DIRICHLET SPACES ASSOCIATED WITH INTEGRO-DIFFERENTIAL OPERATORS. PART I

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1. Introduction

Let E be a domain in n-dimensional Euclidean space R^n, $n \geq 2$. For u a function with support in E and α a number such that $0 < \alpha < 2$, we define

$$B_\alpha u(P) = \Delta(u * r^{2-n-\alpha})(P) + \int_{\partial E} u(Q)\nu(P, dQ) \tag{1.1}$$

where $*$ denotes convolution, r the polar distance from the origin, Δ the Laplacian in the sense of distributions on the open set E, and ν a positive measure on the boundary ∂E of E for each $P \in E$. The expression

$$\Delta(u * r^{2-n-\alpha})$$

is, except for a multiplicative constant, the fractional Riesz potential $I^{-\alpha}u$ of order $-\alpha$. Operators of the form (1.1) arise also in the theory of stochastic processes; we shall say more about this later.

A. Beurling has suggested that the theory of Dirichlet spaces, developed by him and J. Deny in [1] and [2], provides a natural framework in which to study operators of the form (1.1). It turns out that this extremely elegant theory makes possible a rapid and simple access to numerous results which would otherwise require much arduous analysis.

In the present note, we restrict ourselves to the case $\nu = 0$. Thus from now on we shall be concerned with the operator

$$A_\alpha u = \Delta \int_E u(Q) \mid PQ \mid^{2-n-\alpha} dQ. \tag{1.2}$$

In a subsequent paper, we shall treat the more general operator (1.1).

Our main interest here is to show how the theory of Dirichlet spaces can be used to give a self-contained and systematic account of the basic properties of (1.2). For this reason we include a number of known results which have been proved by other methods. Many of the statements we prove appear, either explicitly or implicitly, in the pioneer work of M. Riesz and O. Frostman, cf. [4], [5] and [11]; others will be found in works motivated by probability theory, cf. Getoor [6].

We sketch briefly the connection between (1.2) and probability theory.

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To each α, $0 < \alpha \leq 2$, there corresponds a symmetric stable process of order α on R^n whose transition densities p_α are determined by the Fourier transforms

$$e^{-t|x|^\alpha} = \int_{R^n} e^{ix\cdot\xi} p_\alpha(t, \xi)\, d\xi. \tag{1.3}$$

The case $\alpha = 2$ corresponds to Brownian motion. Each of these processes determines positive contraction semi-groups

$$T_t f(x) = \int_{R^n} p_\alpha(t, x - y) f(y)\, dy \tag{1.4}$$

either from $C_0(R^n)$ to itself, from $L(R^n)$ to itself, or from $L^2(R^n)$ to itself. In each case, when $0 < \alpha < 2$, the infinitesimal generator of the semi-group agrees on a dense set with a constant multiple of the operator (1.2) for $E = R^n$.

Suppose that E is a bounded domain in R^n. Corresponding to each stable α-process on R^n, there is an "absorbing barrier" process on E which can be described roughly as follows: a symmetric stable process $\{X(t); t \geq 0\}$ with starting point $X(0) \in E$ continues until the path leaves E, at which moment the process terminates. The associated transition probabilities again determine positive contraction semi-groups on various spaces of functions on E. The generators agree with a constant multiple of (1.2) on dense sets.

Although we restrict ourselves here to dimension $n \geq 2$, alterations can easily be made at the appropriate points to cover the case $n = 1$. We shall, however, omit these details. For articles which deal with problems connected with the one-dimensional case, cf. [3], [8], [9], [10] and [12].

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2. Preliminaries

Let E be a bounded domain in R^n. We shall assume that E is a Greenian domain, i.e. one for which Green's Theorem holds; for certain results later on we shall impose other more restrictive conditions on the boundary of E.

As mentioned above, we wish to study semi-groups generated by the operator A_α of (1.2). According to the Hille-Yosida theorem this is equivalent to studying the resolvent equation

$$\begin{aligned} \lambda u(P) - A_\alpha u(P) &= \lambda u(P) - \Delta \int_E u(Q) \mid PQ \mid^{2-n-\alpha} dQ \\ &= f(P) \qquad (\lambda > 0) \end{aligned} \tag{2.1}$$

where f is a given function in the space under consideration. We shall in fact start with $\lambda = 0$ and construct the "Green's function" G_α which gives a solution

$$u(P) = \int_E G_\alpha(P, Q) f(Q)\, dQ \tag{2.2}$$

to (2.1) with $\lambda = 0$, unique under certain auxiliary conditions to be specified later. This Green's kernel was studied by Riesz and Frostman, and in [11] Riesz gave an explicit formula for G_α in the case where E is a spherical ball.

In Section 3 we show how a Dirichlet space can be associated with the operator A_α. In Section 4 we study the properties of the solutions of (2.1) for $\lambda = 0$ and obtain the Green kernel (2.2). Finally, in Section 5 the semigroups associated with A_α in several Banach spaces are constructed.

3. The Dirichlet space associated with $A\alpha$

We first recall the following two basic definitions from [2, Section 1].

DEFINITION 3.1. *A normalized contraction of the complex plane C is a transformation T of C into itself such that*

(i) $T(0) = 0$ *and*

(ii) $| T(z_1) - T(z_2)| \leq |z_1 - z_2|$ *for each pair* z_1, z_2 *in* C.

DEFINITION 3.2. *Given a locally compact space X and a positive Radon measure ξ everywhere dense on X (non-empty open sets have positive measure), a Dirichlet space relative to X and ξ is any Hilbert space $D = D(X, \xi)$ whose elements are complex-valued functions*[2]*, locally summable for ξ, and satisfying the following axioms*:

I. *For every compact $K \subset X$, there exists a finite number $A(K)$ such that*

$$\int_K | u | \leq A(K) \; \| u \|_D .$$

III. *If $\mathcal{C}$ denotes the complex-valued functions, continuous on X with compact support, then $\mathcal{C} \cap D$ is dense in $\mathcal{C}$ and in D.*

III. *If T is a normalized contraction of the complex plane, then $u \in D$ implies $T(u) \in D$ and $\| T(u)\| \leq \| u \|$.*

A real Dirichlet space is one whose elements u are real-valued functions satisfying I–III.

We shall now define the real Dirichlet space associated with the operator A_α. Suppose that u and v are infinitely differentiable with compact support contained in the domain E. An application of Green's formula yields

$$(3.1) \qquad \begin{aligned} -\int_E v(P) A_\alpha u(P)\, dP &= (u, v) \\ &= \int_E vum + \frac{C_\alpha}{2} \int_E \int_E \frac{[v(P) - v(Q)][u(P) - u(Q)]}{| PQ |^{n+\alpha}} \, dP \, dQ, \end{aligned}$$

where $0 < \alpha < 2$, $C_\alpha = \alpha(n + \alpha - 2)$, and

[2] We take the usual liberty of speaking of functions when actually equivalence classes are meant.

$$(3.2) \qquad m(P) = C_\alpha \int_{CE} | PQ |^{-n-\alpha} dQ,$$

with CE denoting the complement of E.

The real-valued functions infinitely differentiable with compact support contained in E form a pre-Hilbert space with scalar product (u, v) defined by (3.1). We complete this pre-Hilbert space to a Hilbert space D_α^0 which, as the next lemma shows, is a Dirichlet space.

LEMMA 3.1. *The Hilbert space D_α^0 formed by completing the pre-Hilbert space of infinitely differentiable functions with compact support contained in E and the scalar product (u, v) of* (3.1) *is a Dirichlet space.*

Proof. Since there exists a constant c such that $m(P) > c > 0$, for $P \in E$, it is clear that Axiom I is satisfied. Axiom II follows from the definition of D_α^0. To show that Axiom III is satisfied, we note that if $u \in D_\alpha^0$ and $\{u_n\}$ is a sequence of functions in the pre-Hilbert space such that $u_n \to u$ in the norm, then for any normalized contraction T, Tu_n converges to Tu in the norm of D_α^0. Hence $Tu \in D_\alpha^0$ and the norm condition $\| T(u) \| \leq \| u \|$ follows immediately from the expression for (u, v) given in (3.1).

The norm in D_α^0 can also be expressed as follows: if

$$\hat{u}(x) = \int_{R^n} \exp(-2i\pi x \cdot y) u(y)\, dy,$$

then

$$(3.3) \qquad \| u \|^2 = d_\alpha \int_{R^n} | \hat{u} |^2 r^\alpha$$

with

$$(3.4) \qquad d_\alpha = 4\pi^{(\alpha+n/2)} \Gamma(1 - \alpha/2)/\Gamma([n + \alpha - 2]/2).$$

The space D_α^0 is a subspace of the special Dirichlet space described in the last paragraph of [2].

4. Potentials in D_α^0

Following [2, Section 2], we have

DEFINITION 4.1. *An element u_μ in a Dirichlet space $D(E, \xi)$ is called a potential if there exists a Radon measure μ on E such that*

$$(4.1) \qquad (u_\mu, v) = \int_E v\, d\mu$$

whenever $v \in \mathfrak{C} \cap D$. If μ is a positive measure, then u_μ is called a pure potential.

In our case (4.1) implies that

$$(4.2) \qquad -\Delta(u_\mu * r^{2-\alpha-n}) = \mu$$

in the sense of distributions on the open set E, the $*$ denoting convolution.

If $f \in L^2_{1/m}(E)$, where m is the function in (3.2), then for every $v \in D^0_\alpha$

$$\left| \int_E vf \right|^2 \leq \| v \|_D^2 \left\{ \int_E f^2/m \right\}. \tag{4.3}$$

Hence, there exists a potential $u_f \in D^0_\alpha$ for which (4.1) is valid for *all* $v \in D^0_\alpha$. Note that $L^2_{1/m}(E)$ contains the bounded measurable functions on E, the functions in $L^2(E)$, and all functions of the form $\Phi \cdot m^{1/2}$ with $\Phi \in L^2(E)$.

The following lemma is proved in [2, Lemma 2]:

LEMMA 4.1. *Pure potentials are positive.*

If $0 < \alpha < 1$, then the function m of (3.2) is in $L(E)$, and $u_m \equiv 1$ is the pure potential associated with m. In this case, for every potential u_f associated with an $f \in L(E)$, we have

$$\int_E u_f m = \int_E f. \tag{4.4}$$

This argument breaks down if $\alpha \geq 1$ because in this case $m \notin L(E)$, and the characteristic function of E is not in D^0_α. We shall now show that (4.4) holds even in the case that $\alpha \geq 1$.

LEMMA 4.2. *If $0 < \alpha < 2$, and if u_f is a potential in D^0_α associated with a function $f \in L(E)$, then $u_f m \in L(E)$, with m the function in* (3.2).

Proof. It suffices to prove the lemma for $f \geq 0$. By Theorem 1 of [2], for each open set S such that $\bar{S} \subset E$, there corresponds an "equilibrium potential" u_S satisfying

$$u_S \equiv 1 \quad \text{a.e. on} \quad S \tag{4.5}$$

$$0 \leq u_S \leq 1 \quad \text{a.e. on} \quad E. \tag{4.6}$$

The associated measure m_S is positive and carried by $\bar{S}$.

We have, for $P \in S$,

$$-\Delta \int_E \frac{u_S(Q)}{| PQ |^{n+\alpha-2}} \, dQ = m(P) + C_\alpha \int_{E-S} \frac{1 - u_S(Q)}{| PQ |^{n+\alpha}} \, dQ \geq m(P), \tag{4.7}$$

where C_α is given after (3.1). Thus

$$\int_S u_f m \leq \int_E u_f \, dm_S = \int_E f u_S \leq \int_E f. \tag{4.8}$$

Since S is an arbitrary open set whose closure is contained in E, it follows that $u_f m \in L(E)$.

LEMMA 4.3. *If $f \in L(E)$ and $| f | \leq m$, and if there exists a potential $u_f \in D^0_\alpha$ associated with f, then $| u_f | \leq 1$ a.e.*

Proof. (Necessary only for $1 \leq \alpha < 2$). We may suppose that $f \geq 0$, since $| u_f | \leq u_{|f|}$. For each $g \geq 0$, bounded and measurable on E,

$$\int_E u_f g = \int_E f u_g \leq \int_E m u_g \leq \int_E g \tag{4.9}$$

by (4.8). But this implies $u_f \leq 1$ a.e.

An immediate corollary of Lemma 4.3 is

COROLLARY 4.1 *If $f \in B(E)$, the space of essentially bounded, measurable functions on E, then there exists a corresponding potential $u_f \in B(E) \cap D_\alpha^0$ and*

$$\| u_f \|_B < \| f \|_B [\inf \{m(p) : P \in E\}]^{-1}. \tag{4.10}$$

LEMMA 4.4. *If $\phi \in B(E)$, then there exists a $\psi \in B(E)$ such that*

$$-\Delta(\psi * r^{2-\alpha-n}) = \phi \cdot m; \tag{4.11}$$

if $0 \leq \phi \leq 1$ a.e., then the same is true of ψ. Furthermore, if $u_f \in D_\alpha^0$ is a potential corresponding to some $f \in L(E)$, then

$$\int_E u_f \phi m = \int_E \psi f. \tag{4.12}$$

(Here, as usual, Δ is to be interpreted as a distribution in the open set E.)

Proof. Let us suppose $0 \leq \phi \leq 1$ a.e. Let $\{\phi_n\}$ be a sequence in $B(E)$ such that $\phi_n \uparrow \phi$ and ϕ_n vanishes outside some compact $K \subset E$. There is a potential $\psi_n \in B(E) \cap D_\alpha^0$ corresponding to $\phi_n \cdot m$ by Corollary 4.1, since $\phi_n \cdot m \in B(E)$. By the same corollary, $0 \leq \psi_n \leq 1$ a.e. In addition $\{\psi_n\}$ is a bounded, increasing sequence (or can be made so by alteration on a set of measure zero), and so there exists a limit function $\psi = \lim_n \psi_n$ in $B(E)$.

Now suppose v is infinitely differentiable with compact support in E; then

$$\int_E \psi A_\alpha v = \lim_{n\to\infty} \int_E \psi_n A_\alpha v = \lim_{n\to\infty} \int_E v\phi_n \cdot m = \int_E v \cdot \phi \cdot m, \tag{4.13}$$

so $-A_\alpha \psi = \phi \cdot m$ in the sense of distributions on E, thus proving (4.11). Since (4.12) holds for ϕ_n and ψ_n, we obtain the relation for ϕ and ψ by a passage to the limit. This completes the proof.

Let Ψ denote the class of functions constructed in the last lemma, that is the functions ψ satisfying (4.11) and (4.12) with ϕ and ψ in $B(E)$. Let $\mathfrak{D}(E)$ be the class of functions which are infinitely differentiable on some open set containing E. We then have the following lemma:

LEMMA 4.5. *If $\Delta(u * r^{2-n-\alpha}) = 0$, and if $u = \bar{u} + v + \psi$, where $\bar{u} \in D_\alpha^0$, $v \in \mathfrak{D}(E)$, and $\psi \in \Psi$, then $u = 0$ a.e. in E.*

Proof. It suffices to note that if $v \in \mathfrak{D}(E)$ and $f \in C(E)$, then

$$\int_E vf = -\int_E u_f \left\{ \Delta \int_E v \cdot | PQ |^{2-n-\alpha} \right\}. \tag{4.14}$$

It then follows from (4.14) and (4.12) that if $f \in B(E)$, and if u satisfies the hypothesis of the lemma then

$$\int_E uf = \int_E u_f A_\alpha u = 0 \tag{4.15}$$

and we thus have $u = 0$ a.e. on E.

We obtain immediately from Lemmas 4.4 and 4.5 the following corollary:

COROLLARY 4.2. *If $f \in B(E)$, then (4.4) holds in D_α^0 for $0 < \alpha < 2$.*

Proof. Take $\phi \equiv 1$ in Lemma 4.4, and let ψ be the corresponding solution constructed in that lemma. Then $1 - \psi = u$ satisfies the conditions of Lemma 4.5, and therefore $\psi \equiv 1$ a.e. on E. Putting this in (4.12), we obtain (4.4).

LEMMA 4.6. *If $f \in B(E)$, then*

$$u_f = J_\alpha(\tilde{f} * r^{\alpha-n}), \tag{4.16}$$

where $\tilde{f}$ is the function in $L(R^n)$ defined by

$$\begin{aligned} \tilde{f}(P) &= f(P), && P \in E \\ &= -C_\alpha \int_E u_f(Q) \mid PQ \mid^{-n-\alpha} dQ, && P \in CE \end{aligned} \tag{4.17}$$

and

$$J_\alpha = (4\pi)^{-n-1} \sin(\alpha\pi/2) \cdot \Gamma[(n + \alpha - 2)/2] \cdot \Gamma[(n - \alpha)/2]. \tag{4.18}$$

Proof. Let v be infinitely differentiable with compact support in R^n. Then

$$-\int_{R^n} v \cdot (u_f * r^{2-n-\alpha}) = -\int_E u_f(\Delta v * r^{2-n-\alpha}). \tag{4.19}$$

Since $v * r^{2-n-\alpha}$ is bounded and continuous on R^n, there is a potential u_g corresponding to its restriction on E. In fact, if ψ is the solution constructed in Lemma 4.4 corresponding to $\phi m = \Delta v * r^{2-n-\alpha}$, then we must have $u_g = v - \psi$, using the uniqueness proved in Lemma 4.5. Thus the right integral in (4.19) can be replaced by

$$\begin{aligned} \int_E fu_g &= \int_E f(v - \psi) \\ &= \int_E vf - \int_E u_f(P) \left\{ \int_{CE} v(Q) \cdot \mid PQ \mid^{-n-\alpha} \right\} dP = \int_{R^n} vf. \end{aligned} \tag{4.20}$$

This proves that

$$-\Delta(u_f * r^{2-n-\alpha}) = \tilde{f} \tag{4.21}$$

in the sense of distributions on R^n. That $\tilde{f} \in L(R^n)$ follows from the fact that

$u_f m \in L(E)$. The representation (4.16) follows from an application of the Fourier transform to (4.21).

From this lemma we obtain

COROLLARY 4.3. *If $f \in B(E)$, then u_f is continuous*[3] *on E.*

LEMMA 4.7. *Suppose that $E_1 \subset E_2$, $f \geq 0$, and $f \in B(E_2)$. If $u_f(\cdot\,; E_i)$ is the potential in $D_\alpha^0(E_i)$ corresponding to the restriction of f to E_i, then*

$$u_f(P; E_1) \leq u_f(P; E_2) \tag{4.22}$$

for almost all P in E_1.

Proof. If we extend $u_f(P; E_1)$ to be 0 in CE_1, then $\{u_f(\cdot\,; E_2) - u_f(\cdot\,; E_1)\}$ is the potential in $D_\alpha^0(E_2)$ of a positive integrable function, and is therefore positive. This completes the proof.

We have shown in Corollary 4.3 that $f \in B(E)$ implies that $u_f \in C(E)$. We shall next show that if the boundary of E consists of regular points, then $u_f \in C_0(\bar{E})$, the space of functions continuous on E vanishing at the boundary of E, or more precisely, that u_f is equivalent to such a function.

Before presenting a proof of this result, we look at a couple of special cases. It can be verified by Fourier transforms that if E is the ball $r < b$ then the potential u_1 corresponding to $f \equiv 1$ is given by

$$u_1 = \gamma_\alpha \cdot (b^2 - r^2)^{\alpha/2} \tag{4.23}$$

where

$$\gamma_\alpha = \Gamma(n/2)[2\cdot\pi^{n/2}(n+\alpha+2)\cdot\Gamma(\alpha/2+1)\cdot\Gamma(1-\alpha/2)]^{-1}. \tag{4.24}$$

This formula follows from the work of M. Riesz, and the calculations are carried out in [6, Theorem 5.2]. Since for any E, $f \in B(E)$ implies

$$| u_f | \leq \text{const}(\| f \|_B \cdot u_1), \tag{4.25}$$

it follows from (4.23) that if E is any ball $r < b$, then

$$\lim_{P\to P_0} u_f(P) = 0 \tag{4.26}$$

for P_0 on ∂E.

Next, we shall show that (4.26) also holds if E is a region between two spheres given by $a < r < b$. It is certainly true for P_0 on the outer sphere $r = b$, since the monotonicity property proved in Lemma 4.7 shows that the potential $u_1(\cdot\,; a, b)$ for the region $a < r < b$ is smaller than the potential $u_1(\cdot\,; b)$ for the ball $r < b$. If we perform an inversion about the inner sphere $r = a$, that is, make a change of variable $r' = a^2 r^{-1}$ in (4.16), we find that

$$u_1(a^2/r'; a, b)(r')^{\alpha-n} = u_g(r'; a^2/b, a) \tag{4.27}$$

[3] When we say that u_f is continuous on E we mean, of course, that there is a representative of the equivalence class u_f which belongs to $C(E)$.

where $g(r') = a^{2\alpha}(r')^{-n-\alpha}$. But from our remarks above,

$$\lim_{r' \to a} u_g(r'; a^2/b, a) = 0, \tag{4.28}$$

and therefore from (4.27) we conclude that (4.26) holds also when P_0 lies on the inner sphere $r = a$.

It is now easy to prove

LEMMA 4.8. *If P_0 is a point on the boundary of E lying on a sphere whose interior is contained in CE, then* (4.26) *holds whenever $f \in B(E)$.*

Proof. We may suppose the sphere at P_0 to have the equation $r = a$. Since E is a bounded domain, it can be enclosed in a ball $r < b$. The result then follows from the Lemma 4.7 and the fact that (4.26) holds for the region $a < r < b$.

COROLLARY 4.4. *If each point P_0 on the boundary of E satisfies the condition of Lemma* 4.8, *then $f \in B(E)$ implies $u_f \in C_0(\bar{E})$.*

Following [2, Section 4], we may represent our potentials u_f by means of a symmetric kernel $G_\alpha(P, Q)$, that is,

$$u_f(P) = \int_E G_\alpha(P, Q)\, f(Q)\, dQ \tag{4.29}$$

and

$$\| u_f \|^2 = \int_E \int_E G_\alpha(P, Q) f(P) f(Q)\, dP\, dQ. \tag{4.30}$$

From (4.16) we obtain the relation

$$\begin{aligned} G_\alpha(P, Q) &= J_\alpha \mid PQ \mid^{\alpha-n} \\ &\quad - C_\alpha J_\alpha \int_E G_\alpha(Q, R) \left[\int_{CE} \{ \mid PT \mid^{n-\alpha} \mid TR \mid^{n+\alpha} \}^{-1} dT \right] dR \end{aligned} \tag{4.31}$$

for P and Q in E.

The solution ψ of $-A_\alpha \psi = \phi \cdot m$ constructed in Lemma 4.4 is given by

$$\psi(P) = \int_E G_\alpha(P, Q) \phi(Q) \cdot m(Q)\, dQ \tag{4.32}$$

and in particular

$$\int_E G_\alpha(P, Q) \cdot m(Q)\, dQ = 1 \qquad \text{a. e.} \tag{4.33}$$

5. Semi-groups associated with A_α

The basis for the results of this section is provided by the following two lemmas from [2], appearing in that paper as Lemmas 3 and 4.

LEMMA 5.1. *Given a Dirichlet space $D(X)$ and a function $f \in L^2(X)$ or in D, then for every $\lambda > 0$, there exists a unique element $S_\lambda f \in D$ which minimizes the quadratic functional*

$$F(u) = \lambda \| u \|_D^2 + \int_X | u - f |^2 \, d\xi; \tag{5.1}$$

$u = S_\lambda f$ is the only element in D such that $u - f \in L^2(X)$ and

$$\lambda(u, v) + \int_X (u - f)v \, d\xi = 0 \tag{5.2}$$

for each $v \in L^2 \cap D$.

LEMMA 5.2. *For each $\lambda > 0$, the operator $R_\lambda = \lambda^{-1} S_{1/\lambda}$, defined on D has the following properties:*

(i) *R_λ is linear, positive, hermitian, and bounded in D and in $L^2(X)$ with $\| \lambda R_\lambda \| \leq 1$ in both spaces; $f \in D$ and $\| R_\lambda f \|_D = \| f \|_D$ implies $f = 0$.*

(ii) $\lim_{\lambda \to \infty} \lambda R_\lambda = I$ *and* $\lim_{\lambda \to 0} \lambda R_\lambda = 0$ *strongly in D and in L^2, where I denotes the identity operator.*

(iii) *If T is a normalized contraction, then $Tf = f$ implies $T\{\lambda R_\lambda f\} = \lambda R_\lambda f$. In particular, if $0 \leq f \leq 1$ a.e., then the same is true of $\lambda R_\lambda f$.*

These two lemmas are proved in [2, Section 3].

Note that (5.2) implies that the function $u_f^\lambda = R_\lambda f$ satisfies

$$\lambda u_f^\lambda - A_\alpha u_f^\lambda = f \tag{5.3}$$

in the sense of distributions on E. Thus the preceding two lemmas combined with the Hille-Yosida theorem show that A_α is the infinitesimal generator of a contraction semi-group on D_α^0 and on L_2.

Our aim is now to show that A_α also generates semi-groups on $L(E)$ and on $C_0(\bar{E})$, the space of functions continuous on E vanishing on the boundary of E.

THEOREM 5.1. *If $f \in C_0(\bar{E})$ with E satisfying the condition of Corollary 4.4, and R_λ^0 denotes the restriction of the operator R_λ of Lemma 5.2 to C_0, then*

$$\lambda R_\lambda^0 f - A_\alpha R_\lambda^0 f = f \tag{5.4}$$

with A_α defined in (1.2). Furthermore, in the norm of $C_0(\bar{E})$,

$$\| \lambda R_\lambda^0 f \| \leq \| f \|, \tag{5.5}$$

and as $\lambda \to \infty$

$$\lambda R_\lambda^0 \to I \tag{5.6}$$

strongly in $C_0(\bar{E})$. Finally,

$$0 \leq f \leq 1 \;\Rightarrow\; 0 \leq R_\lambda^0 f \leq \lambda^{-1}. \tag{5.7}$$

Thus $\{R_\lambda^0\}$ *is the family of resolvents of a positive contraction semi-group from* C_0 *to itself with infinitesimal generator* A_α.

Proof. We have already noted in (5.3) that property (5.4) must hold. Property (5.7) follows from Lemma 5.2(iii). Thus if $f \in C_0(\bar{E})$, then $R_\lambda f$ is the potential in D_α^0 of the bounded function $f - \lambda R_\lambda f$ and is therefore in $C_0(\bar{E})$ by Corollary 4.4. From (5.7) we conclude that (5.5) holds.

If we can show that the range of R_λ^0 is dense in $C_0(\bar{E})$, then the two properties (5.4) and (5.5) will imply (5.6). Let G_α denote the integral transformation in (4.29). The range of R_λ^0 is identical with the range of G_α, since

$$(5.8) \qquad R_\lambda^0 = G_\alpha(I - \lambda R_\lambda^0)$$

and the range of $I - \lambda R_\lambda^0$ is equal to $C_0(\bar{E})$. But the range of G_α is dense in $C_0(\bar{E})$, since every infinitely differentiable function with compact support in E is in this range. This completes our proof.

In any Dirichlet space there is defined a "generalized Laplacian", cf. [2, Sec. 4], as follows: if u is a potential in D generated by a function f, then we define $\Delta u = f$ and call f the Laplacian of u. In the classical case, this reduces to a negative constant times the ordinary Laplacian. The operator $-\Delta$ thus defined is the infinitesimal generator of a positive contraction semi-group from D to itself of which $\{R_\lambda\}$, the family of operators constructed in Lemma 5.2, is the resolvent family. In our case, of course, $-\Delta = A_\alpha$.

The next result is a direct corollary of Lemma 5.2. We include it for the sake of completeness:

COROLLARY 5.1. *If* $D = D(X, \xi)$ *is a Dirichlet space with* ξ *a totally finite measure such that* $u \in D$ *implies* $u \in L(X)$ *and*

$$(5.9) \qquad \| u \|_L = \int_X | u | \, d\xi \leq A \, \| u \|_D$$

for some constant A *independent of* u, *then the operator* R_λ *of* Lemma 5.2 *can be extended to a bounded linear operator* R_λ' *from* $L(X)$ *to itself so that* $\{R_\lambda'\}$ *is the family of resolvents of a positive contraction semi-group from* $L(X)$ *to itself. The infinitesimal generator of this semi-group is a closed extension of* $-\Delta$.

Proof. Let us suppose that $f \in \mathcal{C} \cap D$. Then since R_λ is positive and hermitian as an operator on $L^2(X)$, we have for $\lambda > 0$ and $f \in L^2$

$$(5.10) \qquad \lambda \int_X | R_\lambda f | \, d\xi \leq \lambda \int_X R_\lambda | f | \, d\xi = \lambda \int_X | f | \, R_\lambda 1 \, d\xi \leq \int_X | f | \, d\xi.$$

Since L^2 is dense in L we conclude that R_λ can be extended to an operator R_λ' from L to L which satisfies

$$(5.11) \qquad \lambda \, \| R_\lambda' \|_L \leq 1.$$

The range of R_λ, and hence of R_λ' is dense in L, since it is dense in L^2. Since

R_λ satisfies

$$\lambda R_\lambda f + \Delta R_\lambda f = f, \tag{5.12}$$

we have the result.

Note that the hypotheses of Corollary 5.1 are satisfied in our case. The infinitesimal generator of the semi-group from L to L agrees with (1.2) on a dense subset of $L(E)$, since for every infinitely differentiable function v with compact support in E and $f \in L(E)$,

$$\int_E A_\alpha\, v \cdot R'_\lambda f = \int_E v(\lambda R'_\lambda f - f). \tag{5.13}$$

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