THE TYPE SET OF A TORSION-FREE GROUP OF FINITE RANK[1]

BY

JOHN E. KOEHLER, S.J.

In this paper, we shall show that the type set of a torsion-free group of finite rank has certain lattices of types and of pure subgroups associated with it. Conversely, if certain lattice requirements are met by a finite set of types T and by associated subspaces, then a torsion-free group A can be constructed having type set T. The construction of A suggests defining a class of groups having a similar construction. For this class of groups, we shall next establish a set of quasi-isomorphism invariants, together with several other properties. Finally, we shall examine the structure both of the groups and of the class.

1. Necessary conditions on the type set

DEFINITION 1.1 Throughout this paper, by "group" we shall mean "torsion-free abelian group of finite rank" unless some further qualification is given. Let $\sim$ denote the usual equivalence relation on the set of heights; and let $[h]$ denote the equivalence class, or type, to which the height h belongs. Let $\leq$, $\cap$, and $\cup$ have their usual meaning for both heights and types. The set of all types then forms a distributive lattice in which the meet and join of the types t and t' are given by $t \cap t'$ and $t \cup t'$ respectively, [4, pp. 146–147].

DEFINITION 1.2 Let A be a group of rank n. Use A^* to denote the minimal divisible group containing A. Without loss of generality, it can be assumed that $A \subseteq R^n$ and $A^* = R^n$, where R^n is an n-dimensional rational vector space. Let $0 \neq x \in A$; $t^A(x)$, or simply $t(x)$, denotes the type of x in A. Let $t^A(0) = t_\infty$, a type defined to be greater than all other types. $T(A) = \{t^A(x) |\ x \in A\}$ is called the (augmented) type set of A. Let $C(A) = T(A) \cup \{$all finite intersections of members of $T(A)\}$. $C(A)$ is countable since A is countable.

DEFINTION 1.3 Let t be a type; define $A_t = \{x \in A \mid t(x) \geq t\}$. A_t is a pure subgroup of A, [4, p. 147]. Let

$$P(A) = \{A_t \mid t \in C(A)\} \quad \text{and} \quad P^*(A) = \{A_t^* \mid t \in C(A)\}.$$

We shall use A_k to denote A_{t_k} if no confusion arises.

LEMMA 1.4 *Let A be a group; let $t_1, t_2 \in C(A)$ such that $t_\infty > t_2 > t_1$. Then* $\text{Rank}(A_1) > \text{Rank}(A_2)$.

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Proof Since $t_1 \in C(A)$, $t_1 = s_1 \cap s_2 \cap \cdots \cap s_k$, where $s_i \in T(A)$. At least one of these, say s_j, is not greater than or equal to t_2, or else

$$s_1 \cap s_2 \cap \cdots \cap s_k \geq t_2 > t_1,$$

a contradiction. There exists $0 \neq x \in A$ such that $t(x) = s_j \geq t_1$. Thus $x \in A_1$, $x \notin A_2$. Since A_2 is pure, this implies $\text{Rank}(A_1) > \text{Rank}(A_2)$.

LEMMA 1.5 *Let A be a group, $t \in C(A)$, and $x_1, x_2, \cdots, x_r$ a maximal independent set of elements in A_t. Then*

$$t = t^A(x_1) \cap t^A(x_2) \cap \cdots \cap t^A(x_r).$$

Proof Let $t' = t^A(x_1) \cap t^A(x_2) \cap \cdots \cap t^A(x_r) \in C(A)$. $t' \geq t$ since $t^A(x_i) \geq t$ for each i. Suppose $t' > t$. Then $\text{Rank}(A_{t'}) < \text{Rank}(A_t)$, contradicting the fact that $x_1, x_2, \cdots, x_r \in A_{t'}$ and they form a maximal independent set in A_t.

THEOREM 1.6 *Let A be a group of rank n. Then $C(A)$ forms a lattice of length at most n in which lattice meet is type intersection. Thus $C(A)$ has a minimum type*

$$t_0 = t(x_1) \cap t(x_2) \cap \cdots \cap t(x_n),$$

where $x_1, x_2, \cdots, x_n$ is any maximal independent set in A.

Proof $C(A)$ forms a semi-lattice in which meet is type intersection by definition. Let $t_\infty > t_k > \cdots > t_1$ be any linearly ordered subset of $C(A)$. Then $0 < \text{Rank}(A_k) < \cdots < \text{Rank}(A_1) \leq n$. Thus $k \leq n$, and the semi-lattice $C(A)$ has length at most n. Since any two elements in $C(A)$ have an upper bound t_∞ in $C(A)$, they have a least upper bound in $C(A)$. Therefore $C(A)$ is a lattice; the rest follows from Lemma 1.5.

Remark 1 Theorem 1.6 answers conjectures 1(b) and 2(d) of [2, p. 40].

Remark 2 $C(A)$ is not necessarily a sublattice of the lattice of all types, since groups exist (see example 1.10) in which $t_1, t_2 \in C(A)$ and the l.u.b. of t_1 and t_2 in $C(A)$ is greater than $t_1 \cup t_2$.

THEOREM 1.7 *Let A be a group.*

1. *$P(A)$ forms a lattice of pure subgroups of A; $P^*(A)$ forms a lattice of subspaces of A^*. As lattices, $P(A)$ is isomorphic to $P^*(A)$, and both are dually isomorphic to $C(A)$.*

2. *In the lattices $P(A)$ and $P^*(A)$, denote lattice meet by $\wedge$ and lattice join by $\vee$. Then, if $A_i, A_j \in P(A)$,*

$$A_i \wedge A_j = A_i \cap A_j, \qquad A_i^* \wedge A_j^* = A_i^* \cap A_j^*,$$

$$A_i \vee A_j \supseteq A_i + A_j, \qquad A_i^* \vee A_j^* \supseteq A_i^* + A_j^*.$$

Proof (1) The correspondence $t_k \to A_k$, $t_k \in C(A)$, $A_k \in P(A)$, is onto by definition. Suppose $A_i = A_j$ and $x_1, x_2, \cdots, x_r$ is a maximal independent set in both A_i and A_j. Then $t_i = t(x_1) \cap t(x_2) \cap \cdots \cap t(x_r) = t_j$ by Lemma

1.5. Thus $t_k \to A_k$ is also one-to-one. If $t_j \leq t_k$ and $x \in A_k$, then by definition, $x \in A_j$; hence $A_j \supseteq A_k$. Thus $P(A)$ forms a lattice dually isomorphic to $C(A)$.The lattice $P(A)$ is isomorphic to $P^*(A)$ since all the members of $P(A)$ are pure subgroups of A.

(2) Let $x \in A_i \cap A_j$. $x \in A_i \Rightarrow t(x) \geq t_i$; $x \in A_j \Rightarrow t(x) \geq t_j$. Hence $t(x) \geq t_i \cup t_j$; thus $t(x) \geq t_i \vee t_j$, the l.u.b. of the lattice $C(A)$. The argument reverses to give $t(x) \geq t_i \vee t_j \Rightarrow x \in A_i \cap A_j$. Thus $A_i \cap A_j = A_{t_i \vee t_j} = A_i \wedge A_j$ by the dual isomorphism of $P(A)$ and $C(A)$ as lattices.

Let $x = y + z \in A_i + A_j$, where $y \in A_i$, $z \in A_j$. Then

$$t(x) \geq t(y) \cap t(z) \geq t_i \cap t_j,$$

and so $x \in A_{t_i \cap t_j}$. Now $A_{t_i \cap t_j} = A_i \vee A_j$ from the dual isomorphism. Thus $A_i + A_j \subseteq A_i \vee A_j$.

The relations in $P^*(A)$ hold because of the isomorphism of the lattices $P(A)$ and $P^*(A)$.

Example 1.10 will show that $A_i \vee A_j \supset A_i + A_j$ is possible.

Lemma 1.8 *If $S_1, S_2, \cdots, S_m$ are proper subspaces of R^n, then there is a basis $x_1, x_2, \cdots, x_n$ of R^n such that $x_i \notin S_j$ for $1 \leq i \leq n$, $1 \leq j \leq m$.*

The proof is by induction on m.

Corollary 1.9 *If $T(A)$ is finite, then $C(A) = T(A)$ and there are* Rank(A_t) *independent elements of type t in A for every $t \in T(A)$.*

Proof Let $t \in C(A)$. Suppose $t_1, t_2, \cdots, t_k$ are all the types in $T(A)$ that are greater than t. By Theorem 1.7, $A_1^*, A_2^*, \cdots, A_k^*$ are all proper subspaces of A_t^*. Thus by Lemma 1.8 there is a basis $x_1, x_2, \cdots, x_r$ of A_t^*, where $r = \text{Rank}(A_t)$, such that $x_i \notin A_j^*$; $i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, k$. Moreover, the x_i can be chosen so that they are in A_t. Since $x_i \notin A_j^*$, then $t(x_i) \neq t_j$. But $t(x_i) \geq t$; hence $t(x_i) = t \in T(A)$. This proves both statements.

Remark Examples have been constructed of groups of rank 2 and infinite type set such that $T(A) \neq C(A)$, [2, p. 30].

Example 1.10 (1) Define h_0, h_1, h_2, h_3 by

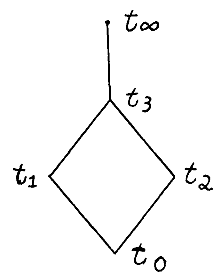
$$h_0(p) = 0 \quad \text{for all } p;$$

$$h_1(2) = \infty; \qquad h_1(p) = 0 \quad \text{otherwise};$$

$$h_2(3) = \infty; \qquad h_2(p) = 0 \quad \text{otherwise};$$

$$h_3(2) = h_3(3) = h_3(5) = \infty; \qquad h_3(p) = 0 \quad \text{otherwise}.$$

Let $t_i = [h_i]$, $i = 0, 1, 2, 3$. In the next section we shall show that there is a rank 3 group A such that $T(A) = \{t_0, t_1, t_2, t_3, t_\infty\}$. Now $C(A) = T(A)$ has a lattice structure as illustrated. Clearly $t_1 \vee t_2 = t_3 > t_1 \cup t_2$. Thus $C(A)$ is not a sublattice of the lattice of all types.



(2) Let A be as in the previous example. Let B be a rank 1 group of type t_0. Let $A' = A \oplus B$. Then $\text{Rank}(A') = 4$ and $T(A) = T(A')$; in $P(A')$, $A'_1 \subseteq A$, $A'_2 \subseteq A$, [4, p. 146]. Hence $A'_1 + A'_2 \subseteq A \subset A' = A'_1 \vee A'_2$. Thus $P(A')$ is not a sublattice of the lattice of all subgroups of A', nor is $P^*(A')$ a sublattice of the lattice of all subspaces of $A'^* = R^4$.

2. A partial converse to Theorems 1.6 and 1.7

THEOREM 2.1 *Let $T = \{t_\infty, t_0, t_1, \cdots, t_N\}$ be a set of distinct types, where t_∞ is a type defined to be greater than all other types. Suppose T forms a lattice under the operations $\wedge$ and $\vee$, where $t_i \wedge t_j = t_i \cap t_j$ and $\vee$ is the l.u.b. in T. Let $L^* = \{0, A_0^*, A_1^*, \cdots, A_N^*\}$ be a lattice of subspaces of $R^n = A_0^*$ under the operations $\wedge$ and $\vee$, where $A_i^* \wedge A_j^* = A_i^* \cap A_j^*$ and $\vee$ is the l.u.b. in L^*. Suppose further that, as lattices, T is dually isomorphic to L^*. Then a group A can be constructed such that $T(A) = T$ and $P^*(A) = L^*$.*

Remark Theorem 2.1 assures the existence of the group A in Example 1.10, since the dual of the lattice of types is clearly realizable in R^3.

Theorems 1.7 and 2.1 show that the problem of finding all the possible finite type sets which are type sets of groups of finite rank is equivalent to the (unsolved) problem of finding all the possible finite lattices, under the operations $\wedge$ and $\vee$, of subspaces of a rational vector space whose dimension is equal to the given rank.

An example of a lattice of types of length 3 may be constructed which has no corresponding lattice of subspaces in 3-space, due to the restrictions on the latter that follow from Desargues' Theorem when we intersect the subspaces by a plane that does not pass through 0.

The actual construction of the group A will occupy the rest of the section.

LEMMA 2.2 *Let $\{t_0, t_1, \cdots, t_N\}$ be a set of types closed under intersection. Let $h_0, h_1, \cdots, h_N$ be arbitrary heights such that $h_i \in t_i$, $i = 0, 1, \cdots, N$. Then there are heights $h'_0, h'_1, \cdots, h'_N$ satisfying, for $0 \le i, j, k \le N$*

(i) $h'_i \sim h_i$;
(ii) $h'_i \le h_i$;
(iii) *if* $t_i \le t_j$, *then* $h'_i \le h'_j$;
(iv) *if* $t_i \cap t_j = t_k$, *then* $h'_i \cap h'_j = h'_k$.

Proof For each $i = 0, 1, \cdots, N$, let $h_i'' = \bigcap\{h_k \mid t_i \leq t_k\}$. It can be shown that $h_0'', h_1'', \cdots, h_N''$ satisfy properties (i), (ii), and (iii).

For a fixed pair of indices i, j, $t_i \cap t_j = t_k$ for some k. Define

$$\pi(i, j) = \{p \mid h_k''(p) \neq \min\{h_i''(p), h_j''(p)\}\}.$$

Each $\pi(i, j)$ is a finite set since $h_k'' \sim h_i'' \cap h_j''$. Therefore $\pi' = \bigcup_{i,j} \pi(i, j)$ is a finite set.

Let $h_0' = h_0''$. For $i = 1, 2, \cdots, N$ define h_i' by

$$\begin{aligned} h_i'(p) &= h_0'(p) \quad \text{if} \quad p \in \pi' \quad \text{and} \quad h_i''(p) < \infty \\ &= h_i''(p) \quad \text{otherwise.} \end{aligned}$$

$h_0', h_1', \cdots, h_N'$ is the desired set of heights.

2.3 THE CONSTRUCTION OF A. Let π denote the primes, Z the integers.

1. Let us first index T so that t_0 is the minimum type in the lattice. Index L^* so that $t_i \to A_i^*$ gives the dual isomorphism $T \to L^*$.
2. Choose a basis $B_0 = \{y_1^0, \cdots, y_n^0\}$ for $A_0^* = R^n$, where $y_i^0 \notin A_k^*$; $i = 1, 2, \cdots, n$; $k = 1, 2, \cdots, N$. This can be done by Lemma 1.8. Applying 1.8 to subspaces, we can choose a basis $y_1^k, y_2^k, \cdots, y_{n_k}^k$ for each A_k^*, $1 \leq k \leq N$, where for each $i = 1, 2, \cdots, n_k$, $y_i^k \notin A_l^*$ if $A_l^* \subset A_k^*$ and where $y_i^k = \sum_{j=1}^n a_{ij}^k y_j^0$, with a_{ij}^k integers such that g.c.d. $\{a_{i1}^k, \cdots, a_{in}^k\} = 1$.
3. Choose heights $h_0, h_1, \cdots, h_N$ such that, for $0 \leq i, j, k \leq N$; $h_i \in t_i$, $h_i \leq h_j$ if $t_i \leq t_j$, and $h_i \cap h_j = h_k$ if $t_i \cap t_j = t_k$ (Lemma 2.2).
4. Let A be the group generated by

$$G = \{p^{-s_k(p)} y_i^k \mid p \in \pi; 0 \leq s_k(p) < h_k(p) + 1; s_k(p) \in Z; \\ k = 0, 1, \cdots, N; i = 1, 2, \cdots, n_k\}.$$

Every element x of A can then be written in the form

$$\sum_{k=0}^N \sum_{i=1}^{n_k} \sum_{s_k(q)} \sum_{q \in \pi} c_i^k(q) q^{-s_k(q)} y_i^k, \tag{1}$$

where $c_i^k(q) \in Z$, $s_k(q) \in Z$, $s_k(q) < h_k(q) + 1$, and the sum has a finite number of terms.

NOTATION 2.4 Define

$$\pi_0 = \{p \mid h_0(p) = h_i(p), i = 1, 2, \cdots, N\},$$

$$\pi_k = \{p \mid h_k = \bigcap\{h_j \mid h_j(p) > h_0(p)\}\}, \qquad k = 1, 2, \cdots, N.$$

It is easy to show that $\pi_0, \pi_1, \cdots, \pi_N$ partition the primes.

Let $A_k = A \cap A_k^*$, $k = 0, 1, \cdots, N$. If $x \in A$, let $A(x) = \bigcap\{A_i \mid x \in A_i\}$. Due to the lattice structure of L^*, $A(x) = A_k$ for some k; in particular, $A(y_i^k) = A_k$.

If $x \in A$, let $H^A(x)$, or simply $H(x)$, denote the height of x in A. Let $h_p^A(x) = H^A(x)(p)$. If $r \in R$, write $r = \prod_p p^{e_p}$, and define $h_p(r) = e_p$.

Remark If $A(x) = A_k$, then we can write $x = \sum_{i=1}^{n_k} a_i y_i^k$, where $a_i \in R$. Now $H(y_i^k) \geq h_k$ by the definition of G. Hence $t(y_i^k) \geq t_k$ and so $t(x) \geq t_k$. To get, as desired, that $t(x) = t_k$, it therefore suffices to show that for some integer $D(x)$, $h_p^A(x) \leq h_k(p) + h_p(D(x))$ for all $p \in \pi$. We now proceed to find this integer $D(x)$ for every x in A.

DEFINITION 2.5 Let $B = \{x_1, x_2, \cdots, x_n\}$ be an arbitrary set of independent elements in A. Let F_B be the free subgroup of A generated by B. We shall say that x is B-reduced if $x \in F_B$ and $h_p^{F_B}(x) = 0$ for all $p \in \pi$.

Let $0 \neq x \in A$; then there is a unique $s \in R$, $s > 0$, such that sx is B_0-reduced, where $B_0 = \{y_1^0, y_2^0, \cdots, y_n^0\}$. Since $t(x) = t(sx)$, $T(A)$ is determined by the B_0-reduced members of A. Let $F_{B_0} = F$.

LEMMA 2.6 *If x is a B_0-reduced element of A, then $h_p^A(x) = h_0(p)$ for every $p \in \pi_0$.*

Proof Write $x = \sum_{j=1}^n b_j y_j^0$. The lemma is obvious if $p \in \pi_0$, $h_0(p) = \infty$. Suppose $p \in \pi_0$, $h_0(p) = s < \infty$, $p^{-s-1}x \in A$. Then we can write $p^{-s-1}x$ in the form (1). Since $p \in \pi_0$, $s_k(p) \leq h_0(p)$ for all i and k. Thus we may write

$$p^{-s-1}x = \sum_{k=0}^N \sum_{i=1}^{n_k} d_i^k p^{-s} y_i^k,$$

where the d_i^k are rationals with denominators prime to p. But then

$$\begin{aligned} x &= \sum_{k=0}^N \sum_{i=1}^{n_k} (p\, d_i^k) y_i^k = \sum_{k=0}^N \sum_{i=1}^{n_k} p\, d_i^k \sum_{j=1}^n a_{ij}^k y_j^0 \\ &= \sum_{j=1}^n \Big(p \sum_{k=0}^N \sum_{i=1}^{n_k} d_i^k a_{ij}^k\Big) y_j^0 = \sum_{j=1}^n b_j y_j^0. \end{aligned}$$

Thus $p \mid b_j$ for each j, contradicting $h_p^F(x) = 0$.

DEFINITION 2.7 Let $p \in \pi$, $0 < r \in Z$, $x \in F$. We can write $x = \sum_{i=1}^n a_i y_i^0$, where $a_i \in Z$. Define $x(p^r) = \sum_{i=1}^n a_i' y_i^0$, where $0 \leq a_i' < p^r$ and $a_i' \equiv a_i \pmod{p^r}$, $i = 1, 2, \cdots, n$. If A' is a subgroup of A, define

$$A'(p^r) = \{x(p^r) \mid x \in A' \cap F\}.$$

LEMMA 2.8 *Let $p \in \pi$, $0 < r \in Z$, $x \in F$, A' be a subgroup of A. Then*

(i) $x(p^r) \in A$;

(ii) *if $x \neq 0$ is B_0-reduced, then $x(p^r) \neq 0$;*

(iii) *$A'(p^r)$ is a group, where addition is defined by*

$$x(p^r) + y(p^r) = (x + y)(p^r);$$

(iv) *if $p \nmid m$, $(mx)(p^r) \in A'(p^r)$, then $x(p^r) \in A'(p^r)$;*

(v) *if $A'' \subseteq A'$, then $A''(p^r) \subseteq A'(p^r)$;*

(vi) $F \cap A' \subseteq \{x \mid x(p^{r+1}) \in A'(p^{r+1})\} \subseteq \{x \mid x(p^r) \in A'(p^r)\}$;

The proof follows easily from the definitions. Note that $y \notin A'$ but $y(p^r) \in A'(p^r)$ is possible as long as $y(p^r) = x(p^r)$ for some $x \in A' \cap F$.

LEMMA 2.9 *Let x be a B_0-reduced element of A. If $p \in \pi_l$ and $h_p^A(x) \geq h_0(p) + r$, where $0 < r \in Z$, then $x(p^r) \in A_l(p^r)$.*

Proof If $p \in \pi_l$, then $h_l = \bigcap\{h_k \mid h_k(p) > h_0(p)\}$. Let

$$I = \{k \mid h_k \geq h_l\}, J = \{k \mid h_k < h_l\}.$$

Then

$$k \in I \Leftrightarrow h_k(p) \geq h_l(p) \Leftrightarrow A_k^* \subseteq A_l^* \Leftrightarrow y_i^k \in A_l^*, \quad i = 1, 2, \cdots, n_k.$$

$$k \in J \Leftrightarrow h_k(p) = h_0(p).$$

Let $s = h_0(p)$; if $h_p^A(x) \geq s + r$, we may write $p^{-s-r}x$ in form (1). Since $p \in \pi_l$, $s_k(p) \leq s$ for $k \in J$, and we may rewrite

$$(2) \qquad p^{-s-r}x = \sum_{k \in I} \sum_{i=1}^{n_k} d_i^k y_i^k + \sum_{k \in J} \sum_{i=1}^{n_k} e_i^k p^{-s} y_i^k,$$

where the d_i^k are rationals, and the e_i^k are rationals with denominators prime to p. Let

$$y = \sum_{k \in I} \sum_{i=1}^{n_k} p^{s+r} d_i^k y_i^k.$$

Then

$$x = y + \sum_{k \in J} \sum_{i=1}^{n_k} p^r e_i^k y_i^k$$
$$= y + \sum_{j=1}^{n} p^r \left(\sum_{k \in J} \sum_{i=1}^{n_k} e_i^k a_{ij}^k\right) y_j^0 = y + z.$$

There is an integer m prime to p such that $mz \in F$. But then $my = mx - mz \in F$; hence $my \in A_l \cap F$ and

$$(my)(p^r) = (mx)(p^r) - (mz)(p^r) = (mx)(p^r) \in A_l(p^r).$$

By Lemma 2.8(iv), $x(p^r) \in A_l(p^r)$.

We now proceed to find necessary conditions on the B_0-reduced elements x of A such that $x(p^r) \in A_k(p^r)$.

LEMMA 2.10 *Let S be a proper subspace of R^n and $u_1, u_2, \cdots, u_m \in R^n - S$. Then there is an $(n-1)$-dimensional subspace S' of R^n containing S and such that $u_1, u_2, \cdots, u_m \in R^n - S'$.*

The proof is by induction on m.

NOTATION 2.11 For the rest of Section 2, let x be a B_0-reduced element of A, $x = \sum a_j y_j^0$. Let i be the first index such that $a_i \neq 0$. Then we define a new basis of A_0^*, $B_x = \{x_1, x_2, \cdots, x_n\}$, where $x_j = y_j^0$ if $j \neq i$ and $x_i = x$.

For $k = 1, 2, \cdots, N$, choose $(n-1)$-dimensional subspaces $A_k'' \supseteq A_k^*$ such that $y_i^0 \notin A_k''$ for all i, and also $x \notin A_k''$ whenever $x \notin A_k^*$ (Lemma 2.10). Extend the basis $y_1^k, y_2^k, \cdots, y_{n_k}^k$ of A_k^* to a basis $y_1^k, y_2^k, \cdots, y_{n-1}^k$ of A_k''. Let A_k'' and this basis be fixed for each B_0-reduced x.

Let m_i^k be the unique positive rationals such that $m_i^k y_i^k$ is B_x-reduced, $1 \leq k \leq N$, $1 \leq i \leq n-1$. Write $m_i^k y_i^k = \sum_{j=1}^{n} b_{ij}^k x_j$, where $b_{ij}^k \in Z$. Let

$M^k = ((b_{ij}^k))$, a matrix whose ij^{th} entry is b_{ij}^k. Let M_i^k be the $(n-1) \times (n-1)$ matrices formed by deleting the i^{th} columns from M^k.

Let δ_i^k be the determinant of M_i^k. $\delta_i^k \in Z$, since all $b_{ij}^k \in Z$. Let $D_i^k = \delta_i^k/\text{g.c.d.}\{\delta_1^k, \cdots, \delta_n^k\} \in Z$. Finally, define

$$u_{ij}^k = D_i^k x_j + (-1)^{i+j+1} D_j^k x_i, \qquad 1 \leq k \leq N, 1 \leq i < j \leq n.$$

LEMMA 2.12

$$D_i^k \neq 0 \iff x_i \notin A_k; \qquad i = 1, 2, \cdots, n, k = 1, 2, \cdots, N.$$

Proof For each index i and k, let N_i^k be the $(n) \times (n)$ matrix whose first $n-1$ rows are those of M^k and whose last row has 1 in the i^{th} place, 0 elsewhere.

By choice of A_k'', and since $x_i \in A$, we have $x_i \notin A_k \Leftrightarrow x_i \notin A_k''$. From vector space theory,

$$\begin{aligned} x_i \notin A_k'' &\iff y_1^k, \cdots, y_{n-1}^k, x_i \text{ form a basis of } A^* \\ &\iff \text{the row vectors of } N_i^k \text{ are independent} \\ &\iff 0 \neq \text{determinant}(N_i^k) = (-1)^{n+i}\delta_i^k \\ &\iff 0 \neq D_i^k. \end{aligned}$$

LEMMA 2.13 *Each* $u_{ij}^k \in A_k'' \cap A$.

Proof $u_{ij}^k \in A$ clearly. The lemma is obvious from 2.12 if x_i or x_j are in A_k. Suppose $x_i, x_j \notin A_k$, where $i < j$; then $x_i, x_j \notin A_k''$. Since A_k'' is $(n-1)$-dimensional and x_i and x_j are independent, $d_i^k x_i + d_j^k x_j \in A_k'' \cap A$ for some non-zero rationals d_i^k, d_j^k.

Thus $y_1^k, \cdots, y_{n-1}^k, d_i^k x_i + d_j^k x_j$ are dependent. Thus the determinant of their coefficients, namely $(-1)^{n+j} d_j^k \delta_j^k + (-1)^{n+i} d_i^k \delta_i^k$, is 0. Hence $d_i^k = (-1)^{i+j+1} d_j^k D_j^k / D_i^k$. Substituting this value for d_i^k into $d_i^k x_i + d_j^k x_j$ and multiplying both coefficients by D_i^k/d_j^k yields u_{ij}^k. Thus $u_{ij}^k \in A_k'' \cap A$.

LEMMA 2.14 *Let* $x = x_i \notin A_k''$. *Suppose there is a* $y \in A_k''$ *such that* $y = \sum b_j x_j$, *where* $h_p(b_i) = 0$ *and* $h_p(b_j) > 0$ *for all* $j \neq i$. *Then*

$$\min_{j \neq i}\{h_p(b_j)\} \leq h_p(D_i^k).$$

Proof $\{u_{ij}^k \mid j \neq i\}$ are independent, and therefore form a basis for A_k''. This is clear since x_j appears with a non-zero coefficient only in the expression for u_{ij}^k, $j \neq i$ (Lemma 2.12). Hence no linear combination of the u_{ij}^k can be 0 unless all coefficients are 0.

Thus we may write $y = \sum_{j \neq i} c_j u_{ij}^k$, where $c_j \in R$. Since

$$b_i = \sum_{j \neq i} (-1)^{i+j+1} c_j D_j^k,$$

then $\min_{j \neq i}\{h_p(c_j)\} \leq 0$ or else $h_p(b_i) > 0$. Since $b_j = c_j D_i^k$, then $h_p(b_j) =$

$h_p(c_j) + h_p(D_i^k)$. Hence

$$\min_{j \neq i}\{h_p(b_j)\} = \min_{j \neq i}\{h_p(c_j)\} + h_p(D_i^k) \leq h_p(D_i^k).$$

LEMMA 2.15 *Let* $x = x_i = \sum_{j \geq i} a_j y_j^0$. *Then* $x(p^r) \in A_k(p^r), 0 < r \in Z$, *only if* $r \leq h_p(a_i D_i^k)$.

Proof If $x \in A_k$, then $D_i^k = 0$ by 2.12 and $h_p(a_i D_i^k) = \infty > r$. If $x \notin A_k$ and $x(p^r) \in A_k(p^r)$, then there is a $y \in A_k \cap F$ such that $x(p^r) = y(p^r)$. Write

$$y = \sum b_j x_j = b_i a_i y_i^0 + \sum_{j \neq i} (b_j + b_i a_j) y_j^0,$$

where each $b_k \in R$. Since $y(p^r) = x(p^r)$, then $b_i a_i \equiv a_i \pmod{p^r}$. If $h_p(a_i) \geq r$, then $h_p(a_i D_i^k) \geq r$. If $h_p(a_i) < r$, then $h_p(b_i) = 0$. Let $s = r - h_p(a_i)$; find the smallest positive integer m such that $mb_i \in Z$. $h_p(m) = 0$ since $h_p(b_i) = 0$. Now $(my)(p^r) = (mx)(p^r)$, yielding

$$mb_i a_i \equiv ma_i$$

and

$$mb_j + mb_i a_j \equiv ma_j \pmod{p^r}.$$

Thus $mb_i \equiv m \pmod{p^s}$. This implies that $mb_j \in Z$ and $mb_j \equiv 0 \pmod{p^s}$ if $j \neq i$. Hence $h_p(b_j) = h_p(mb_j) \geq s > 0$ if $j \neq i$. Thus by Lemma 2.14, $h_p(D_i^k) \geq \min_{j \neq i}\{h_p(b_j)\} \geq s$. Therefore $r = s + h_p(a_i) \leq h_p(a_i D_i^k)$.

COROLLARY 2.16 *If* $A(x) = A_0$, *then there is an integer* $D(x)$ *such that, for all* $p \in \pi$,

$$h_0(p) + h_p(D(x)) \geq h_p^A(x) \geq h_0(p);$$

thus $t(x) = t_0$.

Proof Write $x = \sum_{j \geq i} a_j y_j^0$, $a_j \in Z$. By Lemma 2.6, if $p \in \pi_0$, then $h_p^A(x) = h_0(p)$. If $p \in \pi_k$ for some $k = 1, 2, \cdots, N$, then, since $x \notin A_k$, we may combine Lemmas 2.9 and 2.15 to get

$$h_p^A(x) \geq h_0(p) + r \Rightarrow x(p^r) \in A_k(p^r) \Rightarrow h_p(a_i D_i^k) \geq r$$

whenever $r > 0$. Thus if $D(x) = a_i \prod_{k=1}^N D_i^k$, then $D(x) \neq 0$ and

$$h_0(p) + h_p(D(x)) \geq h_p^A(x) \geq h_0(p)$$

for all $p \in \pi$. $t(x) = t_0$ follows at once.

LEMMA 2.17 *If* $A(x) = A_{k_0}$, *then* $t(x) = t_{k_0}$.

Proof Define $\pi_0^{k_0} = \{p \mid h_k(p) \leq h_{k_0}(p) \text{ for all } k\}$, and if $k > 0$,

$$\pi_k^{k_0} = \{p \mid h_k = \cap \{h_j \mid h_j(p) > h_{k_0}(p)\}\}.$$

Note that $\pi_k^{k_0}$ is empty unless $t_k > t_{k_0}$, and that $\pi_0^{k_0}, \pi_1^{k_0}, \cdots, \pi_N^{k_0}$ partition π.

If $p \in \pi_0^{k_0}$, then $h_p^A(x) \leq h_{k_0}(p)$ following the same proof as in Lemma 2.6, letting now $s = h_{k_0}(p)$.

If $p \in \pi_i^{k_0}$, then, defining I and J as in 2.9, we get

$$k \in J \iff h_k(p) \leq h_{k_0}(p).$$

This is sufficient to obtain the conclusion of Lemma 2.9, that $h_p^A(x) \geq h_{k_0}(p) + r$, where $0 < r \in Z$, only if $x(p^r) \in A_l(p^r)$. By 2.15, $x(p^r) \in A_l(p^r)$ only if $r \leq h_p(a_i D_i^l)$. By 2.12, $D_i^l \neq 0$ since $A(x) = A_{k_0} \supset A_l$, implying that $x \notin A_l$.

Let $S = \{k \mid t_k > t_{k_0}\}$ and $D(x) = a_i \prod_{k \in S} D_i^k$. We have just showed that $h_{k_0}(p) + h_p(D(x)) \geq h_p^A(x)$ for all p; therefore $t(x) = t_{k_0}$.

COROLLARY 2.18 *$T(A) = T$, $P^*(A) = L^*$ and therefore Theorem 2.1 is proved. For each $k = 0, 1, \cdots, N$, the elements $y_1^k, y_2^k, \cdots, y_{n_k}^k$ demonstrate explicitly* Rank(A_k) *independent elements in A of type t_k. For each $x \in A$ and $p \in \pi$, an upper bound of $h_p^A(x)$ may be found by calculating the integer $D(rx)$ as defined above, where rx is B_0-reduced.*

3. Quasi-essential groups

Following the construction of the previous section, we define a class of groups as follows:

DEFINITION 3.1 (1) Let A be a group. We shall call A an essential group if A has for a set of generators

$$\{p^{-s_k(p)} y_i^k \mid p \in \pi; 0 \leq s_k(p) < h_k(p) + 1; \quad k = 0, 1, \cdots, N; i = 1, 2, \cdots, n_k\},$$

where

(a) $h_0, h_1, \cdots, h_N$ are heights satisfying

$$[h_i] = t_i,$$
$$h_i \leq h_j \quad \text{if} \quad t_i \leq t_j,$$
$$h_i \cap h_j = h_k \quad \text{if} \quad t_i \cap t_j = t_k; \quad 0 \leq i, j, k \leq N;$$

(b) $n_k = \text{Rank}(A_k)$, $k = 0, 1, \cdots, N$;

(c) $B_0 = \{y_1^0, y_2^0, \cdots, y_{n_0}^0\}$ is a basis of A^* such that

$$y_i^0 \notin A_k^*, \qquad 1 \leq k \leq N, 1 \leq i \leq n_0;$$

(d) for each $k = 1, 2, \cdots, N$, $\{y_1^k, y_2^k, \cdots, y_{n_k}^k\}$ is a basis of A_k^* such that y_i^k is B_0-reduced and $y_i^k \notin A_j^*$ if $A_j^* \subset A_k^*$.

(2) B is a quasi-essential (q.e.) group if B is quasi-isomorphic to some essential group A.

Remark If A is the essential group constructed above, then it is clear from Corollary 2.18 that

$$T(A) = \{t_\infty, t_0, t_1, \cdots, t_N\}$$

and

$$P^*(A) = L^* = \{0, A_0^*, A_1^*, \cdots, A_N^*\}.$$

NOTATION 3.2 Let y_γ be in R^n and let h_γ be corresponding heights, where $\gamma \in \Gamma$, Γ some indexing set; by $A = \{(y_\gamma, h_\gamma) \mid \gamma \in \Gamma\}$ we shall mean that A is the group generated by

$$\{p^{-s_\gamma(p)} y_\gamma \mid p \in \pi; 0 \leq s_\gamma(p) < h_\gamma(p) + 1; \gamma \in \Gamma].$$

Thus in 3.1, $A = \{(y_i^k, h_k)\}$.

4. Quasi-isomorphism invariants for q.e. groups

DEFINITION 4.1 Let A and B be groups; define

(1) $A \mathrel{\dot\subseteq} B$ if there is some $0 < n \in Z$ such that $nA \subseteq B$;

(2) $A \doteq B$ (A is quasi-equal to B) if $A \mathrel{\dot\subseteq} B$, $B \mathrel{\dot\subseteq} A$;

(3) $A \sim B$ (A is quasi-isomorphic to B) if there are subgroups A' of A and B' of B such that $A' \doteq B'$, $A \mathrel{\dot\subseteq} A'$, $B \mathrel{\dot\subseteq} B'$, [1, p. 62].

LEMMA 4.2 *Let A and B be groups; then the following are equivalent:*

(i) $A \sim B$.

(ii) *There is a subgroup B' of B and a monomorphism ϕ from B' to A such that $A \mathrel{\dot\subseteq} \phi(B')$ and $B \mathrel{\dot\subseteq} B'$.*

(iii) *There is a monomorphism ϕ from B to A such that $A \mathrel{\dot\subseteq} \phi(B) \subseteq A$.*

(iv) *There is a subgroup A' of A such that $B \cong A' \doteq A$.*

(v) *There are non-singular linear transformations L_1 and L_2 of R^n such that $L_1(A) \subseteq B$ and $L_2(B) \subseteq A$.*

The proofs are routine.

COROLLARY 4.3 *Let A and B be quasi-isomorphic subgroups of R^n. Then*

(i) $\text{Rank}(A) = \text{Rank}(B)$;

(ii) $T(A) = T(B)$;

(iii) $A_t \sim B_t$, *for all types t;*

(iv) *there is a non-singular linear transformation L of R^n such that $L(B_t^*) = A_t^*$ for all t;*

(v) *if $A \doteq B$, then $A_t = B_t$, $A_t^* = B_t^*$ for all t.*

Proof That $\text{Rank}(A) = \text{Rank}(B)$ is obvious. For the rest, let $\phi : B \to A$ be a monomorphism such that $NA \subseteq \phi(B) \subseteq A$ for some integer $N > 0$. Then for every $x \in B$,

$$H^B(x) \sim H^B(Nx) = H^{\phi(B)}(N\phi(x)) \leq H^A(N\phi(x)) \sim H^{NA}(N\phi(x)) \leq H^{\phi(B)}(N\phi(x)).$$

Thus $t^B(x) = t^A(\phi(x))$ and so

$$T(B) \subseteq T(A) \quad \text{and} \quad A_t \mathrel{\dot\subseteq} \phi(B_t) \subseteq A_t.$$

The argument reverses to get $T(A) \subseteq T(B)$. ϕ extends naturally to a non-singular linear transformation L of R^n, yielding

$$A_t^* = (NA_t)^* \subseteq L(B_t^*) \subseteq A_t^*.$$

Remark The converse to this corollary is not true in general, as may be seen from the theory of rank 2 groups [2]. However, in the case of q.e. groups, we get

THEOREM 4.4 *Let A and B be q.e. groups. Then $A \approx B$ if and only if* (i) $T(A) = T(B)$; (ii) *there exists a non-singular linear transformation L of R^n such that $t \in T(B) \Rightarrow L(B_t^*) = A_t^*$.*

Proof If $A \approx B$, then (i) and (ii) follow from 4.3.

Conversely, assume that A and B are essential groups. Then

$$A = \{(y_j^k, h_k) \mid k = 0, 1, \cdots, N; j = 1, 2, \cdots, n_k\}.$$

Similarly,

$$B = \{(x_j^k, h_k') \mid k = 0, 1, \cdots, N; j = 1, 2, \cdots, m_k\},$$

where all the conditions of Definition 3.1 are satisfied.

Let L be a non-singular linear transformation of R^n such that $L(B_t^*) = A_t^*$ for every $t \in T(B)$. This implies that $m_k = \text{Dim}(B_k^*) = \text{Dim}(A_k^*) = n_k$ for every $t_k \in T(B) = T(A)$. For each j and k, $L(x_j^k) = \sum_i r_{ij}^k y_i^k$, where the $r_{ij}^k \in R$. Let M be the product of the denominators of all the r_{ij}^k. Find integers J_k such that $J_k h_k(p) \leq h_k'(p)$ for all p; this can be done since $h_k \sim h_k'$. Let $J = J_0 J_1 \cdots J_N$. $(JM)L$ is also non-singular. A simple computation shows that $(JM)L(p^{-s_k(p)} x_j^k) \in A$ for every generator $p^{-s_k(p)} x_j^k$ of B. Hence $(JM)L(B) \subseteq A$. Similarly, there are non-zero integers J' and M' such that $(J'M')L^{-1}(A) \subseteq B$. Hence $A \approx B$ by 4.2.

Finally, if A and B are q.e., then there are essential groups A' and B' such that $A \doteq A'$, $B \doteq B'$. By 4.3, $T(A') = T(A) = T(B) = T(B')$ and $A_t^* = A_t'^*$, $B_t^* = B_t'^*$ for all types t. Hence $L(B_t'^*) = A_t'^*$ for every $t \in T(B)$. By the above argument, $A' \approx B'$; hence $A \approx B$.

COROLLARY 4.5 *If A and B are q.e. groups, then $A \doteq B$ if and only if* (i) $T(A) = T(B)$; (ii) $P^*(A) = P^*(B)$.

DEFINITION 4.6 Let A' be an essential subgroup of A. We shall call A' a maximal essential subgroup if, whenever $A' \dot{\subseteq} B \subseteq A$, where B is an essential subgroup of A, then $A' \doteq B$. Similarly define a maximal q.e. subgroup.

THEOREM 4.7 *Let A be a group with finite type set.*

(1) *A has a maximal essential subgroup A' such that $T(A') = T(A)$ and $P^*(A') = P^*(A)$. A' is unique up to quasi-equality.*

(2) *If $x \in A$, there is a maximal essential subgroup A' of A containing x.*

(3) *A is q.e. if and only if A/A' is a finite group for every maximal essential subgroup A' of A.*

(4) *If A' is a maximal essential subgroup of A, then A/A' is a torsion group.*

Proof (1) and (2). Assume Rank$(A) = n$, $T(A) = \{t_\infty, t_0, t_1, \cdots, t_N\}$; assume also that $x \neq 0$. There is an independent set $\{x, y_2^0, \cdots, y_n^0\}$ where the y_i^0 are of type t_0, the minimal type in $T(A)$. These elements can always be found since $T(A)$ is finite (Corollary 1.9). If $t(x) = t_0$, let $y_1^0 = x$. If $t(x) > t_0$, then consider the pure subgroup P in A generated by $\{x, y_2^0\}$. P has finite type set, since $t^P(y) = t^A(y)$ for all $y \in P$. In particular, $t^P(y_2^0) = t_0$. For some $m \in Z$,

$$t^P(x + my_2^0) = t_0 = t^A(x + my_2^0)$$

[2, p. 27]. Let $y_1^0 = x + my_2^0$; $B_0 = \{y_1^0, y_2^0, \cdots, y_n^0\}$. x is B_0-reduced, since $x = y_1^0 - my_2^0$.

For each $t_k \in T(A)$, $t_k \neq t_0$, we can find $n_k = \text{Rank}(A_k)$ independent B_0-reduced elements of type t_k in A, $y_1^k, y_2^k, \cdots, y_{n_k}^k$. Let $h_k = \bigcap_j H^A(y_j^k)$, $k = 0, 1, \cdots, N$. Find heights $h_0', h_1', \cdots, h_N'$ such that, for $0 \leq i, j, k \leq N$, (i) $h_i' \leq h_i$; (ii) $h_i' \sim h_i$; (iii) if $t_i \leq t_j$, then $h_i' \leq h_j'$; (iv) if $t_i \cap t_j = t_k$, then $h_i' \cap h_j' = h_k'$ (Lemma 2.2).

Let $A' = \{(y_j^k, h_k') \mid k = 0, 1, \cdots, N; j = 1, 2, \cdots, n_k\}$; A' is an essential group. Since $h_k' \leq h_k \leq H^A(y_j^k)$, all the generators of A' are in A and A' is a subgroup of A. $T(A') = T(A)$ and $P^*(A') = P^*(A)$ (Corollary 2.18). If B is any other essential subgroup of A with $A' \dot{\subseteq} B \subseteq A$, then it is clear that $T(B) = T(A')$, $P^*(B) = P^*(A')$. Hence by 4.5, $A' \doteq B$. Thus A' is maximal essential, contains x, and by 4.5 is unique up to quasi-equality.

(3) A is q.e. $\Leftrightarrow A \doteq A'$ for any maximal essential subgroup A' of $A \Leftrightarrow NA \subseteq A' \subseteq A$ for some $0 < N \in Z \Leftrightarrow A/A'$ is a finite group (A being of finite rank).

(4) This is obvious. Thus a maximal essential subgroup A' furnishes a "large" subgroup of A that is also "standard" since A' is unique up to quasi-equality. The problem of finding quasi-isomorphism invariants for torsion-free groups A with finite rank and finite type set could possibly be solved by examining the groups A/A', where A' is a maximal essential subgroup of A.

5. The structure of q.e. groups

THEOREM 5.1 *Let* $A = \{(y_k, h_k) \mid k = 1, 2, \cdots, N\}$, *where the* h_k *are arbitrary heights and* $y_k \in R^n$. *Then* A *is a q.e. group and* $T(A)$ *and* $P(A)$ *may be found in a natural way.*

Proof In (1) we shall describe this "natural way". Then we shall show that this method does yield $T(A)$ and $P(A)$. Finally, we shall show that A is q.e.

(1) Assume that $\text{Rank}(A) = n$. For each h_i, $1 \leq i \leq N$, let A_i^* be the subspace of R^n generated by all the y_k such that $h_k \geq h_i$. Clearly every $x \in A \cap A_i^*$ will have type $t(x) \geq [h_i]$. Let F be the (finite) set of all subsets of the indices $\{1, 2, \cdots, N\}$. For each $f \in F$, $f \neq \phi$, define $A_f^* = \sum_{i \in f} A_i^*$ and $t_f = \bigcap_{i \in f} [h_i]$. Define $A_\phi^* = 0$, $t_\phi = t_\infty$.

If $x \in A \cap A_f^*$, then $x = \sum_{i \in f} a_i x_i$, where $a_i \in R$, $x_i \in A_i^*$. Hence

$$t(x) \geq \bigcap_{i \in f} t(x_i) \geq \bigcap_{i \in f} [h_i] = t_f .$$

If $x \in A$, define $t_x = \bigcup\{t_f \mid x \in A_f^*\}$. By the above remarks, $t(x) \geq t_x$, $t(0) = t_\infty = t_0$. We shall show eventually that $t_x = t(x)$. Let $T = \{t_x \mid x \in A\} \cup \{\text{all finite intersections of members of } \{t_x \mid x \in A\}\}$. T is finite since F is finite, and forms a lattice $\{t_\infty, t_0, t_1, \cdots, t_K\}$.

(2) $\{x \in A \mid t_x \geq t_k\}$ is a pure subgroup B_k of A for each $k = 0, 1, \cdots, K$.

Proof The only difficult part is to show closure, since $t_0 = t_\infty \geq t_k$, $t_{-x} = t_x$, $t_{rx} = t_x$ if $rx \in A$.

First note that, if $f, g \in F$, then

$$t_f \cap t_g = \bigcap_{i\epsilon f}[h_i] \cap \bigcap_{j\epsilon g}[h_j] = \bigcap_{i\epsilon f \cup g}[h_i] = t_{f\cup g}\,.$$

Since the lattice of all types is distributive,

$$(\bigcup_\alpha t_\alpha) \cap (\bigcup_\beta t_\beta) = \bigcup_{\alpha,\beta}(t_\alpha \cap t_\beta)$$

if α, β are finite sets. If $x \in A_f^*$, $y \in A_g^*$, then

$$x + y \in A_f^* + A_g^* = A_{f\cup g}^*\,;$$

thus

$$\{f \cup g \mid x \in A_f^*, y \in A_g^*\} \subseteq \{h \mid x + y \in A_h^*\}.$$

Now let $x, y \in B_k$; that is, $x, y \in A$ and $t_x, t_y \geq t_k$. Combining the above properties, we get

$$\begin{aligned} t_k \leq t_x \cap t_y &= \bigcup\{t_f \mid x \in A_f^*\} \cap \bigcup\{t_g \mid y \in A_g^*\} \\ &= \bigcup\{t_f \cap t_g \mid x \in A_f^*, y \in A_g^*\} \\ &= \bigcup\{t_{f\cup g} \mid x \in A_f^*, y \in A_g^*\} \leq \bigcup\{t_h \mid x + y \in A_h^*\} = t_{x+y}\,. \end{aligned}$$

(3) $P = \{0, B_0, B_1, \cdots, B_K\}$ forms a lattice dually isomorphic to the lattice T. In P, the meet of B_i, B_j is $B_i \cap B_j$ and the join of B_i, B_j is the member of P that corresponds to $t_i \cap t_j$ in the dual isomorphism.

Proof Let $t_r, t_s \in T$. If $t_r \geq t_s$, then

$$B_r = \{x \in A \mid t_x \geq t_r \geq t_s\} \subseteq \{x \in A \mid t_x \geq t_s\} = B_s\,.$$

If $B_r \subseteq B_s$, then

$$t_r = \bigcap\{t_x \mid x \in B_r\} \geq \bigcap\{t_x \mid x \in B_s\} = t_s\,,$$

where the equalities hold because T is a finite lattice closed under $\bigcap$ and because of the definition of B_k. Therefore P forms a lattice dually isomorphic to T and lattice join in P is as asserted. That lattice meet is group intersection is an easy computation (or see Theorem 1.7).

(4) Clearly $P^* = \{0, B_0^*, B_1^*, \cdots, B_K^*\}$ forms a lattice isomorphic to P. Following 2.1 and 3.1, let B be an essential group with $T(B) = T$ and $P^*(B) = P^*$.

If $x \in A \cap B$, then $t^B(x) = t_x$. For $t_x = t_k$ for some k, $1 \leq k \leq K$, by definition of T. Hence $x \in B_k$ and $t^B(x) \geq t_k = t_x$. If $t^B(x) = t_j > t_k$, then $x \in B_j \subset B_k$, implying $t_x \geq t_j > t_k$, a contradiction. Also if $x \in A \cap B$, then $t^A(x) \geq t_x = t^B(x)$. Hence $t^{A \cap B}(x) = t^B(x)$. Since $A^* = B^*$, some integral multiple of every element in A or B is in $A \cap B$. Hence $T(A \cap B) = T(B) = T$ and $P^*(A \cap B) = P^*$. By Theorem 4.7, there is an essential subgroup A' of $A \cap B$ such that $T(A') = T(A \cap B)$ and $P^*(A') = P^*(A \cap B)$. By Corollary 4.5, $A' \doteq B$.

(5) $A' \doteq A$.

Proof Let M_k be integers such that $M_k y_k \in A'$, where the y_k are as in the statement of the theorem. Then

$$t^{A'}(M_k y_k) = t^B(M_k y_k) = t_{M_k y_k} = t_{y_k} \geq [h_k].$$

Thus there are integers N_k such that

$$h_p^{A'}(M_k y_k) + h_p(N_k) \geq h_k(p)$$

for all p. Thus $M_k N_k p^{-s} y_k \in A'$ for all p and k, where $s < h_k(p) + 1$. If $M = \prod M_k N_k$, then $Mu \in A'$ for every generator u of A. Therefore $MA \subseteq A' \subseteq A \cap B \subseteq A$ and $A' \doteq A$.

Thus A is a q.e. group, and $t^A(x) = t^{A'}(Mx) = t_{Mx} = t_x$ for every $x \in A$. This completes the proof of the theorem.

COROLLARY 5.2 *Let T and L^* be as in* 2.1. *For each* $k = 0, 1, \cdots, N$, *let* $n_k = \mathrm{Dim}(A_k^*)$ *and let* $y_1^k, y_2^k, \cdots, y_{n_k}^k$ *be arbitrary independent members of* A_k^*, $h_1^k, h_2^k, \cdots, h_{n_k}^k$ *be arbitrary heights in the equivalence class* t_k. *Then*

$$A = \{(y_i^k, h_i^k) \mid k = 0, 1, \cdots, N; i = 1, 2, \cdots, n_k\}$$

is a q.e. group, with $T(A) = T$, $P^*(A) = L^*$.

COROLLARY 5.3 *Let A be a group with* $T(A) = \{t_\infty, t_0, t_1, \cdots, t_N\}$. *For each k, let* $n_k = \mathrm{Rank}(A_k)$ *and let* $y_1^k, y_2^k, \cdots, y_{n_k}^k$ *be independent elements in* A_k. *Then*

$$B = \{(y_i^k, H^A(y_i^k)) \mid k = 0, 1, \cdots, N; i = 1, 2, \cdots, n_k\}$$

is a maximal q.e. subgroup of A such that $T(B) = T(A)$ *and* $P^*(B) = P^*(A)$. *B is unique up to quasi-equality.*

THEOREM 5.4 *If A is a q.e. group, then there are elements* $y_1, y_2, \cdots, y_N$ *of* R^n *and heights* $h_1, h_2, \cdots, h_N$ *such that*

$$A = \{(y_k, h_k) \mid k = 1, 2, \cdots, N\}.$$

Proof Let A' be a maximal essential subgroup of A,

$$A' = \{(y_k, h_k) \mid k = 1, 2, \cdots, M\}.$$

By Theorem 4.7, A/A' is a finite group, generated by $y_{M+1} + A', \cdots, y_N + A'$.

Then

$$A = \{(y_k, h_k) \mid k = 1, 2, \cdots, N\},$$

where $h_k(p) = 0$ for all p if $M + 1 \leq k \leq N$.

COROLLARY 5.5 *If A and B are q.e. groups, then so is $A + B$.*

LEMMA 5.6 *If A and B are q.e., then so is $A \cap B$.*

Proof (1) Let

$$A = \{(y_j, h_j) \mid j = 1, 2, \cdots, N\},$$

$$B = \{(u_j, k_j) \mid j = 1, 2, \cdots, M\}.$$

We proceed by induction on $N + M$. The lemma is certainly true if $N + M \leq 3$, since then $A \cap B$ is 0 or of Rank 1. If we let

$$A_i = \{(y_j, h_j) \mid j = 1, 2, \cdots, N; j \neq i\}$$

and define B_i similarly, then for all i, $A_i \cap B$, $B_i \cap A$ are q.e. by the induction hypothesis.

Let $D = A \cap B$. Since $T(A)$ and $T(B)$ are both finite, so is $T(D)$ because each $x \in D$ has type $t^A(x) \cap t^B(x)$. For each $t \in T(D)$, there is a maximal independent set $B_t = \{z_i^t\}$ in D such that

$$z_i^t = \sum_{j=1}^{N} r_{ij}^t y_j = \sum_{j=1}^{M} s_{ij}^t u_j$$

for each i, where $0 \neq r_{ij}^t, s_{ij}^t \in Z$ and where all the z_i^t have the same height in D. Let $A_0 = \{(z_i^t, H^D(z_i^t))\}$. We shall show that

$$C = A_0 + \sum_{i=1}^{N} A_i \cap B + \sum_{i=1}^{M} B_i \cap A \doteq D.$$

Since C is q.e. by Corollary 5.5, this will prove the lemma.

(2) Since $C \subseteq D$, $H^C(y) \leq H^D(y)$ for all $y \in C$. As a corollary of the induction hypothesis, there is $0 < K \in Z$ such that $H^C(Ky) \geq H^D(y)$ if $y \in A_i \cap B$, $B_i \cap A$. Thus we need only show that $H^C(Kx) \geq H^D(x)$ if $x = \sum_{j=1}^{N} a_j y_j = \sum_{j=1}^{M} b_j u_j \in C$, where $a_j, b_j \neq 0$.

Let us now fix p and assume that $\min_j\{h_p^A(z_j^t)\} \leq h_p^B(z_i^t)$ for all i. If $\min_j\{h_p^B(z_j^t)\} \leq h_p^A(z_i^t)$ for all i, a similar process to that described below, with the roles of A and B interchanged, will give us the same results. If $t^D(x) = t$, we may assume that x is B_t-reduced. We may further assume that $p^{-k} a_j y_j \in A$ for every j and every $k < h_p^A(x) + 1$; for if this condition does not hold, then x is in some A_i by another representation $x = \sum_{j \neq i} a_j' y_j$ and $h_p^{A_i}(x) = h_p^A(x)$, implying that $h_p^C(Kx) \geq h_p^D(x)$.

Let

$$x = \sum_i c_i z_i^t = \sum_{j=1}^{N} \sum_i c_i r_{ij}^t y_j = \sum_{j=1}^{N} a_j y_j,$$

where each $c_i \in Z$, $\min_i\{h_p(c_i)\} = 0$ for all p, $a_j \neq 0$ for all j. By our assumptions on x,

$$h_p^D(x) \leq h_p^A(x) = \min_j\{h_p(a_j) + h_p^A(y_j)\}$$
$$= \min_j\{\min_i\{h_p(r_{ij}^t)\} + h_p^A(y_j)\} \leq \min_i\{h_p^A(z_i^t)\} \leq h_p^D(z_i^t)$$

for all i, and therefore $h_p^D(x) \leq h_p^C(x)$, unless $h_p(a_j) > \min_i\{h_p(r_{ij}^t)\}$ for some j.

(3) Suppose $r = \max_j\{h_p(a_j) - m_j\} > 0$ for some j, where

$$m_j = \min_i\{h_p(r_{ij}^t)\}.$$

For simplicity's sake, suppose $j = 1$ and $h_p(r_{11}^t) = m_1 = h$. Then find $m \in Z$ such that

$$-m(r_{11}^t/p^h) \equiv 1 \pmod{p^r}.$$

Since

$$\textstyle\sum_i c_i r_{i1}^t = a_1 \equiv 0 \pmod{p^{r+h}},$$

then

$$\textstyle\sum_{i>1} c_i r_{i1}^t/p^h \equiv -c_1 r_{11}^t/p^h \pmod{p^r}.$$

Thus

$$m \textstyle\sum_{i>1} c_i r_{i1}^t/p^h \equiv -mc_1 r_{11}^t/p^h \equiv c_1 \pmod{p^r}.$$

Hence we may rewrite x as $x = x_1 + x_2$, where

$$x_2 = m\ ((\textstyle\sum_{i>1} c_i r_{i1}^t/p^h)z_1^t - \sum_{i>1} (c_i r_{i1}^t/p^h)z_i^t)$$

and

$$x_1 = p^r(\textstyle\sum d_i z_i^t), \qquad d_i \in Z.$$

Since $h_p(a_j) \leq r + h$ for each j, then

$$h_p^C(x_1) \geq \min_i\{r + h_p^D(d_i z_i^t)\}$$
$$\geq \min_j\{h_p(a_j) + h_p^A(y_j)\} = h_p^A(x) \geq h_p^D(x).$$

$x_2 = \sum_{j>1} a_j' y_j$ since the coefficient of y_1 is 0 in the expression for x_2. Hence $x_2 \in A_1$ and $h_p^C(Kx_2) \geq h_p^D(x_2)$. Now

$$h_p^A(x_2) \geq \min\{h_p^A(x), h_p^A(x_1)\} \geq \min[h_p^D(x), h_p^C(x_1)] \geq h_p^D(x).$$

Similarly, $h_p^B(x_2) \geq h_p^D(x)$. Thus $h_p^D(x_2) \geq h_p^D(x)$. Therefore

$$h_p^C(Kx) \geq \min\{h_p^C(Kx_1), h_p^C(Kx_2)\} \geq h_p^D(x).$$

Continuing this process for all p, we get $H^C(Kx) \geq H^D(x)$. Hence $K(A \cap B) = KD \subseteq C \subseteq A \cap B$; $A \cap B \doteq C$ is q.e.

COROLLARY 5.7 *Every pure subgroup of a q.e. group is q.e.*

Proof Let P be a pure subgroup of the q.e. group A. P^*, being a rational vector space, is q.e. $P = A \cap P^*$ is therefore q.e.

COROLLARY 5.8 *If A and B are direct sums of a finite number of rank 1 groups, then $A \cap B$ is q.e.*

Remark (1) Thus, although even pure subgroups of A or B are not completely decomposable in general [4, p. 166], they are at least q.e. groups. To see what $A \cap B$ looks like, we give the following construction:

Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$, where each $A_j = \{(u_j, h_j)\}$ and each $B_j = \{(v_j, k_j)\}$. Let F and G be, respectively, the set of all subsets of the indices $\{1, 2, \cdots, n\}$ and $\{1, 2, \cdots, m\}$. For each $f \in F$ and $g \in G$, there is a maximal independent set $B_{fg} = \{z_i^{fg}\}$ in $A \cap B$, where for each i,

$$z_i^{fg} = \sum_{j \in f} r_{ij}^{fg} u_j = \sum_{j \in g} s_{ij}^{fg} v_j, \qquad 0 \neq r_{ij}^{fg}, s_{ij}^{fg} \in Z.$$

Let $C = \{(z_i^{fg}, H^{A \cap B}(z_i^{fg})) \mid f \in F, g \in G, \text{all } i\}$. By a proof much the same as that of Lemma 5.6, it can be shown that $C \doteq A \cap B$.

(2) If A, B, C, D are groups, $A \doteq B$, $C \doteq D$, then $A \cap C \doteq B \cap D$, $A + C \doteq B + D$. Thus if $\mathcal{E}$ is the set of equivalence classes of quasi-equal subgroups of R^n, then $\mathcal{E}$ forms a lattice with meet $\wedge$ and join $\vee$ defined as follows: let $E, F \in \mathcal{E}$, define $E \wedge F = [A \cap B]$ and $E \vee F = [A + B]$, where $A \in E$, $B \in F$.

COROLLARY 5.9 *The set of equivalence classes of quasi-equal q.e. subgroups of R^n form a sublattice of $\mathcal{E}$, the set of all equivalence classes of quasi-equal subgroups of R^n.*

6. Quotient divisible groups

DEFINITION 6.1 Let A be a torsion-free group. Then A is called quotient divisible (q.d.) if A contains a free subgroup F such that A/F is a torsion group $D \oplus B$, where D is divisible and B is of bounded order. (If A is of finite rank, then B is necessarily a finite group.)

Q.d. groups are of importance in the study of rings over torsion-free groups [1]. We shall prove a few facts concerning the types of the elements in such groups.

LEMMA 6.2 (i) *If A is q.d. and $A \sim A'$, then A' is q.d.* (ii) *If A is q.d., then there is a free subroup F of A such that A/F is divisible.* (iii) *Any torsion-free homomorphic image of a q.d. group of finite rank is also q.d.*

The proofs are given in [1].

DEFINITION 6.3 A height H is said to be non-nil if $H(p) = 0$ or ∞ for all but a finite number of primes p.

A type t is said to be non-nil if $t = [H]$, where H is a non-nil height. If t is non-nil, then there is a unique $H \in t$ such that $H(p) = 0$ or ∞ for all p.

THEOREM 6.4 *Let A be a q.d. group of finite rank and let*

$$C(A) = T(A) \cup \{\textit{all finite intersections of members of } T(A)\}$$

(see 1.2). *Then t_0, the minimal type in $C(A)$, is non-nil.*

Proof Let A be of rank n, F a free rank n subgroup such that $A/F = D$, where D is divisible. Let $x_1, x_2, \cdots, x_n$ be independent generators of F. Then for each prime p, either $h_p^A(x_i) = \infty$ for all i, or $h_p^A(x_i) = 0$ for some x_i.

For let p be a prime such that $\infty > h_p^A(x_j) = h > 0$ for some generator x_j of F. Since $p^{-h}x_j \notin F$, it follows that $p^{-h}x_j + F \neq 0$ in $A/F = D$. Hence there is a $y \in A$ such that $y + F = p^{-h}x_j + F$ and $p^{-1}y \in A$. Write

$$p^{-h}x_j = y + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n ,$$

where each $a_i \in Z$. Since $p^{-1}y \in A$ and $p^{-1}p^{-h}x_j \notin A$, we must have

$$p^{-1}(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \notin A.$$

Hence $p^{-1}x_i \notin A$ for some i, that is, $h_p^A(x_i) = 0$ as we asserted.

Thus $\min_i\{h_p^A(x_i)\} = 0$ or ∞. Hence

$$t_0 = \bigcap_i t(x_i) = \bigcap_i[H(x_i)] = [\min_i\{h_p^A(x_i)\}]$$

is non-nil.

Lemma 6.5 *Let*

$$A = \{(y_i^k, h_k) \mid k = 0, 1, \cdots, N; i = 1, 2, \cdots, n_k\},$$

be an essential group, where $h_k(p) = 0$ *or* ∞ *for all* p *and all* k. *Let* F *be the free group generated by* $\{y_1^0, y_2^0, \cdots, y_{n_0}^0\}$. *Then* A/F *is divisible.*

Proof By the definition of an essential group, the y_i^k and h_k satisfy the conditions of Definition 3.1. The added condition above on the h_k in no way conflicts with these conditions. To show that A/F is divisible, it is sufficient to show that, if $x \in A - F$ and $px \in F$, then $x + F$, as an element of A/F, is divisible. If $h_0(p) = \infty$, then $h_p^A(x) = \infty$ and so $x + F$ is divisible. If $h_0(p) = 0$, then $h_p^A(px) \geq 1$ implies that $px = y + pz$ where $z \in F$ and $y \in A_k \cap F$ for some k such that $h_k(p) \geq 1$, (Lemmas 2.6, 2.9). But then $h_k(p) = \infty$; therefore $h_p^A(y) = \infty = h_p^A(p^{-1}y)$. Hence $x = p^{-1}y + z$ and $x + F = p^{-1}y + F$ is divisible.

Lemma 6.6 *Let*

$$A = \{(y_i^k, h_k) \mid k = 0, 1, \cdots, N; i = 1, 2, \cdots, n_k\}$$

be an essential group, where some h_k *is not non-nil. Then* A *is not a q.d. group.*

Proof Let h_k be a minimal not non-nil height among all the h_j. If $h_k = h_0$, then A is not q.d. by Theorem 6.4. If $h_k > h_0$, let

$$\pi' = \{p \mid 0 = h_0(p) < h_k(p) = h_p^A(y_1^k) = \cdots = h_p^A(y_{n_k}^k) < \infty\}.$$

π' is infinite since h_0 is non-nil, h_k is not non-nil, and

$$H(y_1^k) \sim \cdots \sim H(y_{n_k}^k) \sim h_k .$$

Since h_k is a minimal non-nil height, then $h_j \cap h_k$ is non-nil unless $h_j \geq h_k$.

Hence for all but a finite number of primes in π', $h_j(p) = 0$. Thus for an infinite set of primes $\pi'' \subseteq \pi'$, $h_j(p) > 0$ only if $h_j \geq h_k$; that is, $y_i^j \in A_k$ for all i.

Let A' be the projection of A upon A_k^*. A' is then a torsion-free homomorphic image of A and hence $H^{A'}(y_i^k) \geq H^A(y_i^k)$ [4, p. 146]. Extend $y_1^k, \cdots, y_{n_k}^k$ to a basis B of A^* by proper choice of members $y_{j'}^0$ of B_0. Let

$$x = ay_i^k + \sum a_j y_{j'}^0$$

be a B_0-reduced member of A, where $\sum a_j y_{j'}^0 \notin A_k$. If $p \in \pi''$ and $h_p^A(x) = r > 0 = h_0(p)$, then $x(p^r) \in A_k(p^r)$ by Lemma 2.9. $ay_i^k(p^r) \in A_k(p^r)$ and therefore $\sum a_j y_{j'}^0 \in A_k(p^r)$ by Lemma 2.8.

Thus there is a $y = \sum a_j y_{j'}^0 + \sum p^r c_j y_j^0 \in A_k$, where $c_j \in Z$ and $h_p^A(y) \geq r$. (This statement is almost equivalent to the definition of $A_k(p^r)$.) Hence $r \leq h_p^A(\sum a_j y_{j'}^0)$ and

$$h_p^A(x) = r \leq h_p^A(x - \sum a_j y_{j'}^0) = h_p^A(ay_i^k).$$

If $a^{-1}x \in A$, then $h_p^A(a^{-1}x) \leq h_p^A(y_i^k)$.

$$h_p^{A'}(y_i^k) = \sup \{h_p^A(x) \mid x = y_i^k + \sum b_j y_{j'}^0 \in A,\ b_j \in R\} = S$$

When x is in the above form, $h_p^A(x) \neq \infty$, since $t^A(x) \leq t_k$, for all $p \in \pi''$. Hence we have just showed that $S \leq h_p^A(y_i^k)$ if $p \in \pi''$. For such p, an infinite set, $0 < h_p^{A'}(y_i^k) = h_p^A(y_i^k) < \infty$. Since the minimal type in A' is given by $[H^{A'}(y_1^k) \cap \cdots \cap H^{A'}(y_{n_k}^k)]$, it cannot be non-nil. Therefore by 6.4, A' is not q.d., and by 6.2, neither is A.

THEOREM 6.7 *Let A be a q.e. group. Then A is q.d. if and only if every type in $T(A)$ is non-nil.*

Proof Necessity was proved in Lemma 6.6. For sufficiency, we may assume that A is essential, since quotient divisibility is a quasi-isomorphism invariant (Lemma 6.2). Thus $A = \{y_i^k, h_k)\}$, where every h_k is non-nil. For each k, let h_k' be the unique height such that $h_k' \sim h_k$ and $h_k'(p) = 0$ or ∞ for all p. It is easy to check that the h_k' satisfy all the conditions of 3.1. Hence $A' = \{(y_i^k, h_k')\}$ is essential, and $A' \doteq A$ by Corollary 4.5. A' is q.d. by Lemma 6.5, and so A is q.d.

COROLLARY 6.8 (1) *If A is a q.d. group and $T(A)$ possesses some type that is not non-nil, then A requires among its generators an infinite number of pairwise independent elements of A.*

(2) *If A is a q.d. group that has a set of generators containing only a finite number of pairwise independent elements of A, then $T(A)$ is finite and every type in $T(A)$ is non-nil.*

Proof Apply Theorem 6.7 and Theorem 5.4.

Remark There are many q.d. groups whose type sets possess some type that is not non-nil [5].

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Seattle University
Seattle, Washington