ON THE TRANSFORMATION OF SEQUENCES AND RELATED CONVERGENCE CRITERIA FOR CONTINUED FRACTIONS

BY

DAVID F. DAWSON

1. Introduction

Lane and Wall [3] investigated convergence of the continued fraction

$$f(a) = \frac{1}{1} + \frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \cdots$$

as related to properties of the sequence $\{h_p\}_{p=1}^{\infty}$ associated with $f(a)$ in the following way. Let $f_0 = 0, f_1 = 1, f_2 = 1/(1 + a_1), \cdots$ denote the sequence of approximants of $f(a)$, and suppose no $a_i = 0$. If $t_p(z) = 1/(1 + a_p z)$, $T_p(z) = t_1 t_2 \cdots t_p(z)$, $p = 1, 2, 3, \cdots$, then $T_p(0) = f_p$, $T_p(\infty) = f_{p-1}$, $T_p(1) = f_{p+1}$, $p = 1, 2, 3, \cdots$, and in case no $f_i = \infty$, $\{h_p\}_{p=1}^{\infty}$ is defined by

$$T_p(h_p) = \infty, \qquad p = 1, 2, 3, \cdots. \tag{1.1}$$

Their investigations led to the result that if the even and odd parts of $f(a)$ converge absolutely, then $f(a)$ converges if and only if either some $a_i = 0$ or else $a_p \neq 0$, $p = 1, 2, 3, \cdots$, and the series $\sum |b_p|$ diverges, where

$$b_1 = 1, \qquad b_{p+1} = 1/a_p b_p, \qquad p = 1, 2, 3, \cdots. \tag{1.2}$$

In case $a_p \neq 0$, $p = 1, 2, 3, \cdots$, and $b = \{b_p\}_{p=1}^{\infty}$ is defined by (1.2), then the continued fraction

$$g(b) = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots$$

is equivalent to $f(a)$ in the sense that if $g_0 = 0, g_1 = 1/b_1, g_2 = 1/(b_1 + 1/b_2), \cdots$ is the sequence of approximants of $g(b)$, then $g_p = f_p$, $p = 0, 1, 2, \cdots$.

In Section 2, a transformation H is given which transforms (under approprate restrictions) the sequence $\{b_1 + b_3 + \cdots + b_{2p+1}\}$ into $\{g_1 - g_{2p+1}\}$, and it is shown that both H and its inverse are convergence preserving if and only if the product $\prod (1 - h_{2p})(1 - h_{2p+2})$ converges absolutely. From this and a similar result, we are able to obtain (Section 3) convergence and divergence criteria for $g(b)$ as related to properties of $\{h_p\}$ and $\{b_p\}$.

2. A class of continued fractions

Suppose $z = \{z_p\}_{p=1}^{\infty}$ is a complex sequence whose terms are distinct from 0 and 1. Let

$$D_1 = 1, \qquad D_{2p+1}/D_{2p-1} = 1 - z_p, \qquad p = 1, 2, 3, \cdots. \tag{2.1}$$

Received September 26, 1963; received in revised form February 24, 1964.

Let A denote the set of all continued fractions $f(a)$ such that no $a_i = 0$ and no $f_i = \infty$.

LEMMA 2.1. *There exists a continued fraction $g(b)$ such that* (1) *the sequence of odd denominators of $g(b)$ is the sequence $\{D_{2p-1}\}_{p=1}^{\infty}$ defined by* (2.1), *and* (2) *$g(b)$ is equivalent to some $f(a) \in A$.*

Proof. Let $z' = \{z'_p\}_{p=1}^{\infty}$ be a complex sequence whose terms are distinct from 0 and 1. Let $D_0 = 1$, $D_{2p}/D_{2p-2} = 1 - z'_p$, $p = 1, 2, 3, \cdots$. Define $\{b_{2p-1}\}_{p=1}^{\infty}$ and $\{b_{2p}\}_{p=1}^{\infty}$ as follows:

$$(2.2) \qquad \begin{aligned} b_1 = 1, \qquad b_{2p+1} &= (D_{2p+1} - D_{2p-1})/D_{2p}, \\ b_{2p} &= (D_{2p} - D_{2p-2})/D_{2p-1}, \end{aligned} \qquad p = 1, 2, 3, \cdots.$$

Then if $b = \{b_p\}_{p=1}^{\infty}$, (1) follows immediately from the fundamental recurrence formulas for $g(b)$ [1]. Since $b_1 = 1$, no $b_i = 0$, and no $D_i = 0$, we note that (2) is true.

Notation. If z is a complex sequence whose terms are distinct from 0 and 1, then $B(z)$ will denote the set of all continued fractions $g(b)$ having properties (1) and (2) of Lemma 2.1.

LEMMA 2.2. *If $g(b) \in B(z)$ and $g(b)$ is equivalent to $f(a)$, then the sequence $\{h_p\}_{p=1}^{\infty}$ defined by* (1.1) *has the property that $h_{2p} = z_p$, $p = 1, 2, 3, \cdots$.*

Proof. By (2.10) of [3], $h_{2p} = -b_{2p+1} D_{2p}/D_{2p-1}$. But from (2.2) and (2.1), $-b_{2p+1} D_{2p}/D_{2p-1} = (D_{2p-1} - D_{2p+1})/D_{2p-1} = z_p$.

LEMMA 2.3. *Suppose $\{w_p\}_{p=1}^{\infty}$ is a complex sequence whose terms are distinct from 1, and suppose $1 - w_p = u_{p+1}/u_p$, $p = 1, 2, 3, \cdots$. Then, if n is a positive integer, the infinite product*

$$(2.3) \qquad \prod_{p \geqq 1} \left(\prod_{i=0}^{n-1} (1 - w_{p+i})\right)$$

converges absolutely if and only if each of the sequences $\{u_{pn+i}\}_{p=0}^{\infty}$, $i = 1, 2, \cdots, n$, converges absolutely to a nonzero limit.

Proof. We note that

$$\prod_{i=0}^{n-1} (1 - w_{p+i}) = u_{p+n}/u_p = 1 - (1 - u_{p+n}/u_p), \quad p = 1, 2, 3, \cdots.$$

Hence (2.3) converges absolutely if and only if the series $\sum |1 - u_{p+n}/u_p|$ converges. Thus (2.3) converges absolutely if and only if each of the series $\sum |1 - u_{(p+1)n+i}/u_{pn+i}|$, $i = 1, 2, \cdots, n$, converges. But from a proof given in [1] it follows that $\sum |1 - u_{(p+1)n+i}/u_{pn+i}|$ converges if and only if $\{u_{pn+i}\}_{p=0}^{\infty}$ converges absolutely to a nonzero limit, $i = 1, 2, \cdots, n$.

THEOREM 2.1. *Suppose $z = \{z_p\}_{p=1}^{\infty}$ is a complex sequence whose terms are distinct from 0 and 1. Then the following two statements are equivalent:*

(1) *If $g(b) \in B(z)$, then $\{g_{2p-1}\}$ and $\sum b_{2p-1}$ both converge or both diverge.*

(2) *The product $\prod (1 - z_p)(1 - z_{p+1})$ converges absolutely.*

Proof. We apply Lemma 2.3 for the case that $n = 2$, $w_p = z_p$, and $u_p = D_{2p-1}$, $p = 1, 2, 3, \cdots$. Thus by Lemma 2.3, the product $\prod (1 - z_p)(1 - z_{p+1})$ converges absolutely if and only if each of the sequences $\{D_{4p+2i-1}\}_{p=0}^{\infty}$, $i = 1, 2$, converges absolutely to a nonzero limit. Let $H = (h_{pq})$ and $H' = (h'_{pq})$ be triangular matrices defined as follows:

$$
(2.4)\qquad
\begin{aligned}
h_{pq} &= 0 && \text{if } q > p\\
&= 1/D_{2p-1}\,D_{2p+1} && \text{if } p = q\\
&= 1/D_{2q-1}\,D_{2q+1} - 1/D_{2q+1}\,D_{2q+3} && \text{if } p > q,\\
h'_{pq} &= 0 && \text{if } q > p\\
&= D_{2p-1}\,D_{2p+1} && \text{if } p = q\\
&= D_{2q-1}\,D_{2q+1} - D_{2q+1}\,D_{2q+3} && \text{if } p > q.
\end{aligned}
$$

Using induction and the formula $g_{2p-1} - g_{2p+1} = b_{2p+1}/D_{2p+1}\,D_{2p-1}$, $p = 1, 2, 3, \cdots$, we can show that H transforms the sequence of partial sums of the series $\sum_{p=1}^{\infty} b_{2p+1}$ into the sequence $\{g_1 - g_{2p+1}\}_{p=1}^{\infty}$, and H' is the inverse of H. Recalling the Silverman-Toeplitz conditions which are necessary and sufficient for a triangular matrix to be convergence preserving, we see that H and H' are both convergence preserving if and only if both of the series

$$(2.5)\qquad \sum \mid 1/D_{2q-1}\,D_{2q+1} - 1/D_{2q+1}\,D_{2q+3} \mid$$

and

$$(2.6)\qquad \sum \mid D_{2q-1}\,D_{2q+1} - D_{2q+1}\,D_{2q+3} \mid$$

are convergent. But (2.5) and (2.6) are both convergent if and only if $\{D_{2q-1}\,D_{2q+1}\}_{q=1}^{\infty}$ converges absolutely to a nonzero limit, and this condition is equivalent to the convergence of the series $\sum \mid 1 - D_{2p+1}\,D_{2p+3}/D_{2p-1}\,D_{2p+1} \mid$ [1]. Thus H and H' are both convergence preserving if and only if each of the sequences $\{D_{4p+2i-1}\}_{p=0}^{\infty}$, $i = 1, 2$, converges absolutely to a nonzero limit, and this condition is equivalent to the absolute convergence of the product $\prod (1 - z_p)(1 - z_{p+1})$, as shown above from Lemma 2.3. Hence (2) implies (1).

We next suppose that (1) is true. This means that H and H' are both convergence preserving over the set of all complex sequences $\{t_p\}_{p=1}^{\infty}$ such that $t_1 \neq 0$ and $t_i \neq t_{i+1}$, $i = 1, 2, 3, \cdots$. Using a slight modification of Corollary 3.6a of [2], we see that H and H' are both convergence preserving, and so (2) must hold. This completes the proof of Theorem 2.1. A similar theorem is obtained if the roles of even and odd indices are interchanged.

THEOREM 2.2. *Suppose $z = \{z_p\}_{p=1}^{\infty}$ is a complex sequence whose terms are distinct from 0 and 1. Then the following two statements are equivalent*:

(1) *If $g(b) \in B(z)$, then $\{D_{2p}\}$ and $\sum b_{2p}$ both converge or both diverge.*

(2) $\sum \mid z_p \mid$ *converges.*

Proof. Let $E = (e_{pq})$ and $E' = (e'_{pq})$ be triangular matrices defined as follows:

$$(2.7) \quad \begin{aligned} e_{pq} &= 0 && \text{if } q > p & e'_{pq} &= 0 && \text{if } q > p \\ &= 1/D_{2p-1} && \text{if } p = q & &= D_{2p-1} && \text{if } p = q \\ &= 1/D_{2q-1} - 1/D_{2q+1} && \text{if } p > q, & &= D_{2q-1} - D_{2q+1} && \text{if } p > q. \end{aligned}$$

Using induction and the fundamental recurrence formulas for $g(b)$ [1], we can show that E transforms the sequence $\{D_{2p} - D_0\}_{p=1}^{\infty}$ into the sequence of partial sums of the series $\sum_{p=1}^{\infty} b_{2p}$, and E' is the inverse of E. We note that E and E' are both convergence preserving if and only if both of the series $\sum | 1/D_{2p-1} - 1/D_{2p+1} |$ and $\sum | D_{2p-1} - D_{2p+1} |$ are convergent, and this condition is equivalent to the convergence of the series $\sum | 1 - D_{2p+1}/D_{2p-1} |$ [1]. Thus from (2.1) we see that E and E' are both convergence preserving if and only if $\sum | z_p |$ converges. Hence (2) implies (1).

We now suppose that (1) holds. Then E and E' are both convergence preserving over the set of all complex sequences $\{t_p\}_{p=1}^{\infty}$ such that $t_1 \neq 0$ and $t_i \neq t_{i+1}$, $i = 1, 2, 3, \cdots$. As in the proof of Theorem 2.1, it follows that E and E' are both convergence preserving, and so (2) must hold. A similar theorem holds if the roles of even and odd indices are interchanged.

3. Theorems on convergence and divergence

Throughout this section it will be assumed that whenever a continued fraction $g(b)$ and a sequence $\{h_p\}$ are mentioned, $g(b)$ is equivalent to some $f(a) \in A$ and $\{h_p\}$ is defined by (1.1). The theorems and remarks of this section remain valid if the roles of even and odd indices are interchanged.

THEOREM 3.1. *If $\sum | h_{2p} |$ converges and either $\sum b_{2p}$ converges or $\sum | b_{2p-1} |$ diverges, then $g(b)$ diverges.*

Proof. From (2.1) and Lemma 2.2, the convergence of $\sum | h_{2p} |$ implies absolute convergence of $\{D_{2p-1}\}$ to a nonzero limit [1]. Suppose $\sum b_{2p}$ converges. Then by Theorem 2.2, $\{D_{2p}\}$ converges. Thus $g(b)$ diverges, since

$$(3.1) \qquad g_{p+1} - g_p = (-1)^p/D_{p+1} D_p, \qquad p = 0, 1, 2, \cdots.$$

Suppose $\sum | b_{2p-1} |$ diverges. From the formula

$$(3.2) \qquad D_{2p+1} - D_{2p-1} = b_{2p+1} D_{2p}, \qquad p = 1, 2, 3, \cdots,$$

and the absolute convergence of $\{D_{2p-1}\}$, it follows that $\sum | b_{2p+1} D_{2p} |$ converges. Hence $\{D_{2p}\}$ contains a subsequence convergent to 0. Therefore by (3.1), $g(b)$ diverges.

Remark 3.1. Theorem 3.1 can be proved by use of formulas of Lane and Wall [3, pp. 370–371] and a theorem of Scott and Wall [4, Theorem B] to the effect that if the series $\sum b_{2p-1}$ and $\sum b_{2p}$ converge, at least one of them absolutely, then $g(b)$ diverges. It is interesting to note that there is *no* theorem to the effect that if the series $\sum h_{2p-1}$ and $\sum h_{2p}$ converge, at least one of them absolutely, then $g(b)$ diverges. We show this by means of the following example. Let $h_{2p-1} = (-1)^p(-p)^{-1/2}$ and $h_{2p} = 2^{-p}$, $p = 1, 2, 3, \cdots$. Clearly $|(h_1 - 1)(h_2 - 1) \cdots (h_n - 1)| \to \infty$ as $n \to \infty$, and by (2.7) of [3], $\{g_{2p-1}\}$ converges. Hence by (2.4) of [3], $g(b)$ converges.

THEOREM 3.2. *If the odd part of $g(b)$ converges absolutely and the even part converges, then $g(b)$ converges if and only if either $\sum | h_{2p} |$ diverges or $\sum h_{2p+1}(1 - h_1)(1 - h_3) \cdots (1 - h_{2p-1})$ diverges.*

Proof. The necessity follows from Theorem 3.1, Theorem 2.2, and the fact that

$$h_{2p+1}(1 - h_1)(1 - h_3) \cdots (1 - h_{2p-1}) = D_{2p} - D_{2p+2}, \quad p = 1, 2, 3, \cdots . \tag{3.3}$$

Convergence of $g(b)$ when $\sum | h_{2p} |$ diverges follows from a theorem of Lane and Wall [3, Theorem 2.2a]. Suppose then that $\sum | h_{2p} |$ converges and $\sum h_{2p+1}(1 - h_1)(1 - h_3) \cdots (1 - h_{2p-1})$ diverges. We have then the absolute convergence of $\{D_{2p-1}\}$ to a nonzero limit, and from (3.3), the divergence of $\{D_{2p}\}$. But since the even and odd parts of $g(b)$ converge and

$$g_{2p+1} - g_{2p} = 1/D_{2p+1} D_{2p}, \qquad p = 1, 2, 3, \cdots, \tag{3.4}$$

we see that $| D_{2n} | \to \infty$ as $n \to \infty$, and so $g(b)$ is convergent.

THEOREM 3.3. *If the product $\prod (1 - h_{2p})(1 - h_{2p+2})$ converges absolutely and $h_{2n} \nrightarrow 0$, then $g(b)$ converges if and only if $\sum b_{2p-1}$ converges.*

Proof. From the proof of Theorem 2.1, we see that each of the sequences $D_1, D_5, D_9, \cdots$ and $D_3, D_7, D_{11}, \cdots$ converges absolutely and neither limit is 0. These limits are distinct since $h_{2p} = 1 - D_{2p+1}/D_{2p-1}$ and $h_{2n} \nrightarrow 0$. From Theorem 2.1 it follows that if $g(b)$ converges, then $\sum b_{2p-1}$ converges. Suppose conversely that $\sum b_{2p-1}$ converges. Then by (3.2), $| D_{2n} | \to \infty$ as $n \to \infty$. By Theorem 2.1, the odd part of $g(b)$ converges, and so by (3.4), $g(b)$ converges.

Remark 3.2. Lane and Wall [3, Theorem 2.3] showed that if $\{g_p\}$ is bounded, then the two series $\sum | h_p |$ and $\sum | b_p |$ converge or diverge together. We can use Theorem 3.3 to show that the two series $\sum | h_{2p} |$ and $\sum | b_{2p-1} |$ need not converge or diverge together whenever $\{g_p\}$ is bounded. Let $z = \{z_p\}_{p=1}^{\infty}$ be a complex sequence such that $z_i \neq 0$, $z_i \neq 1$, $i = 1, 2, 3, \cdots$, and such that $\prod (1 - z_p)(1 - z_{p+1})$ converges absolutely, but $z_n \nrightarrow 0$. Let $g(b) \in B(z)$ such that $\sum | b_{2p-1} |$ converges. Then by Theorem 3.3, $g(b)$ converges. By Lemma 2.2, $h_{2p} = z_p$, $p = 1, 2, 3, \cdots$, and so $\sum | h_{2p} |$ diverges. Thus the convergence of $\sum | b_{2p-1} |$ need not imply convergence of $\sum | h_{2p} |$ even when $\{g_p\}$ is convergent. It follows easily from the formula on the bottom of page 371 of [3], however, that the convergence of $\sum | h_{2p} |$ implies convergence of $\sum | b_{2p-1} |$ when $\{g_p\}$ is bounded.

Remark 3.3. It is easy to show that if both of the matrices H and H' defined by (2.4) are convergence preserving, then the sequences $\{g_{2p-1}\}$ and $\{b_1 + b_3 + \cdots + b_{2p-1}\}$ are either both bounded or both unbounded. Similarly, if both of the matrices E and E' defined by (2.7) are convergence preserving, then the sequences $\{D_{2p}\}$ and $\{b_2 + b_4 + \cdots + b_{2p}\}$ are either both bounded or both unbounded. Hence if $\sum | h_{2p} |$ converges,

$$\limsup | b_1 + b_3 + \cdots + b_{2p-1} | < \infty,$$

and

$$\limsup | b_2 + b_4 + \cdots + b_{2p} | = \infty,$$

then $\{D_{2p-1}\}$ converges to a nonzero limit, $\{g_{2p-1}\}$ is bounded, and $\{D_{2p}\}$ is unbounded. Thus by (3.4), $\liminf | g_{2p+1} - g_{2p} | = 0$, and so there exists a finite point v, every neighborhood of which contains infinitely many even and infinitely many odd approximants of $g(b)$.

References

1. D. F. Dawson, *Convergence of continued fractions of Stieltjes type*, Proc. Amer. Math. Soc., vol. 10 (1959), pp. 12–17.
2. ———, *On certain sequence to sequence transformations which preserve convergence*, Proc. Amer. Math. Soc., vol. 14 (1963), pp. 542–545.
3. R. E. Lane and H. S. Wall, *Continued fractions with absolutely convergent even and odd parts*, Trans. Amer. Math. Soc., vol. 67 (1949), pp. 368–380.
4. W. T. Scott and H. S. Wall, *On the convergence and divergence of continued fractions*, Amer. J. Math., vol. 69 (1947), pp. 551–561.

North Texas State University
Denton, Texas