THE SPACE OF HOMEOMORPHISMS ON A TORUS[1]

BY

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There have been several recent results concerning homotopy properties of the space of homeomorphisms on a manifold. Most of these properties have been local. In [4], Eldon Dyer and I proved that the space of homeomorphisms on a 2-manifold is locally contractible and in [5] and [6] it is proved that the space of homeomorphisms on a 3-manifold is locally homotopy connected in all dimensions. Global properties appear to be more difficult. A well known result of Alexander's [1] states that the space of homeomorphisms on an n-cell leaving its boundary pointwise fixed is contractible and locally contractible. In a recent paper [7] it is proved that the *identity component* of the space of homeomorphisms on a disc with holes leaving its boundary pointwise fixed is homotopically trivial. In the present paper, the identity component of the space of homeomorphisms on a torus is considered and it is proved that its homotopy groups are the same as those for the torus. For related results, see [2], [11], [12], and [13].

THEOREM 1. *If k is an integer greater than* 1, *then the identity component of the space H of homeomorphisms of a torus T onto itself has the property that* $\pi_k(H) = 0$.

Proof. Let C denote a meridian simple closed curve on T and P a point of C. A covering space of T is $C \times E^1$, where E^1 is the real line and the covering map π is such that $\pi(x, 0) = x$ for each x in C and, in general, $\pi(x, t) = \pi(y, t')$ if and only if $x = y$ and $t - t'$ is an integer. If n is a non-negative integer, S^n denotes an n-sphere and will be considered as the boundary of the $(n + 1)$-cell, R^{n+1}.

Let F denote a mapping of S^k into H and g the mapping of S^k into T defined by $g(x) = F(x)(P)$. There exists a mapping G of S^k into $C \times E^1$ such that $\pi G(x) = g(x)$ and for each x in S^k, there is a unique mapping $f(x)$ of C into $C \times E^1$ such that $f(x)(P) = G(x)$ and for y in C, $\pi f(x)(y) = F(x)(y)$. The existence of G is a consequence of the various lifting properties of fiber spaces. (See [10, p. 63, Th 3.1.].) To see that $F(x) \mid C$ can be lifted, note that $F(x) \mid C$ is homotopic to the identity in T, since F is in the identity component of H. In particular, there is a mapping φ of $C \times I$ into T such that $\varphi \mid C \times 0$ is a homeomorphism onto a meridian of T, $\varphi \mid C \times 1 = F(x)$ and $\varphi(P, t) = g(x)$. (See Lemma A.) Since $C \times 0$ is a strong deformation retract of $C \times I$ and there is clearly a mapping $\bar{\varphi}$ of $C \times 0$ into $C \times E^1$ such

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that $\pi\tilde{\varphi} = \varphi \mid C \times 0$ and $\tilde{\varphi}(P, 0) = G(x)$, another form of the lifting property mentioned in [10] implies the existence of an extension of $\tilde{\varphi}$ to a map Φ of $C \times I$ into $C \times E^1$ such that $\pi\Phi(x) = \varphi(x)$. Since $\varphi(P, t) = g(x)$ for each t, $\Phi(P, t) = G(x)$. Then $f(x)$ is the mapping $\Phi \mid C \times 1$ and it is obviously a homeomorphism.

The mapping f can be obtained in another instructive way. Coordinatize C by the reals mod 1, letting P have coordinate 0 and let $k(x)$ be the mapping of $I(= [0, 1])$ into T such that $k(x)(y) = F(x)(y)$. Then the mapping $k^*(x)$ of I into $C \times E^1$ such that $k^*(x)(y) = f(x)(y)$ is the unique "lifting" of $k(x)$ that takes 0 onto $G(x)$. Note that $k^*(x)(0) = k^*(x)(1)$. Now consider $S^k \times I$. Let ψ be the mapping of this into T such that $\psi(x, y) = F(x)(y)$. For each x, $\psi(x, 0) = \psi(x, 1)$. But $S^k \times 0$ is a strong deformation retract of $S^k \times I$. Thus there is a mapping ψ^* of $S^k \times I$ into $C \times E^1$ such that $\pi\psi^* = \psi$ and $\psi^*(x, 0) = G(x)$. Since $k^*(x)$ above is unique, $\psi^*(x, y) = k^*(x)(y) = f(x)(y)$ and $\psi^*(x, 1) = \psi^*(x, 0)$. This demonstrates the continuity of the mapping f of S^k into $C \times E^1$.

Since $k > 1$, the mapping g is homotopic to 0 in T. It thus follows from the theorems of [8] that F is homotopic in H to a mapping F' such that $F'(x)(P) = P$ for each x in S^k. In what follows it will be assumed that $F(x)(P)$ does not vary with x.

Let $N^+(F)$ denote the largest integer n such that there exist an x in S^k and a y in C such that the E^1 coordinate of $f(x)(y)$ is in the half-open number interval $[n, n + 1)$ and let $N^-(F)$ denote the least integer m for which there exist such x and y such that the E^1 coordinate of $f(x)(y)$ is in $(m - 1, m]$. Denote by A_j the annulus $C \times [j, j + 1]$. Suppose that there exist an x and an x' such that $f(x)(C)$ meets A_n and $f(x')(C)$ meets A_{m-1} but that for no x does $f(x)(C)$ meet A_{m-2} or A_{n+1}. An upper semicontinuous decomposition of A_n will be constructed that will be used to deform F in H to a mapping F' for which $N^+(F') - N^-(F') < N^+(F) - N^-(F)$ unless this last number is already -1, the least it can be.

For each x in S^k, denote by C_x, C'_x, J^+ and J^- the sets $A_n \cap f(x)(C)$, $A_{m-1} \cap f(x)(C)$ and the right and left boundary curves of A_n. Note that C_x does not intersect J^+. Translate C'_x to the right through $n + 1 - m$ units, i.e., take the point (a, b) of C'_x onto $(a, b + n + 1 - m)$, to obtain C^*_x. Then C^*_x does not intersect $C_x \cup J^-$. Let G_x denote the collection whose elements are (1) the union of J^-, C_x and the components of $A_n - C_x$ whose closures do not intersect J^+, (2) the union of J^+, C^*_x and the components of $A_n - C^*_x$ whose closures do not intersect J^- and (3) the remaining points of A_n. It is seen that G_x is an upper semicontinuous decomposition of A_n whose decomposition space is homeomorphic to S^2.

In $S^k \times A_n$, let G be the decomposition consisting of those sets (x, g), where g is an element of G_x. Since the convergence of the sequence $\{x_i\}$ of points of S^k to a point x implies the convergence of $\{f(x_i)(C)\}$ to $f(x)(C)$, the collection G is upper semicontinuous. From [9] it follows that the decomposition space X associated with G is homeomorphic to $S^k \times S^2$. If T represents the associated mapping of $S^k \times A_n$ onto X, or the homeomorphism of X onto $S^k \times S^2$ and α the projection map of $S^k \times S^2$ onto S^k, then if $(x, y) \in S^k \times A_n$, $\alpha rT(x, y) = x$. Note that there exist points p, q of S^2 such that for each x in S^k, $(rT)^{-1}(x, p)$ and $(rT)^{-1}(x, q)$ are nondegenerate and that if $a \neq p, q$, then $(rT)^{-1}(x, a)$ is degenerate.

Let K be a simple closed curve in S^2 separating p from q. Then for each x, $(rT)^{-1}(x, K)$ is a simple closed curve in (x, A_n) separating $(x, C_x \cup J^-)$ from $(x, C_x^* \cup J^+)$ in (x, A_n) and there is a homeomorphism β of $\bigcup (x, (rT)^{-1}(x, K))$ onto $S^k \times K$ such that the diagram,

$$\begin{array}{ccc} \bigcup(x, (rT)^{-1}(x, K)) & \xrightarrow{\beta} & S^k \times K \\ & \underset{\alpha'}{\searrow} & \downarrow \alpha \\ & & S^k, \end{array}$$

where α' is the projection map of $S^k \times A_n$ onto S^k, is commutative.

If K is coordinatized, as is C, by the reals mod 1, the mapping $z(x)$, $x \in S^k$, that takes each point y of C onto the second coordinate of $\beta^{-1}(x, y)$ is a homeomorphism and z maps S^k continuously into G_C, the space of homeomorphisms of C into int A_n. Each $z(x)(C)$ separates $C_x \cup J^-$ from $C_x^* \cup J^+$. The homeomorphism β may be chosen so that πz maps S^k into H_C, the space of orientation-preserving homeomorphisms of C into curves of T isotopic to meridian curves. Let Z denote the mapping of $C \times S^k$ into $T \times S^k$ such that $Z(y, x) = (\pi z(x)(y), x)$ and let A_x denote the annulus in (T, x) bounded by (C, x) and $Z(C, x)$ (specifically, that annulus which, in T, would be the image under π of the annulus in A_n bounded by J^{-1} and $\beta^{-1}(x, K)$). By Theorem 2.9 of [9], there is a homeomorphism η of $C \times [0, 1] \times S^k$ into $T \times S^k$ such that

$$\eta(C \times [0, 1] \times x) \subset T \times x, \qquad \eta(y, 0, x) = (y, x), \qquad \eta(y, 1, x) \in Z(C, x);$$

by [8, Th. 1.2], there is a homeomorphism γ of $T \times [0, 1] \times S^k$ onto itself such that if $y \in C$, $\gamma(y, t, x) = [\eta(y, t, x), t, x]$ and, for each y, $\gamma(y, 0, x) = (y, 0, x)$. Hence, by a projection of $T \times [0, 1] \times S^k$ onto T, there is obtained a mapping γ^* of $I \times S^k$ into H such that $\gamma^*(1, x)(C) = \pi z(x)(C)$ and $\gamma^*(0, x) = i$.

For each x in S^k, denote by $Q(t, x)$ the mapping $\gamma^*(t, x)[\gamma^*(1, x)]^{-1}$. Then Q is a mapping of $I \times S^k$ into H, $Q(1, x) = i$ and $Q(0, x) = [\gamma^*(1, x)]^{-1}$. Then if $F^*(t, x) = Q(t, x)F(x)$,

$$F^*(1, x) = F(x) \quad \text{and} \quad F^*(0, x) = [\gamma^*(1, x)]^{-1}F(x).$$

Note that since $\gamma^*(1, x)(C) = \pi z(x)(C)$,

$$N^+[F^*(0, x)] - N^-[F^*(0, x)] < N^+(F) - N^-(F)$$

unless the latter number is -1. Precautions could have been made, by using the theorems of [8], to keep $F^*(0, x)(P)$ independent of x or these theorems could be used now to achieve this result without changing $N^+[F^*(0, x)] - N^-[F^*(0, x)]$.

This process can be repeated until F is homotopic in H to a mapping F_1 such that for each x in S^k, $F_1(x)(C)$ does not intersect C. The same reasoning yields a homotopy in H of F_1 to a mapping F_2 such that $F_2(x)$ leaves C pointwise fixed. Since H is the *identity* component, the *angle change*, as defined in [4], along $F_2(x)(C')$, where C' is a longitudinal simple closed curve, is 0. Therefore, the techniques of [4] (see page 526) demonstrate that F_2 is homotopic to F_3 in H, where for each x, $F_3(x)$ is the identity homeomorphism on T. This proves that $\pi_k(H) = 0$ if $k > 1$.

LEMMA A. *Suppose that f is a member of H that leaves P fixed. Then f is isotopic to the identity in such a way that each homeomorphism in the isotopy leaves P fixed.*

Proof. Let f_t, $0 \leq t \leq 1$ be an isotopy such that $f_1 = f$ and $f_0 = i$. Denote by g the mapping of $I \times I$ into T taking (t, s) onto $f_{t+s(1-t)}(P)$. There is a mapping G of $C \times I$ into T such that

$$G(x, 0) = x, \qquad G(x, 1) = x, \qquad G(P, t) = f_t(P),$$

and $G \mid C \times t$ is a homeomorphism. For each t, $G \mid C \times t$ can be constructed by rigidly moving P to $f_t(P)$ and taking C along with it. It is then easy to extend $G \mid C \times t$ to $T \times t$ so that there is a mapping G^* of I into H such that $G^*(t) \mid C = G \mid C \times t$ and $G^*(0) = G^*(1) = i$.

In $T \times I \times I$, let Z be a homeomorphism of

$$(T \times I \times 0) \cup (T \times I \times 1) \cup (T \times 0 \times I)$$

onto itself such that $Z(x, t, 1) = (f_1(x), t, 1)$, $Z(x, t, 0) = (G^*(t)(x), t, 0)$ and $Z(x, 0, s) = (f_s(x), 0, s)$. Also, there is a homeomorphism z of $P \times I \times I$ into $T \times I \times I$ such that $z(P, t, s) = (g(t, s), t, s)$. Note that

$$z(P, 1, s) = (g(1, s), 1, s) = (f_1(P), 1, s) = (P, 1, s)$$

and that where Z is defined, Z extends z. It thus follows from Theorem 1.3 of [8] that there is a homeomorphism Z^* of $T \times I \times I$ onto itself that extends z and Z and carries each (T, t, s) onto itself. If $Z^*(x, 1, s) = (y, 1, s)$, let $f_s^*(x) = y$. It is seen that $f_s^*(P) = P$, $f_1^*(x) = f_1(x) = f(x)$ and $f_0^*(x) = G^*(1)(x) = x$. Then f_s^* is the required homotopy.

LEMMA B. *If f is an orientation preserving map of $C \times I$ onto itself such that $f \mid C \times (0 \cup 1) = i$ and for each $t \neq 0, 1$, $f \mid C \times t$ is a homeomorphism into* int $(C \times I)$ *that leaves (P, t) fixed, then there is a homotopy f_s such that* (1) $f_0 = f$, (2) $f_1 = i$, *and* (3) *for each s, f_s maps $C \times I$ onto itself,*

$$f_s \mid C \times (0 \cup 1) = i,$$

$f_s \mid C \times t$ *is a homeomorphism into* int $(C \times I)$ *for each* $t \neq 0, 1$ *and* $f_s(P, t) = (P, t)$.

Proof. For each $t \geq \frac{1}{2}$, let g_t be the mapping of $C \times I$ into itself that takes (x, s) onto $(x, s/2t)$. If $t \leq \frac{1}{2}$, let g_t take (x, s) onto

$$(x, 1 - (1 - s)/2(1 - t)).$$

For each t, $g_t(P, t) = (P, \frac{1}{2})$ and $g_t f(C, t) \subset$ int $(C \times I)$. Also, $g_1(x, 1) = (x, \frac{1}{2}) = g_0(x, 0)$ and $g_{1/2}(x, s) = (x, s)$.

Let ϕ be the mapping of S^1 into the space H' of orientation-preserving homeomorphisms of C into int $(C \times I)$ that takes t into the homeomorphism mapping the point x of C into $g_t f(x, t)$. It follows from Theorem 3.1 of [8] that there is a mapping Φ of $S^1 \times I$ into H' such that $\Phi(t, 0) = \phi(t)$, $\Phi(t, 1)(x) = (x, \frac{1}{2})$, $\Phi(t, s)(P) = (P, \frac{1}{2})$ for each t, s and x, and $\Phi(1, s)(x) = (x, \frac{1}{2}) = \Phi(0, s)(x)$. Then if f_s maps $C \times I$ into itself in such a way that $f_s(x, t) = g_t^{-1}\Phi(t, s)(x)$, f_s is the required homotopy. The computations that demonstrate this are easily made.

THEOREM 2. *The group* $\pi_1(H)$ *is isomorphic to* $\pi_1(T)$.

Proof. Coordinatize C and S^1 by the reals mod 1, consider T as $C \times C$, identify $0 \times C$ with C and suppose $\pi(x, t) = (x, t)$. Let F be a mapping of S^1 into H. Since H is the identity component, there is a mapping Z of I into H such that $Z(0) = F(0)$ and $Z(1) = i$. Then $F(x)[Z(1 - t)]^{-1}$ is a homotopy of F to a mapping taking 0 onto the identity. Hereafter, it will be assumed of F that $F(0) = F(1) = i$. Consider the mapping g of S^1 into T such that $g(x) = F(x)(0, 0)$. There is a *unique* mapping G of I into $C \times E^1$ such that $\pi G(x) = g(x)$ and $G(0) = (0, 0)$. Note that $G(1) = (0, r)$, where r is some integer. There is, for x in I, a unique mapping $f(x)$ of C into $C \times E^1$ such that $f(x)(0) = G(x)$ and $\pi f(x)(y) = F(x)(0, y)$. Note that $f(1)(C)$ is merely a translation of $f(0)(C)$ and that, as in the proof of Theorem 1, f is a continuous mapping of I into the space of homeomorphisms of C into $C \times E^1$.

Consider the homeomorphisms α and β of S^1 into T such that $\alpha(x) = (0, x)$ and $\beta(x) = (x, 0)$. Then g is homotopic in T relative to 0 to $r\beta + s\alpha$, where r and s are integers, and this mapping may be assumed to "lift" under π to an arc in $C \times E^1$ that, if $r > 0$, goes along $0 \times [0, r - 1]$ and then wraps around $C \times [r - 1, r]$ s times, meeting each $C \times x$ exactly once. If $r < 0$, a similar remark holds. If $r = 0$, then $s\alpha$ takes each x of S^1 onto the point $(0, sx)$.

Case 1. $r > 0$. By the theorems of [8], F may be assumed to be such that g actually is $r\beta + s\alpha$ and lifts into $C \times E^1$ as described above. Let $0 = t_0 < t_1 < \cdots < t_r = 1$ be such that $G(t_j)$ has coordinates $(0, j)$. Note that $F(t_j)(0, 0) = (0, 0)$. In fact, it may be assumed that the second coordinate of $g(t)$ is $(t - t_{j-1})/(t_j - t_{j-1})$ if $t_{j-1} \leq t \leq t_j$. It then follows from

Lemma A that in H there is an arc connecting $F(t_j)$ to a map $F_1(t_j) = i$ and that each homeomorphism in this arc leaves (0, 0) fixed. These arcs carry a partial homotopy of F in H which may be extended to a homotopy of F to a mapping F_1 of S^1 into H such that $F_1(t_j) = i$. Define g_1, G_1, f_1 as g, G, f were defined.

The proof of Theorem 1 may now be followed almost word for word to get a sequence of homotopies leaving $F_1(t)$ fixed if $t_1 \leq t \leq 1$. The first takes F_1 to a mapping F_1' such that $F_1'(t)(C)$ doesn't intersect C if $0 < t < t_1$. Since g_1' is homotopic to g under a homotopy leaving $g_1'(0) = g_1'(t)$ fixed, the second homotopy of the sequence takes F_1' to F_1'', where $F_1''(t)(0, 0) = (0, t/t_1)$. The third homotopy takes F_1'' to F_1''', where $F_1'''(t)(x) = (x, t/t_1)$ for each x in C (see Lemma B). The fourth takes F_1''' to F_2 where $F_2(t)(x, a) = (x, a + t/t_1)$ (see the final remarks on the proof of Theorem 1.)

Similarly, F_2 is homotopic to F_3 under a homotopy leaving $F_2(t)$ unchanged unless $t_1 < t < t_2$, in which case, $F_3(t)(x, a) = (x, a + (t - t_1)/(t_2 - t_1))$. Repeat this process until F_r is obtained by means of a homotopy leaving $F_{r-1}(t)$ unchanged unless $t_{r-2} < t < t_{r-1}$, in which case,

$$F_r(t)(x, a) = (x, a + (t - t_{r-2})/(t_{r-1} - t_{r-2}).$$

Finally, F_r is homotopic to F_{r+1} under a homotopy leaving $F_r(t)$ unchanged unless $t_{r-1} < t < t_r$, in which case,

$$F_{r+1}(t)(x, a) = (x + y, a + (t - t_{r-1})/(t_r - t_{r-1}),$$

where $g(t) = (y, (t - t_{r-1})/(t_r - t_{r-1})$.

If F is homotopic to F' in H, g and g' represent the same element of the fundamental group of T so that g' may also be taken as $r\beta + s\alpha$. Hence $F_{r+1} = F'_{r+1}$. Clearly $F_{r+1} = F'_{r+1}$ implies that F is homotopic to F' in H. Hence it follows that the function that maps the homotopy class of F onto that of g is well defined and one to one.

Case 2. $r < 0$ or $r = 0$ but $s \neq 0$. The same argument applies.

Case 3. $r = 0 = s$. In this case, g is homotopic to 0 in T and the argument for Theorem 1 may be applied to obtain the fact that F is homotopic to 0 in H, since in this case $G(0) = G(1)$.

The three cases combine to show that the function mapping the homotopy class of F onto that of g is an isomorphism of $\pi_1(H)$ onto $\pi_1(T)$.

THEOREM 3. *If M is a torus from which the interiors of a finite (positive) number of disjoint discs have been removed, then the identity component of the space H of homeomorphisms of M onto itself that leave the boundary of M pointwise fixed is homotopically trivial.*

Proof. The proof is essentially that of the Theorem of [8], which states a similar fact for discs with holes. Suppose that M is obtained by removing a disc D from a torus T and that f maps S^k into H. Let $f(x)$ be extended to

$f^*(x)$, a homeomorphism of T onto itself leaving D pointwise fixed. The mapping g^* of S^k into T associated with f^* as in the preceding arguments is, if P is considered to be in D, homotopic to 0 in an obvious way. Hence f^* is homotopic to 0 in the identity component of the space of homeomorphisms of T onto itself and the argument for the theorem of [8] now applies to prove that f is homotopic to 0 in H. As in the proof of the theorem of [8] an induction argument may now be applied.

These arguments may also be applied to obtain the

COROLLARY. *If the mappings of H above are also required to leave fixed the points of some finite set, then H is homotopically trivial.*

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