(n)}$, where $\lambda(i) \in \Lambda$, $\varepsilon(i) = \pm 1$ and $n \geq 1$. Suppose that $n = 1$; then $[h_{\lambda(1)}, x] \in R$ and

$$[h_{\lambda(1)}^{-1}, x] = [h_{\lambda(1)}, x]^{-h_{\lambda(1)}^{-1}},$$

so $[h_{\lambda(1)}^{-1}, x] \in R$. Let $n > 1$; then in view of the equation

$$[h, x] = [h_{\lambda(1)}^{\varepsilon(1)} \cdots h_{\lambda(n-1)}^{\varepsilon(n-1)}, x]^{h_{\lambda(n)}^{\varepsilon(n)}} [h_{\lambda(n)}^{\varepsilon(n)}, x],$$

$[h, x] \in R$, by induction on n and the case $n = 1$.

COROLLARY. *Let H and K be subgroups of a group and let $(h_\lambda)_{\lambda\epsilon\Lambda}$ and $(k_\mu)_{\mu\epsilon M}$ be sets of generators for H and K respectively. Then*

(i) $[H, K] = \{[h_\lambda, K]^H : \lambda \epsilon \Lambda\}$.

(ii) $[H, K] = \{([h_\lambda, k_\mu]^K)^H : \lambda \epsilon \Lambda, \mu \epsilon M\}$.

(These can be proved by a straight-forward application of Lemma 3.1.)

LEMMA 3.2. *Let H and K be subgroups of a group and let $(k_\mu)_{\mu\epsilon M}$ be a set of generators for K. Then for each integer $m > 1$,*

$$[H, K] = \{[H, k_{\mu(1)}, \cdots, k_{\mu(r)}], [H, k_{\mu(1)}, \cdots, k_{\mu(m-1)}, K] : \mu(i) \epsilon M, r = 1, \cdots, m-1\}.$$

Proof. By the corollary to Lemma 3.1,

$$[H, K] = \{[H, k_\mu]^K : \mu \epsilon M\} = \{[H, k_\mu], [H, k_\mu, K] : \mu \epsilon M\}.$$

Let $m > 2$ and suppose that the result has been proved for $m - 1$. Then

$$[H, K] = \{[H, k_{\mu(1)}, \cdots, k_{\mu(r)}], [H, k_{\mu(1)}, \cdots, k_{\mu(m-2)}, K] : \mu(i) \epsilon M, r = 1, \cdots, m-2\}.$$

Now, by the case $m = 2$ already dealt with, $[H, k_{\mu(1)}, \cdots, k_{\mu(m-2)}, K]$ is generated by the subgroups $[H, k_{\mu(1)}, \cdots, k_{\mu(m-2)}, k_\mu]$ and $[H, k_{\mu(1)}, \cdots, k_{\mu(m-2)}, k_\mu, K]$, where μ runs over M. By substituting these generating subgroups in the last expression for $[H, K]$, we obtain the required result.

LEMMA 3.3. *Let H be a subgroup and let $x_1, x_2, \cdots, x_n$ be a finite set of elements of a group. Then for each integer $m \geq 0$ the subgroup generated by the $N = 1 + n + n^2 + \cdots + n^m$ conjugates of H, $H^{x_{i(1)}x_{i(2)}\cdots x_{i(r)}}$, $1 \leq i(1), i(2), \cdots, i(r) \leq n$, $r = 0, 1, \cdots, m$, coincides with the subgroup generated by the N commutator groups $[H, x_{i(1)}, x_{i(2)}, \cdots, x_{i(r)}]$, $1 \leq i(1), \cdots, i(r) \leq n$, $r = 0, 1, \cdots, m$. (We must point out here that according to our convention if $r = 0$, $H^{x_{i(1)}\cdots x_{i(r)}}$ and $[H, x_{i(1)}, \cdots, x_{i(r)}]$ are both interpreted as H.)*

Proof. If $m = 0$, the lemma is trivially true, so suppose that $m > 0$. Now let $C(m)$ and $K(m)$ denote respectively the subgroups generated by the N conjugates of H and the N commutator subgroups. Then

$$C(m) = \{H, \{H^{x_{i(1)}\cdots x_{i(r)}} : 1 \leq i(1), \cdots, i(r) \leq n, r = 0, 1, \cdots, m-1\}^{x_j} : j = 1, 2, \cdots, n\}.$$

By induction hypothesis on m,

$$\{H^{x_{i(1)}\cdots x_{i(r)}} : 1 \leq i(1), \cdots, i(r) \leq n, r = 0, 1, \cdots, m-1\}$$
$$= \{[H, x_{i(1)}, \cdots, x_{i(r)}] : 1 \leq i(1), \cdots, i(r) \leq n, r = 0, 1, \cdots, m-1\}.$$

By combining these two results we obtain

$$(12)\qquad C(m) = \left\{ H, [H, x_{i(1)}, \cdots, x_{i(r)}]^{x_j} : \begin{array}{c} j = 1, \cdots, n, \\ 1 \leq i(1), \cdots, i(r) \leq n, \\ r = 0, 1, \cdots, \quad m - 1. \end{array} \right\}$$

Now $[H, x_{i(1)}, \cdots, x_{i(r)}]$, $(0 \leq r < m)$, is contained in the subgroup $C(m-1)$, generated by all the conjugates $H^{x_{i(1)}\cdots x_{i(s)}}$, $(0 \leq s < m)$, by induction hypothesis, and $C(m-1) \leq C(m)$. Hence it is contained in the right side of (12). In view of the equation

$$\{[H, x_{i(1)}, \cdots, x_{i(r)}], [H, x_{i(1)}, \cdots, x_{i(r)}]^{x_j}\} = \{[H, x_{i(1)}, \cdots, x_{i(r)}], [H, x_{i(1)}, \cdots, x_{i(r)}, x_j]\},$$

$C(m) = K(m)$ follows from equation (12).

Lemmas 3.2 and 3.3 constitute the main step in the proof of the key Lemma 3.4.

LEMMA 3.4. *Let H and K be subgroups of a group and suppose that K has a normal subgroup N such that K/N can be generated by a finite number n of its elements. Then for any positive integer m,*

$$H^K = \{\{H^{k_1}, \cdots, H^{k_t}\}^N, \gamma^m HK^m\}$$

for certain elements k_i in K and $t = 1 + n + n^2 + \cdots + n^{m-1}$.

Proof. If $m = 1$, the statement of the lemma is trivial, so let $m > 1$. By hypothesis there exist elements $x_1, x_2, \cdots, x_n$ in K such that $K = \{x_1, \cdots, x_n, N\}$. Let $K_1 = \{x_1, \cdots, x_n\}$; then $K = K_1 N$, since $N \lhd K$.

By lemma 3.2, $[H, K_1]$ is generated by all the subgroups

$$[H, x_{i(1)}, \cdots, x_{i(r)}] \quad \text{and} \quad [H, x_{i(1)}, \cdots, x_{i(m-1)}, K_1],$$

$1 \leq i(1), \cdots, i(r) \leq n$, $r = 1, 2, \cdots, m-1$. Since $H^{K_1} = \{H, [H, K_1]\}$, H^{K_1} is generated by all the $t = 1 + n + n^2 + \cdots + n^{m-1}$ conjugates of H of the form $H^{x_{i(1)}x_{i(2)}\cdots x_{i(r)}}$, $1 \leq i(1), i(2), \cdots, i(r) \leq n$, $r = 0, 1, \cdots, m-1$, (with our usual convention if $r = 0$), and by $\gamma^m HK_1^m$; here we are applying Lemma 3.3. Hence for suitable elements $k_1, k_2, \cdots, k_t$ of K_1,

$$\begin{aligned} H^K = (H^{K_1})^N &= \{H^{k_1}, \cdots, H^{k_t}, \gamma^m HK_1^m\}^N \\ &\leq \{\{H^{k_1}, \cdots, H^{k_t}\}^N, \gamma^m HK^m\} \leq H^K, \end{aligned}$$

since $\gamma^m HK^m$ is normalized by K and $N \leq K$.

4. Some sufficient conditions in the join problem

Notation. Let A be a subgroup and B a non-empty subset of a group. By $\text{cn}(A, B)$ we shall mean the set of all subgroups of the form

$$\{A^{b_1}, \cdots, A^{b_n}\},$$

and by $\mathrm{cm}(A, B)$ the set of all subgroups of the form

$$\{[A, b_1], \cdots, [A, b_n]\},$$

where n is a positive integer and $b_1, \cdots, b_n$ are elements of B.

LEMMA 4.1. *Let H be a subnormal subgroup and let K be any subgroup of a group G. Then the following are equivalent.*

(i) $\mathrm{cn}(H, K) \leq \mathrm{sn}(G)$.
(ii) $\mathrm{cm}(H, K) \leq \mathrm{sn}(G')$.

Proof. Suppose that $\mathrm{cn}(H, K) \leq \mathrm{sn}(G)$ and let $\{[H, k_1], \cdots, [H, k_n]\}$ be any member of the set $\mathrm{cm}(H, K)$. This subgroup is normal in

$$\{H, [H, k_1], \cdots, [H, k_n]\} = \{H, H^{k_1}, \cdots, H^{k_n}\},$$

which is subnormal in G. Conversely let $\mathrm{cm}(H, K) \leq \mathrm{sn}(G')$ and let $\{H^{k_1}, \cdots, H^{k_n}\}$ belong to $\mathrm{cn}(H, K)$. Without loss of generality we may assume that $k_1 = 1$. Then

$$\{H, H^{k_2}, \cdots, H^{k_n}\} = \{H, [H, k_2], \cdots, [H, k_n]\},$$

both H and $\{[H, k_1], \cdots, [H, k_n]\}$ are subnormal in G and H normalizes $\{[H,k_1], \cdots, [H,k_n]\}$. Thus we conclude via Lemma 2.2 that $\{H, H^{k_2}, \cdots, H^{k_n}\}$ is subnormal in G.

We can now state our first main result.

THEOREM 4.2. *Let H and K be two subnormal subgroups of a group G and let $J = \{H, K\}$. Then either of the following two conditions is sufficient to make J subnormal in G.*

(i) $\mathrm{cn}(H, J)$ *satisfies Max.*
(ii) $\mathrm{cm}(H, K^H)$ *satisfies Max.*

Proof. We will show that (ii) implies (i) and that (i) implies the subnormality of J in G.

Let $\mathrm{cm}(H, K^H)$ satisfy Max. First we note the obvious fact $\mathrm{cn}(H:J) = \mathrm{cn}(H:K^H)$. Let $H_1 \leq H_2 \leq \cdots$ be an ascending chain in the set $\mathrm{cn}(H, K^H)$. Without loss of generality we can assume that

$$H_i = \{H, H^{x_2}, \cdots, H^{x_{m(i)}}\},$$

where $x_i \in K^H$ and $m(2) \leq m(3) \leq \cdots$. Let $L_i = \{[H, x_2], \cdots, [H, x_{m(i)}]\}$. Then

$$L_2 \leq L_3 \leq \cdots$$

is an ascending chain in $\mathrm{cm}(H, K^H)$ and $H_i = \{H, L_i\}$. Hence there exists a positive integer n such that $L_n = L_{n+1} = L_{n+2} = \cdots$ and therefore such that $H_n = H_{n+1} = H_{n+2} = \cdots$. Thus we have proved that $\mathrm{cn}(H, K^H)$, (or $\mathrm{cn}(H, J)$), satisfies Max.

Next we will show by induction on $r = S(G:H)$ that J is subnormal in G. If $r = 0$, $H = G = J$, so we can suppose that $r > 0$. Let $L = \{H^{x_1}, \cdots, H^{x_n}\}$ be any member of the set $\mathrm{cn}(H, J)$. Let $n > 1$ and suppose that $M = \{H^{x_2}, \cdots, H^{x_n}\}$ has been shown to be subnormal in G. Let $\bar{H} = H^G$. Then M is subnormal in $\bar{H}$ and so is H^{x_1}; also $S(\bar{H}:H^{x_1}) = r - 1$. $\mathrm{cn}(H^{x_1}, L)$ satisfies Max, since

$$\mathrm{cn}(H^{x_1}, L) \leq \mathrm{cn}(H, J).$$

Our induction hypothesis on r permits us to conclude that $L = \{H^{x_1}, M\}$ is subnormal in $\bar{H}$ and hence in G. Thus we have proved that

$$\mathrm{cn}(H, J) \leq \mathrm{sn}(G).$$

Since $\mathrm{cn}(H, J)$ satisfies Max, $H^K \in \mathrm{cn}(H, J)$. Hence H^K and therefore J is subnormal in G (by Lemma 2.3).

Corollary. *Let H and K be subnormal in G. If $[H, K]$ satisfies Max, then $J = \{H, K\}$ is subnormal in G.*

For if $L \in \mathrm{cm}(H, K^H)$, $L \leq [H, K^H] = [H, K]$; so the given condition implies that $\mathrm{cm}(H, K^H)$ satisfies Max. The result follows at once from the theorem.

Theorem 4.3. *Let H and K be subnormal in G and let $J = \{H, K\}$. Then either of the following two conditions is sufficient to make J subnormal in G.*
(i) *G' satisfies Max-s.*
(ii) *J' satisfies Max-s.*

Proof. (i) Let G' satisfy Max-s and let $r = S(G:H)$. Assume that $r > 0$. Let $\bar{H} = H^G$; now $\bar{H}'$ satisfies Max-s, so by the induction hypothesis and the argument of the second part of the proof of Theorem 4.2, $\mathrm{cn}(H, J) \leq \mathrm{sn}(G)$. This implies that $\mathrm{cm}(H, J) \leq \mathrm{sn}(G')$, by Lemma 4.1. But by hypothesis $\mathrm{sn}(G')$ satisfies Max and so $\mathrm{cm}(H, J)$ does too. That J is subnormal in G follows via Theorem 4.2.

(ii) Let J' satisfy Max-s. By (i) it follows that $\mathrm{cn}(H, J) \leq \mathrm{sn}(J)$ and by Lemma 4.1 that $\mathrm{cm}(H, J) \leq \mathrm{sn}(J')$. Hence $\mathrm{cm}(H, J)$ satisfies Max and so J is subnormal in G, by Theorem 4.2.

The next theorem has been proved by Baer [1].

Theorem 4.4. *Let H and K be two subnormal nilpotent subgroups of a group G and suppose that $J = \{H, K\}$ can be finitely generated. Then J is nilpotent and subnormal.*

This may of course be proved by means of the Hirsch-Plotkin theorem. However we shall give a different proof based on Lemma 3.4. First a lemma is needed.

Lemma 4.5. *If H and K are two subnormal nilpotent subgroups of a group, $J = \{H, K\}$ and $H \lhd J$, then J is nilpotent.*

Proof. Let $K = K_s < K_{s-1} < \cdots < K_0 = J$ be the standard series of K in J, where $s = S(J:K)$. We can assume that $s > 0$. Let $H_i = K_i \cap H$: then $K_i = H_i K$, since $J = HK$, and $K_i = H_i K_{i+1}$. Now let $i < s$ and suppose that K_{i+1} is nilpotent. H_i is contained in H and hence is nilpotent. $K_{i+1} \lhd K_i$ and $H_i \lhd K_i$ because $H \lhd J$. Hence $K_i = H_i K_{i+1}$ is nilpotent by Fitting's theorem (which asserts that the product of two normal nilpotent subgroups is nilpotent). It follows that $J = K_0$ is nilpotent and that if $C(H)$, $C(K)$ and $C(J)$ are respectively the nilpotent classes of H, K and J, then

$$C(J) \leq C(K) + S(J:K)C(H). \tag{13}$$

Proof of Theorem 4.4. Since J is finitely generated, it can be generated by two finitely generated subgroups, one contained in H and the other in K. Now any subgroup of H or K is subnormal in G and nilpotent, so we may as well assume that H and K are finitely generated.

Let $r = S(G:H)$ and $s = S(G:K)$; we assume that $r > 0$ and $s > 0$ and use induction on r. If d is the nilpotent class of K, $\gamma^{s+d}HK^{s+d} = 1$ and

$$H^K = \{H^{x_1}, \cdots, H^{x_t}\} \tag{14}$$

by Lemma 3.4, where $x_1, \cdots, x_t \in K$, $t = 1 + n + n^2 + \cdots + n^{s+d-1}$ and K can be generated by n elements. Since $S(H^G:H^{x_i}) = r - 1$, H^K is subnormal in G and nilpotent, by (14) and the induction hypothesis. By Lemmas 2.3 and 4.5, J is subnormal in G and nilpotent.

In conclusion we remark that if H and K can be generated by m and n elements respectively, it is possible by the methods of the proof to find upper bounds for $S(G:J)$ and $C(J)$ in terms of $S(G:H)$, $S(G:K)$, $C(H)$, $C(K)$, m and n; the inequalities (10) and (13) are needed. One would of course expect such upper bounds to be rather crude.

5. The classes $\mathfrak{S}$, $\mathfrak{S}_1$, $\mathfrak{S}_2$

We have already defined $\mathfrak{S}$ to be the class of all groups in which each pair of subnormal subgroups generates a subnormal subgroup. It is convenient to introduce two further classes of groups connected with the join problem.

Definitions. (a) A group G belongs to the *class* $\mathfrak{S}_1$ if and only if given a group X with two subgroups Y, Z both of which are subnormal in X, $W = \{Y, Z\}$ is subnormal in X if W can be embedded in G.

Thus $\mathfrak{S}_1$ is the largest s-closed class of groups with the property that, in any group, an $\mathfrak{S}_1$-subgroup which is the join of a pair of subnormal subgroups is itself subnormal.

(b) A group G belongs to the *class* $\mathfrak{S}_2$ if and only if G belongs to the class $\mathfrak{S}$ and, given a group X with Y, Z subnormal subgroups of X, $W = \{Y, Z\}$ is subnormal in X, if W can be embedded subnormally in G, (i.e., if W is isomorphic with a subnormal subgroup of G).

Thus $\mathfrak{S}_2$ is the largest s_n-closed subclass of $\mathfrak{S}$ with the property that an

$\mathfrak{S}_2$-subgroup (of any group) which is the join of a pair of subnormal subgroups is itself subnormal.

LEMMA 5.1. (i) $s_n \mathfrak{S} = \mathfrak{S}$, $s_n \mathfrak{S}_2 = \mathfrak{S}_2$, $s\mathfrak{S}_1 = \mathfrak{S}_1$.
(ii) $\mathfrak{S}_1 < \mathfrak{S}_2 < \mathfrak{S}$.

Proof. The s_n-closure of $\mathfrak{S}$ and $\mathfrak{S}_2$ and the s-closure of $\mathfrak{S}_1$ are immediate. So are the inclusions $\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \mathfrak{S}$. Obviously $\mathfrak{S}$ contains the class $\mathfrak{N}$, but $\mathfrak{S}_2$ does not even contain the class $\mathfrak{N}_2$, by Theorem 6.1 below; hence $\mathfrak{S}_2 \neq \mathfrak{S}$. We will show that $s\mathfrak{S}_2$ contains $\mathfrak{S}_2$ properly; this of course implies that $\mathfrak{S}_1 \neq \mathfrak{S}_2$.

To establish the assertion it is sufficient to show that any countable group can be embedded in a simple group. For $\mathfrak{S}_2$ contains the class of all simple groups (by Theorem 4.3 (ii)), but by the results of the next section $\mathfrak{S}_2$ does not contain the class of all countable groups. One way of proving this is to make use of a theorem of Schreier and Ulam [6] on the normal structure of the group Σ_∞ of all permutations of a countably infinite set. These authors have shown that a proper normal subgroup of Σ_∞ consists entirely of finite permutations (i.e., of permutations fixing all but a finite number of the elements of the set). Let $\bar{\Sigma}_\infty$ be the subgroup of all finite permutations; then $\Sigma_\infty/\bar{\Sigma}_\infty$ is simple. Now let G be any countable group: if G is finite, it can easily be embedded in a countably infinite group, so we can assume that G is infinite. Σ_∞ may be regarded as permuting the elements of G. Thus G can be embedded in Σ_∞ by means of its regular representation $x \rightarrow x^*$, $(x \in G)$, x^* being the permutation of G in which $g \rightarrow gx$, $(g \in G)$. Let G^* be the subgroup of all x^*, $(x \in G)$. Then, since $\bar{\Sigma}_\infty \cap G^* = 1$, G is effectively embedded in $\Sigma_\infty/\bar{\Sigma}_\infty$ as the subgroup $G^*\bar{\Sigma}_\infty/\bar{\Sigma}_\infty$.

The main theorem on the classes $\mathfrak{S}$, $\mathfrak{S}_1$, $\mathfrak{S}_2$.

THEOREM 5.2. *Let $\mathfrak{X}$ denote one of the three classes $\mathfrak{S}$, $\mathfrak{S}_1$, $\mathfrak{S}_2$. Then an extension of an $\mathfrak{X}$-group by a group satisfying Max is an $\mathfrak{X}$-group. In other words $\mathfrak{X}\mathfrak{M} = \mathfrak{X}$.*

For the classes $\mathfrak{S}$, $\mathfrak{S}_2$ there is a slightly more general result.

THEOREM 5.2*. *Let $\mathfrak{X}$ denote one of the two classes $\mathfrak{S}$, $\mathfrak{S}_2$. Then an extension of an $\mathfrak{X}$-group by a group in which each subnormal subgroup is finitely generated is an $\mathfrak{X}$-group.*

Proof. We will prove the more general result for the classes $\mathfrak{S}$ and $\mathfrak{S}_2$. Each proof makes use of the key Lemma 3.4.

(i) Let G be a group with a normal $\mathfrak{S}$-subgroup N such that each subnormal subgroup of G/N is finitely generated; we have to show that G is an $\mathfrak{S}$-group. Let H and K be two subnormal subgroups of G and let $J = \{H, K\}$. The proof that J is subnormal in G is by induction on $r = S(G:H)$; we can assume that $r > 0$ and $s = S(G:H) > 0$. Since KN/N is subnormal in G/N, K/A is finitely generated, where $A = K \cap N$. By Lemma 3.4

$$H^K = \{\{H^{k_1}, \cdots, H^{k_t}\}^A, \gamma^s HK^s\}, \tag{15}$$

where $k_1, \cdots, k_t$ are elements of K and t is finite. Let

$$L = \{H^{k_1}, \cdots, H^{k_t}, \gamma^s HK^s\}. \tag{16}$$

Since $s = S(G:K)$, $\gamma^s HK^s$ is contained in K and $\gamma^s HK^s \lhd K$. Hence $\gamma^s HK^s$ is subnormal in G. Let $\bar{H} = H^G$; $\bar{H} \cap N$ is an $\mathfrak{S}$-group, every subnormal subgroup of $\bar{H}/\bar{H} \cap N$ is finitely generated and also $S(\bar{H}:H^{k_i}) = r - 1$. Therefore, by the induction hypothesis on r, L is subnormal in $\bar{H}$ and hence in G. Let $l = S(G:L)$ and let $B = L \cap N$. By Lemma 3.4

$$A^L = \{A_1^B, \gamma^l AL^l\}, \tag{17}$$

where A_1 belongs to $\operatorname{cn}(A, L)$, (because LN/N is subnormal in G/N and hence L/B is finitely generated). Now $\gamma^l AL^l$ is contained in L and therefore is subnormal in G; $A = K \cap N$ and $B = L \cap N$ are subnormal in G. Hence $M = \{A_1, B, \gamma^l AL^l\}$ is the join of a finite set of subnormal subgroups of the $\mathfrak{S}$-group N. It follows that M is subnormal in G. $B \leq L$, so B normalizes $\gamma^l AL^l$ and $A^L \lhd M$, by equation (17). Thus we have proved that A^L is subnormal in G. But both A and L are subnormal in G and therefore L^A is subnormal in G, by Lemma 2.3. By equations (15) and (16), $L^A = H^K$, and so J is subnormal in G.

(ii) Let H and K be subnormal subgroups of a group G and let $J = \{H, K\}$. Suppose that J has a normal $\mathfrak{S}_2$-subgroup N such that each subnormal subgroup of J/N is finitely generated. It has to be shown that J is subnormal in G. The argument is by induction on $r = S(G:H)$; let $r > 0$ and $s = S(G:K)$. Since $\mathfrak{S}_2 < \mathfrak{S}$, $J \in \mathfrak{S}$, by (i). Let $A = K \cap N$; then K/A is finitely generated, so that

$$H^K = \{H_1^A, \gamma^s HK^s\},$$

where H_1 belongs to $\operatorname{cn}(H, K)$. As before let us write

$$L = \{H_1, \gamma^s HK^s\}$$

and $B = L \cap N$. Since $J \in \mathfrak{S}$, L is subnormal in J, from which we conclude that B belongs to $\mathfrak{S}_2$ and each subnormal subgroup of L/B is finitely generated. The induction hypothesis on r may now be applied to L, as the join of a finite set of subnormal subgroups of $\bar{H} = H^G$, and our conclusion is that L is subnormal in G. Let $l = S(G:L)$; since L/B is finitely generated,

$$A^L = \{A_1^B, \gamma^l AL^l\},$$

where A_1 belongs to $\operatorname{cn}(A, L)$. The group $M = \{A_1, B, \gamma^l AL^l\}$ is subnormal in J, (since $J \in \mathfrak{S}$), and hence in N. Therefore M is subnormal in G, because it is the join of a finite number of subnormal subgroups of G (since $A \lhd K$ and $B \lhd L$) and is subnormal in the $\mathfrak{S}_2$-group N. It follows just as in (i) that J is subnormal in G.

(iii) Let H and K be two subnormal subgroups of G and let $J = \{H, K\}$. Suppose that J has a normal $\mathfrak{S}_1$-subgroup N such that J/N satisfies Max. Then by the argument used in (ii) one can show that J is subnormal in G.

This completes the proof of Theorems 5.2 and 5.2* and we will now mention some special cases.

COROLLARY 1. $\mathfrak{N}(\mathfrak{G} \cap \mathfrak{N}) \leq \mathfrak{S}$. *So, in particular, all finitely generated metanilpotent (i.e., nilpotent-by-nilpotent) groups belong to the class* $\mathfrak{S}$.

COROLLARY 2. $\mathfrak{A}\mathfrak{M}\mathfrak{N} \leq \mathfrak{S}_1$.

COROLLARY 3. *If the join of a pair of subnormal subgroups is finitely generated and metabelian, then it is subnormal.*

The second corollary follows because $\mathfrak{A} \leq \mathfrak{S}_1$, by Lemma 2.2. By Theorem 6.1, $\mathfrak{A}\mathfrak{N}_2$ is not $\leq \mathfrak{S}$ and $\mathfrak{A}^2$ is not $\leq \mathfrak{S}_2$; hence the "finitely generated" cannot be omitted in either Corollary 1 or 3.

Various more complicated results can be read off from Theorem 5.2. For example, $\mathfrak{M}\mathfrak{N}_s \mathfrak{A} \leq \mathfrak{S}_2$, by Theorem 4.3, and hence

$$(\mathfrak{M}\mathfrak{N}_s \mathfrak{A})\mathfrak{M}\mathfrak{N} \leq \mathfrak{S}_2 .$$

There are two further results of the same type, but rather easier.

THEOREM 5.3. (i) *An extension of a group with a composition series of finite length by an* $\mathfrak{S}$*-group is an* $\mathfrak{S}$*-group.*

(ii) *An extension of a nilpotent group by an* $\mathfrak{S}_2$*-group is an* $\mathfrak{S}$*-group.*

Proof. (i) Let G have a normal subgroup N such that G/N is an $\mathfrak{S}$-group. Let us for the moment assume that N is finite and write $C = C_G(N)$. If L and M are subnormal subgroups of C, $\{L, M\} \lhd \{L, M, N\}$ and $\{L, M, N\}$ is subnormal in G, since $G/N \in \mathfrak{S}$. Hence $C \in \mathfrak{S}$. Also G/C is finite and so $G \in \mathfrak{S}\mathfrak{F} = \mathfrak{S}$, by Theorem 5.2.

Now suppose that N has a composition series of finite length m,

$$1 = N_0 < N_1 < \cdots < N_m = N.$$

Let $m > 0$, (if $m = 0$, $N = 1$ and $G \in \mathfrak{S}$), and let $\bar{N}_1 = N_1^G$. $\bar{N}_1 \leq N$ and $N/\bar{N}_1$ has a composition series of length $\leq m - 1$. Thus by induction on m we can assume that $G/\bar{N}_1 \in \mathfrak{S}$. Now N_1 is simple; if it has prime order, $\bar{N}_1$, as the join of subnormal finite subgroups of a group with a composition series of finite length, is finite, [7, Satz 10]; therefore $G \in \mathfrak{F}\mathfrak{S} = \mathfrak{S}$, by the first part. So we can assume that N_1 is a non-Abelian simple group. By a theorem of Wielandt, [8], in any group a subnormal non-Abelian simple subgroup normalizes every other subnormal subgroup. Hence if H and K are subnormal in G and $J = \{H, K\}$, $J \lhd J\bar{N}_1$. However $G/\bar{N}_1 \in \mathfrak{S}$, so $J\bar{N}_1$ is subnormal in G. It follows that $G \in \mathfrak{S}$.

(ii) Let G have a nilpotent normal subgroup N such that G/N is an

$\mathfrak{S}_2$-group. Let c be the nilpotent class of N; if $c = 0$, $N = 1$ and $G \in \mathfrak{S}_2 < \mathfrak{S}$, so we may assume that $c > 0$. Let Z be the centre of N; then Z is characteristic in N and so normal in G. Since N/Z is nilpotent of class $c - 1$, $G/Z \in \mathfrak{S}$ by induction hypothesis on c. Now let H and K be two subnormal subgroups of G and let J be their join. Then JZ is subnormal in G, so what we have to show is that J is subnormal in JZ. Since $N \lhd G$ and Z is the centre of N, $J \cap N \lhd JZ$. Further $H(J \cap N)/J \cap N$ and $K(J \cap N)/J \cap N$ are subnormal in $JZ/J \cap N$ and their join is $J/J \cap N$. But $J/J \cap N \cong JN/N$ which is subnormal in G/N, since $G/N \in \mathfrak{S}_2 < \mathfrak{S}$. By definition of the class $\mathfrak{S}_2$, $J/J \cap N$ is subnormal in $JZ/J \cap N$ and J is subnormal in JZ.

Corollary. $\mathfrak{N}(\mathfrak{G} \cap \mathfrak{A}^2) \leq \mathfrak{S}$. *In particular all finitely generated soluble groups of derived length at most equal to* 3 *belong to the class* $\mathfrak{S}$.

This follows at once from Theorem 5.3 (ii) and Corollary 3 to Theorem 5.2. We remark that in Section 6 it is shown that soluble groups of derived length 3 are not necessarily in $\mathfrak{S}$ if they cannot be finitely generated; also finitely generated soluble groups of derived length 4 do not in general belong to $\mathfrak{S}$ (see Theorems 6.1 and 6.2).

6. Counterexamples

We shall now show how to construct examples of groups not in the class $\mathfrak{S}$. I am grateful to Professor P. Hall for drawing my attention to this construction. A similar example is given in the book by Zassenhaus [9, p. 235].

Let Z denote the set of all integers and let $\mathcal{S}$ be the set of all subsets X of Z such that there exist integers $l = l(X)$ and $L = L(X)$, $l \leq L$, with the property that X contains all integers $\leq l$ and no integers $> L$. Roughly speaking, X contains all large negative integers but no large positive integers.

Let A and B be two elementary Abelian 2-groups with sets of basis elements respectively

$$(a_X)_{X \in \mathcal{S}} \quad \text{and} \quad (b_X)_{X \in \mathcal{S}}\,. \tag{18}$$

For each $n \in Z$ two automorphisms of $M = A \times B$, u_n and v_n, are defined by the rules

$$\begin{gathered}[A, u_n] = 1 = [B, v_n] \\ [b_X\,, u_n] = a_{X+n} \quad \text{and} \quad [a_X\,, v_n] = b_{X+n}\end{gathered} \tag{19}$$

for each $X \in \mathcal{S}$. Our notation here is as follows: if $n_1, n_2, \cdots, n_r$ are integers, (r being finite), and $X \in \mathcal{S}$, $a_{X+n_1+\cdots+n_r}$ is to mean a_Y, where

$$Y = X \cup (n_1) \cup (n_2) \cup \cdots \cup (n_r)$$

if the n_i's are all different and none of them belongs to X; otherwise $a_{X+n_1+\cdots+n_r} = 1$. Similar remarks apply to $b_{X+n_1+\cdots+n_r}$. Also $[b_X\,, u_n]$ is used to denote $b_X^{-1}b_X^{u_n}$. It is easy to check that these definitions make u_n and v_n automorphisms of M.

It follows at once from (19) that for $m, n \in Z$

$$[a_X, u_m, u_n] = [b_X, u_m, u_n] = 1 = [a_X, v_m, v_n] = [b_X, v_m, v_n].$$

Hence

$$(20) \qquad [u_m, u_n] = 1 = [v_m, v_n] \quad \text{and} \quad u_n^2 = 1 = v_m^2 .$$

Let $H = \{u_n : n \in Z\}$ and $K = \{v_n : n \in Z\}$. Then H and K are elementary Abelian 2-groups of rank $\aleph_0$. Let $z_{mn} = [u_m, v_n]$; by the identity

$$[u_m, v_n^{-1}, a_X]^{v_n}[v_n, a_X^{-1}, u_m]^{a_X}[a_X, u_m^{-1}, v_n]^{u_m} = 1$$

and equations (19) and (20), we get

$$(21) \qquad [a_X, z_{mn}] = a_{X+m+n} \quad \text{and similarly} \quad [b_X, z_{mn}] = b_{X+m+n}.$$

For all $l, m, n \in Z$,

$$(22) \qquad z_{mn}^2 = 1 \quad \text{and} \quad [z_{mn}, u_l] = 1 = [z_{mn}, v_l].$$

The first part follows from

$$[a_X, z_{mn}^2] = [a_X, z_{mn}]^2[a_X, z_{mn}, z_{mn}] = [a_X, z_{mn}, z_{mn}] = 1.$$

For the second part one has the identity

$$[b_X, z_{mn}^{-1}, u_l]^{z_{mn}}[z_{mn}, u_l^{-1}, b_X]^{u_l}[u_l, b_X^{-1}, z_{mn}]^{b_X} = 1.$$

By equations (19), (20) and (21), $[b_X, [z_{mn}, u_l]] = a_{X+l+m+n}^2 = 1$. Similarly, $[a_X, [z_{mn}, u_l]] = 1$.

Let $J = \{H, K\}$. Then it follows easily from (22) and Corollary (ii) to Lemma 3.1 that J is nilpotent of class 2 and has exponent 4.

Next it will be shown that

$$(23) \qquad [H, B] = A \quad \text{and} \quad [K, A] = B.$$

Of course it is clear that $[H, B] \leq A$, by (19). Let $X \in \mathcal{S}$ and let n be the largest integer in X. Let $Y = X - (n)$; clearly $Y \in \mathcal{S}$. Then $[H, B]$ contains the element

$$[b_Y, u_n] = a_{Y+n} = a_X .$$

Hence $[H, B] = A$; the other equation follows by symmetry.

Let G be the split extension of M by the group of automorphisms $J = \{H, K\}$.

$$G = JM, \quad M \lhd G \quad \text{and} \quad J \cap M = 1.$$

Now $H^G = (H^M)^K = (H^B)^K = \{H, A\}^K = \{H^K, M\}$, by (23). Hence

$$H^{G,2} = H^{H^G} = (H^{H^K})^M = H^M = H^B = \{H, A\},$$

since H commutes elementwise with $[H, K]$. Thus $H \neq H^{G,2}$. However

$$H^{G,3} = H^{H^{G,2}} = H,$$

since $[H, A] = 1$. Thus we have proved that H is subnormal in G and that $S(G:H) = 3$ (similarly of course for K).

But $J = \{H, K\}$ is not subnormal G. Indeed, J has two much stronger properties than non-subnormality, namely J is neither ascendent nor descendent in G. A subgroup of a group is said to be *ascendent* (*descendent*) if it occurs in an ascending (descending) series in G. (Here we use the term "series" in the sense of Hall [3].) A subnormal subgroup is both ascendent and descendent.

J is not descendent in G because $J^G = G$ and so J is not contained in any proper normal subgroup of G. For $J^G \geq \{H^B, K^A\} = \{H, A, K, B\} = G$, by (23).

J is not ascendent in G because J coincides with its normalizer $N_G(J)$ in G. For if $J < N_G(J)$, then $C_M(J) \neq 1$, since $J \cap M = 1$. But let x be any element of J different from the unit element; then x is a finite product of a_X's and b_X's, $(X \in \mathcal{S})$. Choose $n \in Z$ to be larger than any integer in any X involved in this finite product. Then x cannot commute with both u_n and v_n. Hence $C_M(J) = 1$.

Hence we have proved

THEOREM 6.1. *There exists a group G with a normal subgroup M such that*

(i) *M is a countably infinite elementary Abelian 2-group.*

(ii) *G is a split extension of M by a group of automorphisms J which is nilpotent of class 2 and has exponent 4.*

(iii) *$J = \{H, K\}$ and $H \cap K = 1$, where H and K are countably infinite elementary Abelian 2-groups.*

(iv) *H and K are both subnormal in G and*

$$S(G:H) = 3 = S(G:K).$$

(v) *$J = N_G(J)$ and $J^G = G$. Thus J is neither ascendent nor descendent in G and so is certainly not subnormal in G.*

A finitely generated counterexample. The group G which has just been constructed is a soluble group with derived length 3 not in the class $\mathfrak{S}$. G cannot be finitely generated. It is rather easy to obtain from G a finitely generated soluble group (of derived length 4) which fails to belong to $\mathfrak{S}$.

Let X belong to the set $\mathcal{S}$; then we will define X^* to be the set of all integers of the form $n + 1$, where $n \in X$. Clearly $X^* \in \mathcal{S}$. An automorphism t of G is defined by the rules

$$a_X^t = a_{X^*}, \qquad b_X^t = b_{X^*} \qquad (X \in \mathcal{S}).$$

$$u_n^t = u_{n+1}, \qquad v_n^t = v_{n+1} \qquad (n \in Z).$$

We omit the easy verification of the fact that t as defined is an automorphism of G. Let $\bar{G}$ be the split extension of G by the infinite cyclic group $T = \{t\}$,

$$\bar{G} = TG, \quad G \lhd \bar{G} \quad \text{and} \quad T \cap \bar{G} = 1.$$

$\bar{G}$ can be generated by five of its elements, namely

$$u_0 , \quad v_0 , \quad a_{X_0} , \quad b_{X_0} \quad \text{and} \quad t,$$

where X_r is the set of all integers $\leq r$. For clearly, if we put

$$L = \{u_0 , v_0 , a_{X_0} , b_{X_0} , t\},$$

$L \geq \{J, T\}$. Let $X \in \mathcal{S}$; we can write

$$X = X_r \cup (n_1) \cup (n_2) \cup \cdots \cup (n_s) \qquad (n_i \neq n_j , i \neq j),$$

for certain integers r, s and $n_j > r$. Now $a_{X_r} = t^{-r} a_{X_0} t^r$, so a_{X_r} belongs to L, as does b_{X_r} by symmetry. Also

$$[a_{X_r} , \underbrace{v_{n_1} , u_{n_2} , v_{n_3} , u_{n_4} , \cdots}_{s}]$$

equals either a_X or b_X (according to whether s is even or odd). Hence L contains a_X and b_X ; thus $L = \bar{G}$.

H and K are subnormal in $\bar{G}$. Since $H^G = H^{\bar{G}}$ and $K^G = K^{\bar{G}}$,

$$S(\bar{G}:H) = 3 = S(\bar{G}:K).$$

But J is neither ascendent nor descendent in $\bar{G}$, since it is neither ascendent nor descendent in G. Thus we can state

THEOREM 6.2. *There exists a finitely generated $\mathfrak{A}\mathfrak{N}_2\mathfrak{A}$-group $\bar{G}$ which contains two subgroups each of which is subnormal in $\bar{G}$ with index of subnormality* 3, *but whose join is neither ascendent nor descendent in $\bar{G}$. In particular* $\mathfrak{G} \cap \mathfrak{A}\mathfrak{N}_2\mathfrak{A}$ *is not* $\leq \mathfrak{S}$.

7. Groups with bounding functions

Let H and K be two subnormal subgroups of a group G and let $J = \{H, K\}$. Even if J is subnormal in G, there need not be a close connection between $S(G:J)$ on the one hand and $S(G:H)$ and $S(G:K)$ on the other. In general it is not possible to find an upper bound for $S(G:J)$ in terms of $S(G:H)$ and $S(G:K)$. Of course if $S(G:H) \leq 2$, then $S(G:J) \leq 2S(G:K)$ (Lemma 2.3, Corollary 3). But even if $S(G:H) = 3 = S(G:K)$, $S(G:J)$ may be arbitrarily large.

For example let $G = JM$ be the group constructed in Section 6. Let n be an integer > 1 and let $H(n)$ and $K(n)$ be generated by $u_0 , u_2 , \cdots , u_{2n-2}$ and $v_1 , v_3 , \cdots , v_{2n-1}$ respectively. It is not difficult to verify that

$$S(G:H(n)) = 3 = S(G:K(n)).$$

Let $J(n) = \{H(n), K(n)\}$. $J(n)$ is subnormal in G, by Theorem 4.4. But

$$S(G:J(n)) > 2n,$$

because, if X is the set of all negative integers,

$$[b_X, u_0, v_1, u_2, v_3, \cdots, u_{2n-2}, v_{2n-1}] = b_Y \neq 1$$

where Y is the set of negative integers with $0, 1, 2, \cdots, 2n-1$ adjoined.

Suppose that we can assign to a group G a function $f^{(G)}$ of two variables, assuming non-negative integral values for non-negative integral values of the variables, such that for each pair of subnormal subgroups H and K, their join J is subnormal in G and

$$S(G:J) \leq f^{(G)}(S(G:H), S(G:K)). \tag{24}$$

Such a function will be called a *bounding function* for G and the class of all groups which possess a bounding function will be denoted by $\bar{\mathfrak{S}}$. Then by definition $\bar{\mathfrak{S}} \leq \mathfrak{S}$.

Let G be a group with a bounding function $f^{(G)}$. Then we can define a function $\bar{f}^{(G)}$ which is monotonic increasing in both variables and which is also a bounding function for G, namely the function defined by

$$\bar{f}^{(G)}(x, y) = \max_{0 \leqq u, v \leqq d} f^{(G)}(u, v), \tag{25}$$

where $d = \max(x, y)$. It is easy to verify that $\bar{f}^{(G)}$ is monotonic increasing in both variables; clearly $f^{(G)}(x, y) \leq \bar{f}^{(G)}(x, y)$, for all (x, y). Hence a group which has a bounding function also has a monotonic one.

LEMMA 7.1. *The class $\bar{\mathfrak{S}}$ is* Q- *and* s_n-*closed.*

Proof. Let $G \in \bar{\mathfrak{S}}$ and let $f^{(G)}$ be a bounding function for G. Let $N \triangleleft G$ and suppose that H/N and K/N are subnormal in G/N. Then H and K are subnormal in G and $J = \{H, K\}$ is subnormal in G with

$$S(G:J) \leq f^{(G)}(S(G:H), S(G:K)). \tag{26}$$

But the indices of subnormality of H/N, K/N and J/N in G/N are the same as those of H, K and J in G, respectively. Hence $f^{(G)}$ is also a bounding function for G/N.

Next let M be any subnormal subgroup of G and let H, K be subnormal in M; let $J = \{H, K\}$. Then H, K and J are all subnormal in G and also $S(G:J) \leq f^{(G)}(S(G:H), S(G:K))$. Choose $f^{(G)}$ to be monotonic (increasing) in both variables. Then, in view of the inequalities,

$$S(G:H) \leq S(G:M) + S(M:H),$$
$$S(G:K) \leq S(G:M) + S(M:K),$$
$$S(M:J) \leq S(G:J),$$

the function $f^{(M)}$ defined by

$$f^{(M)}(x, y) = f^{(G)}(S(G:M) + x, S(G:M) + y) \tag{27}$$

is a bounding function for M. Clearly $f^{(M)}$ is a bounding function for any subnormal subgroup of G whose index of subnormality does not exceed $S(G:M)$.

THEOREM 7.2. *Let G be a group with a normal subgroup N such that*

(i) *N belongs to the class $\bar{\mathfrak{S}}$;*

(ii) *each subnormal subgroup of G/N can be generated by n elements, where n is a fixed integer.*

Then to each monotonic bounding function $f^{(N)}$ of N we can assign a monotonic function $F_{f^{(N)}}$ of two variables, taking non-negative integral values for non-negative integral values of the variables, such that for each pair of subnormal subgroups H, K of G, their join J is subnormal in G and

$$S(G:J) \leq F_{f^{(N)}}(S(G:H), S(G:K)).$$

The form of the function $F_{f^{(N)}}$ for a given $f^{(N)}$ depends on G only through n. Hence G belongs to $\bar{\mathfrak{S}}$.

Proof. In the first place G belongs to the class $\mathfrak{S}$ by Theorem 5.2*. The proof is essentially that of Theorem 5.2* with some extra argument.

We will define $F_{f^{(N)}}(x, y)$ by induction on x and show that it has the required property. If either x or $y = 0$, let $F_{f^{(N)}}(x, y) = 0$. Suppose that $F_{f^{(N)}}(x, y)$ has been defined for all x, y and $f^{(N)}$ with $0 \leq x < r$ and $y \geq 0$. We have to show how to define $F_{f^{(N)}}(r, s)$ for any $s > 0$.

Let H and K be two subnormal subgroups of G such that $S(G:H) = r$ and $S(G:K) = s$. Write $J = \{H, K\}$. Let $A = K \cap N$; by hypothesis K/A can be generated by n elements. Hence, by Lemma 3.4,

$$H^K = \{\{H^{k_1}, \cdots, H^{k_t}\}^A, \gamma^s HK^s\} \tag{28}$$

where $k_1, \cdots, k_t \in K$ and $t = 1 + n + n^2 + \cdots + n^{s-1}$. Let

$$L = \{H^{k_1}, \cdots, H^{k_t}, \gamma^s HK^s\}.$$

If $\bar{H} = H^G$, $H^{k_i} \lhd^{r-1} \bar{H}$ and $\gamma^s HK^s \lhd^{s+1} \bar{H}$. Now it is clear that $\bar{H}$ inherits the properties (i) and (ii) from G, so we can apply the induction hypothesis on r. Now $F_{f^{(N)}}$ is monotonic in the region where it has been defined; so by induction on r,

$$\begin{aligned} &S(\bar{H}:\{H^{k_1}, \cdots, H^{k_i}, \gamma^s HK^s\}) \\ &\qquad \leq F_{f^{(\bar{H}\cap N)}}(r-1, S(\bar{H}:\{H^{k_1}, \cdots, H^{k_{i-1}}, \gamma^s HK^s\})), \end{aligned}$$

for $i = 1, 2, \cdots, t$, and any monotonic bounding function $f^{(\bar{H}\cap N)}$ of $\bar{H} \cap N$. Now $\bar{H} \cap N \lhd N$ so, by the proof of the second part of Lemma 7.1, we can choose for $f^{(\bar{H}\cap N)}$ the function determined by

$$f^{(\bar{H}\cap N)}(x, y) = f^{(N)}(x + 1, y + 1).$$

In view of the last inequality and the monotonicity of $F_{f^{(\bar{H}\cap N)}}$, $S(\bar{H}:L) \leq s_t$, where s_t is determined recursively by the relations

$$s_0 = s + 1, \qquad s_i = F_{f^{(H \cap N)}}(r - 1, s_{i-1}) \qquad (i = 1, \cdots, t).$$

It is clear that s_t depends on H and K through r and s only. Let $l = 1 + s_t$; then

$$(29) \qquad s(G:L) \leq l.$$

Let $B = L \cap N$; L is subnormal in G, so L/B can be generated by n elements. By Lemma 3.4

$$(30) \qquad A^L = \{\{A^{y_1}, \cdots, A^{y_u}\}^B, \gamma^l AL^l\},$$

where $y_1, \cdots, y_u \in L$ and $u = 1 + n + n^2 + \cdots + n^{l-1}$. Since $A = K \cap N$, $A^{y_i} \lhd^s N$; also $B \lhd^l N$ and $\gamma^l AL^l \lhd^{l+1} G$. Let $M = \{A^{y_1}, \cdots, A^{y_u}, B, \gamma^l AL^l\}$. $M \leq N$ and $f^{(N)}$ is a monotonic bounding function for N. Therefore $S(N:M) \leq s'_u$, where s'_u is determined recursively by the relations

$$s'_0 = f^{(N)}(l, l + 1), \qquad s'_i = f^{(N)}(s, s'_{i-1}) \qquad (i = 1, \cdots, u).$$

$A^L \lhd M$, so $S(G:A^L) \leq s'_u + 2$. Moreover $H^K = L^A$. Hence $S(G:H^K) \leq (s'_u + 2)l + 1$, by Lemma 2.2; by the same result

$$S(G:J) \leq (s'_u + 2)ls + s.$$

We now define $F_{f^{(N)}}(r, s) = (s'_u + 2)ls + 2$ and observe that this depends on H and K through r and s only. If (for a given $f^{(N)}$) $F_{f^{(N)}}(x, y)$ now fails to be monotonic in the region $x \leq r$, we can easily increase its value at each integral point (r, s), so as obtain a function, identical with $F_{f^{(N)}}(x, y)$ in $x \leq r - 1$, which is monotonic in $x \leq r$ and which of course is still a bounding function for G. This completes the proof of the theorem.

Corollary 1. $\bar{\mathfrak{S}}\mathfrak{F} = \bar{\mathfrak{S}}$.

Corollary 2. $\mathfrak{N}\mathfrak{F} \leq \bar{\mathfrak{S}}$.

Corollary 3. *$\bar{\mathfrak{S}}$ contains the class of polycyclic groups.*

By a polycyclic group we mean a soluble group satisfying Max or equivalently, (cf. [4]), a group which has a series of finite length with each factor cyclic. Let G be a polycyclic group with a series of finite length n with each factor cyclic. Then it is very easy to show that each subgroup of G can be generated by n elements. Hence G belongs to $\bar{\mathfrak{S}}$ by Theorem 7.2.

It seems likely that $\bar{\mathfrak{S}}$ is a proper subclass of $\mathfrak{S}$, but we have not been able to find a group which would show this.

8. The join of an arbitrary collection of subnormal subgroups

We shall now consider what conditions may be imposed in order that an arbitrary collection of subnormal subgroups should generate a subnormal subgroup. Let $\mathfrak{S}^\infty$ be the class of groups in which joins of subnormal subgroups are always subnormal. It is clear that

$$_{Q}\mathfrak{S}^{\infty} = \mathfrak{S}^{\infty} = s_n\,\mathfrak{S}^{\infty},$$

though of course $\mathfrak{S}^{\infty}$ is not s-closed (see the proof of Lemma 5.1).

The following lemma gives a simple characterisation of the class $\mathfrak{S}^{\infty}$.

LEMMA 8.1. *A group G belongs to the class $\mathfrak{S}^{\infty}$ if and only if the union of every ascending chain of subnormal subgroups of G is subnormal in G.*

Proof. Only the sufficiency of the condition is in question, so suppose that it is satisfied. First of all we prove that G belongs $\mathfrak{S}$. Let H and K be subnormal in G and let $J = \{H, K\}$; we argue by induction on $r = S(G:H)$ that J is subnormal in G. Let $r > 0$. Let the elements of J be well-ordered as $(x_\alpha)_{\alpha<\gamma}$, for some ordinal γ, and let $L_\beta = \{H^{x_\alpha} : \alpha < \beta\}$, $\beta \leq \gamma$. Then $L_\gamma = H^J$ and, for any limit ordinal $\mu \leq \gamma$, $L_\mu = \bigcup_{\beta<\mu} L_\beta$. If H^J is not subnormal in G, there exists a least ordinal α such that L_α is not subnormal in G. By our hypothesis α cannot be a limit number. Hence

$$L_\alpha = \{L_{\alpha-1}, H^{x_{\alpha-1}}\};$$

now $L_{\alpha-1}$ and $H^{x_{\alpha-1}}$ are subnormal in H^G, $S(H^G:H^{x_{\alpha-1}}) = r - 1$ and H^G inherits from G the chain property. Hence L_α is subnormal in G by induction hypothesis. This contradiction shows that H^J, and therefore J, is subnormal in G. Thus $G \in \mathfrak{S}$.

Now suppose that we are given a set $\mathfrak{A}$ of subnormal subgroups of G; we have to show that the join of all the subgroups in $\mathfrak{A}$ is subnormal. Since $\mathrm{sn}(G)$ is closed with respect to finite joins and unions of ascending chains, we may without loss of generality assume that $\mathfrak{A}$ also enjoys these properties. By use of Zorn's lemma we conclude that $\mathfrak{A}$ has a unique maximal member which is none other than the join of all the subgroups in $\mathfrak{A}$. Hence $G \in \mathfrak{S}^{\infty}$.

One immediate consequence of Lemma 8.1 is that all groups satisfying Max-s belong to $\mathfrak{S}^{\infty}$. Obviously $\mathfrak{S}^{\infty}$ is a subclass of $\mathfrak{S}$: in fact, as the next lemma shows, it is a much narrower class than $\mathfrak{S}$.

LEMMA 8.2. *For each prime p there exists a countably infinite metabelian p-group which has a set of subnormal cyclic subgroups whose join, though Abelian, is not subnormal.*

Proof. Let A be an elementary Abelian p-group of order p^n, $(n > 1)$, and let $a_1, \cdots, a_n$ be a basis for A. An automorphism t of A is determined by the rules

$$a_1^t = a_1 \quad \text{and} \quad a_i^t = a_i a_{i-1}, \text{ if } i > 1.$$

Let p^m be the least power of p which is not less than n. Then $m > 0$ and the order of t is p^m. For, by a simple induction argument,

$$a_r^{t^l} = a_r a_{r-1}^{l(1)} a_{r-2}^{l(2)} \cdots a_{r-l}^{l(l)} \quad (r = 1, 2, \cdots, n, l > 0) \tag{31}$$

where $l(i) = \binom{l}{i}$. Our convention here is that $a_s = 1$ if $s < 1$.

For $0 < i < p^m$, p divides $\binom{p^m}{i}$; also $r \leq n \leq p^m$, so $t^{p^m} = 1$. But $t^{p^{m-1}}$ carries a_n into $a_n a_{n-p^{m-1}}$ which is not equal to a_n, since $p^{m-1} < n$.

Let X be the split extension of A by $\{t\}$; then X is a metabelian group of order p^{m+n}. Since $[a_i, t] = a_{i-1}$, the $(n+1)^{\text{th}}$ term of the lower central series of X is

$$\gamma_{n+1}(X) = [A, \underbrace{t, \cdots, t}_{n}] = 1.$$

On the other hand

$$[a_n, \underbrace{t, \cdots, t}_{n-1}] = a_1 \neq 1,$$

so X is nilpotent of class n and $\{t\}$ is subnormal in X with index of subnormality n.

For a given prime p and each integer $n > 1$ take a group X_n of the type just constructed, $X_n = \{t_n, A_n\}$ say, with $S(X_n : \{t_n\}) = n$. Define G to be the direct product of the X_n's,

$$G = \mathrm{Dr}_{n>1} X_n .$$

G is a countably infinite metabelian p-group and $\{t_n\}$ is subnormal in G with $S(G:\{t_n\}) = n$, $(n = 2, 3, \cdots)$. But $H = \{t_2, t_3, \cdots\}$ is not subnormal in G. For if it were, there would exist an integer r such that $\gamma^r X_n\{t_n\}^r = 1$ for all $n > 1$ (since H is Abelian). Hence $S(X_n : \{t_n\}) \leq r$ for all n, which is impossible.

Remark. It follows that $\mathfrak{A}^2$ is not $\leq \mathfrak{S}^\infty$, whereas $\mathfrak{A}^2 \leq \mathfrak{S}$ by Corollary 1 to Lemma 2.3.

It is perhaps surprising that for the class $\mathfrak{S}^\infty$ there is a theorem analogous to Theorem 5.2.

THEOREM 8.3. *An extension of an $\mathfrak{S}^\infty$-group by a group satisfying Max-s is an $\mathfrak{S}^\infty$-group. That is*

$$\mathfrak{S}^\infty \mathfrak{M}_s = \mathfrak{S}^\infty.$$

Proof. Let G have a normal $\mathfrak{S}^\infty$-subgroup N and suppose that G/N satisfies Max-s. By Lemma 8.1 it is enough to prove that the union of any ascending chain of subnormal subgroups of G is subnormal. Let

$$H_1 \leq H_2 \leq \cdots \leq H_\alpha \leq \cdots,$$

$(\alpha < \mu$, μ a limit ordinal), be an ascending chain in $\mathrm{sn}(G)$ and let H be its union. Since G/N satisfies Max-s,

$$H = \{H_\alpha, H \cap N\} \tag{32}$$

for some $\alpha < \mu$. Also

$$H \cap N = \bigcup_{\beta<\mu} (H_\beta \cap N). \tag{33}$$

Each $H_\beta \cap N$ is subnormal in N and hence $H \cap N$ is subnormal in N, since $N \in \mathfrak{S}^\infty$. H is the join of two subnormal subgroups, H_α and $H \cap N$, by (32), and thus H itself is subnormal in G, since $H \cap N \triangleleft H$

COROLLARY. $\mathfrak{N}(\mathfrak{G} \cap \mathfrak{N}) \leq \mathfrak{S}^\infty$.

In particular, $\mathfrak{S}^\infty$ contains all finitely generated metabelian groups, though not, as we have seen, all metabelian groups.

THEOREM 8.4. *An extension of a group with a composition series of finite length by an $\mathfrak{S}^\infty$-group is an $\mathfrak{S}^\infty$-group.*

The proof of this result runs on the same lines as that of Theorem 5.3(i) (we appeal in this case to Theorem 8.3 instead of to Theorem 5.2).

Finally we will prove

THEOREM 8.5. *An extension of an $\mathfrak{S}^\infty$-group by an Abelian group is an $\mathfrak{S}$-group, that is $\mathfrak{S}^\infty\mathfrak{A} \leq \mathfrak{S}$.*

Proof. Let $G \in \mathfrak{S}^\infty\mathfrak{A}$; then G' is an $\mathfrak{S}^\infty$-group. Let H and K be subnormal in G and let $J = \{H, K\}$. We have to prove that J is subnormal in G. Let $r = S(G:H) > 0$. By induction on r and the usual argument, it follows that $\mathrm{cn}(H, J) \leq \mathrm{sn}(G)$. By Lemma 4.1 $\mathrm{cm}(H, J) \leq \mathrm{sn}(G')$ and in particular, for each $k \in K$, $[H, k]$ is subnormal in G'. But $[H, K] = \{[H, k] : k \in K\}$ and $G' \in \mathfrak{S}^\infty$, so $[H, K]$ is subnormal in G'. The theorem now follows easily.

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