ON FINITE GROUPS WHICH CONTAIN A FROBENIUS FACTOR GROUP

BY

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1. Introduction

In [2] R. Brauer and the author obtained rather detailed information about the irreducible characters of finite groups G of order g which satisfy the following condition:

(∗)   *There exists a prime p dividing g such that if $y \neq 1$ is an element of a p-Sylow subgroup P of G then the centralizer of y in G coincides with the centralizer of P in G.*

W. Feit [4] has studied the characters of groups satisfying a more general condition, stated in the present paper as Hypothesis II in §3. In [5] he abstracted from this a generalization of Brauer and Suzuki's results on exceptional characters of finite groups.

Here we extend Feit's results on the characters of groups satisfying Hypothesis II. It is pointed out (§2) that Feit's proof [5] yields a more general theorem on exceptional characters than he stated, and two corollaries are derived. These facts are applied (§3) to the characters of groups G satisfying Hypothesis II. Finally we obtain lower bounds for the degrees of the irreducible characters and of the faithful characters of G (§4), in a sense made explicit there. These results include many of the results obtained in [2] for groups satisfying condition (∗).

For any subset T of a group G, we shall denote the centralizer, normalizer, and number of elements of T by $C(T)$, $N(T)$, and $|T|$, respectively. By a *character* we shall mean a (possibly reducible) character over the complex number field. By the *kernel* of a character is meant the kernel of the corresponding representation. A *generalized character* is a linear combination of characters with integer coefficients. The inner product of two generalized characters α and β of a group G is

$$(\alpha, \beta) = (1/|G|) \sum_G \alpha(x)\overline{\beta(x)}.$$

A subscript G will be attached to the inner product when it is desirable to emphasize which group is involved.

2. Exceptional characters

Let G be a finite group, let L be a subgroup, and let $\hat{L}$ be a subset of L. Consider the following conditions:

(Ia)   *For every conjugacy class K of G which contains elements of $\hat{L}$, $K \cap \hat{L}$ is a class of L.*

(Ib)   *For every element y of $\hat{L}$, the centralizer $C(y)$ of y in G lies in L.*

It is easily seen that these two conditions are equivalent to the single condition:

HYPOTHESIS I.   *For every element x of G,*

$$\hat{L} \cap x\hat{L}x^{-1} = \hat{L} \quad if \quad x \in L,$$
$$= \emptyset \quad if \quad x \notin L,$$

where $\emptyset$ denotes the empty set.

If α is a generalized character of L, denote by α^* the generalized character of G induced by α. The proof of the lemma in [5] yields the following stronger statement.[2]

LEMMA 2.1.   *Assume G L, and $\hat{L}$ satisfy Hypothesis I. If α is a generalized character of L which vanishes at all elements of $L - \hat{L}$ which are conjugate relative to G to elements of $\hat{L}$, then for every $y \in \hat{L}$*

(2.1a) $$\alpha^*(y) = \alpha(y).$$

Furthermore, for every generalized character β of L which vanishes on $L - \hat{L}$,

(2.2a) $$(\alpha^*, \beta^*)_G = (\alpha, \beta)_L.$$

The Frobenius Reciprocity Theorem makes possible another formulation of this lemma. Denote the column of numbers $(\chi \mid L, \alpha)$, as χ ranges over the irreducible characters of G, by A_α. If x is a fixed member of G, write $X(x)$ for the column $(\chi(x))$. Then

$$A_\alpha = \frac{1}{|L|} \sum_{y \in L} X(y)\overline{\alpha(y)} = \frac{1}{|G|} \sum_{x \in G} X(x)\overline{\alpha^*(x)}.$$

Let inner products of columns of complex numbers have the usual meaning. Then $\alpha^*(x) = (X(x), A_\alpha)$ and $(\alpha^*, \beta^*) = (A_\alpha, A_\beta)$. If α and β satisfy the conditions in the lemma, then (2.1a) and (2.2a) become, respectively,

(2.1b) $$(X(y), A_\alpha) = \alpha(y) \qquad\qquad (y \in \hat{L}),$$

(2.2b) $$(A_\alpha, A_\beta) = (\alpha, \beta).$$

Direct proofs of (2.1b) and (2.2b) without the use of induced characters can be given quite easily, as was done in [2] in a special case. The equations (2.1a), (2.2a), (2.1b), and (2.2b) are also useful when the hypotheses of the next theorem are not fulfilled.

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[2] Professor Curtis has informed me that in the corresponding Lemma (38.15) in [3], an additional assumption was intended: in their notation that β vanishes outside S.

Feit's proof of his theorem on exceptional characters [5] yields the following stronger result.

THEOREM 2.1. *Let G, L, and $\hat{L}$ satisfy Hypothesis I. Consider a family*

$$\mathfrak{L} = \{\lambda_{is} : i = 1, 2, \cdots, k; s = 1, 2, \cdots, n_i\}$$

of distinct irreducible characters of L. Assume there exist positive integers l_i, $i = 1, 2, \cdots, k$, with $l_1 = 1$, such that for all $y \in L - \hat{L}$

$$(2.3) \qquad l_i \lambda_{jt}(y) - l_j \lambda_{is}(y) = 0$$

for $1 \leq i \leq k$, $1 \leq j \leq k$, $1 \leq s \leq n_i$, $1 \leq t \leq n_j$, and that

$$(2.4) \qquad \sum_{i=1}^{j-1} n_i l_i^2 > 2l_j \qquad (j = 2, \cdots, k).$$

Then there exist irreducible characters χ_{is} of G and a sign $\varepsilon = \pm 1$ such that

$$(2.5) \qquad (l_i \lambda_{jt} - l_j \lambda_{is})^* = \varepsilon(l_i \chi_{jt} - l_j \chi_{is}),$$

for $1 \leq i \leq k$, $1 \leq j \leq k$, $1 \leq s \leq n_i$, $1 \leq t \leq n_j$.

Notice that if $L \neq G$, then (Ib) implies $1 \in L - \hat{L}$, and (2.3) implies $\lambda_{jt}(1) = l_j \lambda_{11}(1)$ for all λ_{jt}. As Feit pointed out, the conclusion (2.5) is equivalent to the statement that

$$\left(\sum_{i,s} a_{is} \lambda_{is}\right)^* = \varepsilon\left(\sum_{i,s} a_{is} \chi_{is}\right)$$

whenever the a_{is} are integers such that $\sum_{i,s} a_{is} \lambda_{is}(1) = 0$.

From this theorem we obtain[3]

COROLLARY 2.1. *Under the conditions of the theorem, there exist integers d_{jt} and characters μ_{jt} of L which are orthogonal to all the λ_{is} such that*

$$(2.6) \qquad \chi_{jt} \mid L = \varepsilon\lambda_{jt} + d_{jt}\sum_{i,s} l_i \lambda_{is} + \mu_{jt}.$$

Furthermore, for $x \in G$

$$(2.7) \qquad \chi_{jt}(x) = l_j \chi_{11}(x) + \begin{cases} \varepsilon(\lambda_{jt} - l_j \lambda_{11})(y) \\ 0, \end{cases}$$

where the first case occurs if x is conjugate to an element y of $\hat{L}$, and the second case otherwise. If α is a generalized character of L which vanishes at all elements of $L - \hat{L}$ which are conjugate relative to G to elements of $\hat{L}$, and which is orthogonal to $\lambda_{jt} - l_j \lambda_{11}$, then

$$(2.8) \qquad (\chi_{jt} \mid L, \alpha) = l_j(\chi_{11} \mid L, \alpha).$$

If λ_{11}, and hence all λ_{is}, vanish on all elements of $L - \hat{L}$ which are conjugate relative to G to elements of $\hat{L}$, then for all χ_{jt}, $d_{jt} = l_j d_{11}$.

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[3] Professor Feit has informed me that in his corollary in [5] the equation for $\Lambda_{jt} \mid \mathfrak{L}$ was intended to apply only to elements of $\hat{\mathfrak{L}}$.

For every other irreducible character χ_j of G, there is an integer d_j and a character μ_j of L orthogonal to all the λ_{is} such that

$$(2.9) \qquad \chi_j \mid L = d_j \sum_{i,s} l_i \lambda_{is} + \mu_j .$$

Proof. By the Frobenius Reciprocity Theorem, (2.5) is equivalent to the assertion that, for every irreducible character χ of G, if $(j, t) \neq (1, 1)$,

$$(2.10) \qquad \begin{aligned} (\chi \mid L, \lambda_{jt} - l_j \lambda_{11}) &= \varepsilon & &\text{if } \chi = \chi_{jt}, \\ &= -\varepsilon\, l_j & &\text{if } \chi = \chi_{11}, \\ &= 0 & &\text{otherwise,} \end{aligned}$$

and this yields (2.6) and (2.9). Equation (2.7) is obtained by applying (2.1a) to (2.5).

If λ_{11}, and hence all λ_{is}, vanish on all elements of $L - \hat{L}$ which are conjugate relative to G to elements of $\hat{L}$, then, for $(j, t) \neq (1, 1)$, we apply the Frobenius Reciprocity Theorem and (2.5) and (2.2a) to

$$((\chi_{jt} - l_j \chi_{11}) \mid L, \lambda_{11})_L ,$$

finding that

$$d_{jt} - l_j(\varepsilon + d_{11}) = -\varepsilon\, l_j .$$

Treating $((\chi_{jt} - l_j \chi_{11}) \mid L, \alpha)_L$ in the same way, we obtain (2.8). This proves the corollary.

The character χ_{is} is called the *exceptional character* of G corresponding to λ_{is}, and all other characters of G are called *non-exceptional* for the family $\mathfrak{L}$, except that when $\mathfrak{L}$ has only one member we shall regard all characters of G as non-exceptional for $\mathfrak{L}$. It is clear that if $\mathfrak{L}$ has more than two members, then for every $\lambda_{is} \in \mathfrak{L}$, the character χ_{is} of G which is exceptional for λ_{is} is unique. If $\mathfrak{L}$ has only two members then $k = 1$ and, although the two characters of G that are exceptional for $\mathfrak{L}$ are uniquely determined, it is not uniquely determined which of the two is exceptional for λ_{11} and which for λ_{12} until ε is specified.

COROLLARY 2.2. *If G satisfies Hypothesis I and $\mathfrak{L}$ and $\mathfrak{M}$ are disjoint families of characters of L, both of which satisfy the conditions of Theorem 2.1, then a character χ of G cannot be the exceptional character both for a member of $\mathfrak{L}$ and for a member of $\mathfrak{M}$.*

Proof.[4] If either $\mathfrak{L}$ or $\mathfrak{M}$ has only one member, this is a matter of definition. Let $\mathfrak{L} = \{\lambda_{is}\}$ and $\mathfrak{M} = \{\mu_{jt}\}$, and let $\{l_i\}$, δ, $\{m_j\}$, and ε be the integers associated with these families. Suppose the corollary is false, and that χ is the exceptional character corresponding both to λ_{is} and to μ_{jt}. Let $\lambda_{hu} \in \mathfrak{L}$ and $\mu_{nv} \in \mathfrak{M}$, and let them be distinct from λ_{is} and μ_{jt}. Let χ_{hu} be the exceptional

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[4] I wish to thank the referee for pointing out that the proof of this corollary becomes simpler if it is based on Corollary 2.1.

character corresponding to λ_{hu}. By (2.8)

$$l_h(\chi \mid L, m_n \mu_{jt} - m_j \mu_{nv}) = l_i(\chi_{hu} \mid L, m_n \mu_{jt} - m_j \mu_{nv}).$$

According to (2.6) the left-hand side is $l_h m_n \varepsilon$. Hence according to (2.10) the right-hand side is $l_i m_n \varepsilon$ and $\chi_{hu} = \chi$. But this contradicts the fact that χ is also exceptional for λ_{is}.

3. Groups which contain a Frobenius factor group

We now study groups which satisfy the following:

HYPOTHESIS II. G is a finite group with a subgroup of the form $M \times H$ such that $M \neq \{1\}$ and

(i) If $y \in M \times H - H$ then $C(y) \subset M \times H$.

(ii) For every $x \notin N(M \times H)$, $(M \times H) \cap x(M \times H)x^{-1} \subset H$.

(iii) H and M are normal subgroups of $N(M \times H)$.

Denote $M \times H$, $N(M \times H)$, and $(M \times H) - H$ by C, L, and $\hat{L}$, respectively. Denote $\mid M \mid$, $\mid H \mid$, and $\mid L/C \mid$ by m, h, and q, respectively. Let the irreducible characters of M be $1 = \zeta_0, \zeta_1, \zeta_2, \cdots, \zeta_{n-1}$, and let those of H be $1 = \kappa_0, \kappa_1, \kappa_2, \cdots$. Denote the character $\zeta_i \kappa_a$ of C by ζ_{ia}, and by $\tilde{\zeta}_{ia}$ the character of L induced by ζ_{ia}. Let $z_i = \zeta_i(1)$, for $0 \le i \le n - 1$.

LEMMA 3.1. Let G be a group which satisfies Hypothesis II. Then $N(M) = L$. Either $q = 1$ or L/H is a Frobenius group whose regular subgroup is C/H, and in the latter case M is nilpotent. The integer q divides $m - 1$. No element of $L - \hat{L}$ is conjugate relative to G to any element of $\hat{L}$.

Proof. This is a generalization of parts of Lemma 2.4 in [4], and the first two statements of the present lemma are proved in the same way. Since the regular subgroup of a Frobenius group is nilpotent [8], M must be nilpotent if $q \neq 1$. Since L/C is a group of automorphisms of M under which each element of $M - \{1\}$ has q images, q must divide $m - 1$.

Now we prove the last statement of the lemma. First, no element of H is conjugate to an element of $\hat{L}$ [4, Lemma 2.4]. Suppose $x \in L - \hat{L}$ and is conjugate relative to G to an element y of $\hat{L}$. Since $y \notin H$, the order of y is not relatively prime to m, and hence the same is true of the order of x. If t is the order of x, then $t = t_1 t_2$, where t_1 is a product of primes which divide m, and t_2 a product of primes not dividing m. We can write $x = x_1 x_2 = x_2 x_1$, where x_1 has order t_1 and x_2 has order t_2. Here x_1 and x_2 are uniquely determined by these conditions. Also $x_1 \neq 1$ since $t_1 \neq 1$. Since L/C has order q and $(q, m) = 1$, $x_1 C = C$, and thus $x_1 \in C$. If $x_1 \in C - H$ then $x_2 \in C$ by (i), and then $x \in C$. But $x \in L - \hat{L}$, so we would have $x \in H$, which would contradict the second sentence of this paragraph. Thus $x_1 \in H$.

Now $y = y_1 y_2 = y_2 y_1$, where y_1 and y_2 have orders t_1 and t_2, respectively. Then $x_1 x_2 = z y_1 z^{-1} z y_2 z^{-1} = z y_2 z^{-1} z y_1 z^{-1}$ for some $z \in G$. The uniqueness of x_1

implies that $x_1 = zy_1 z^{-1}$. Clearly $y_2 \epsilon H$ and hence $y_1 \epsilon C - H = \hat{L}$. Again we have a contradiction. This proves the lemma.

By the definition of induced characters,

(3.1)
$$\tilde{\zeta}_{ia}(y) = 0 \qquad \text{if} \quad y \notin C,$$
$$= \sum_x \zeta_{ia}^x(y) \quad \text{if} \quad y \epsilon C,$$

where x ranges over a complete residue system R of L (mod C) and $\zeta_{ia}^x(y) = \zeta_{ia}(xyx^{-1})$. If $i \neq 0$ then the inertia group of ζ_{ia} is C (that is, $\zeta_{ia}^x \neq \zeta_{ia}$ for $x \epsilon L - C$), and $\tilde{\zeta}_{ia}$ is irreducible [4, Lemma 2.2].[5] From (3.1) we see that $\tilde{\zeta}_{ia} = \tilde{\zeta}_{jb}$ if and only if $\zeta_j = \zeta_i^x$ and $\kappa_b = \kappa_a^x$ for some $x \epsilon L$. In particular, to obtain all the characters $\tilde{\zeta}_{ia}$ it is sufficient to let κ_a range over a full system $\mathcal{S}$ of irreducible characters of H no two of which are associated in L.

Let $F(\kappa_a)$ denote the inertia group of κ_a. Then $C \subset F(\kappa_a)$. We have $\tilde{\zeta}_{ia} = \tilde{\zeta}_{ja}$ if and only if $\zeta_j = \zeta_i^x$ for some $x \epsilon F(\kappa_a)$. If $\kappa_a \epsilon \mathcal{S}$, let

$$\mathcal{L}(\kappa_a) = \{\tilde{\zeta}_{ia} : i \neq 0\}.$$

The sets $\mathcal{L}(\kappa_a)$ are disjoint and together contain every irreducible character of L whose kernel does not contain M. According to the notation above, the number of classes of conjugate elements of M is n. Let

(3.2)
$$f_a = (F(\kappa_a):C).$$

Then each character $\tilde{\zeta}_{ia}$ with $i \neq 0$ is obtained from f_a different characters $\zeta_j = \zeta_i^x$ of M. Hence $\mathcal{L}(\kappa_a)$ contains $(n - 1)/f_a$ distinct characters.

We summarize some of these results.

LEMMA 3.2. *Let G be a group which satisfies Hypothesis II. Let $\mathcal{S}$ be a full system of irreducible characters of H no two of which are associated in L. The irreducible characters λ of L whose kernel does not include M are distributed in the disjoint families $\mathcal{L}(\kappa_a)$, $\kappa_a \epsilon \mathcal{S}$. Here $\mathcal{L}(\kappa_a)$ consists of the characters $\tilde{\zeta}_{ia}$ of L induced by characters $\zeta_i \kappa_a$ of C where ζ_i is a non-principal irreducible character of M. The number of members of $\mathcal{L}(\kappa_a)$ is $(n - 1)/f_a$ where n is the class number of M, $f_a = (F(\kappa_a):C)$, and $F(\kappa_a)$ is the inertia group of κ_a in L.*

Every other irreducible character of L is a constituent of the character $\tilde{\zeta}_{0a}$ induced by $\zeta_0 \kappa_a$ for some $\kappa_a \epsilon \mathcal{S}$, where $\zeta_0 = 1$.

Next we show

LEMMA 3.3. *For every a,*

$$\sum_{\tilde{\zeta}_{ia} \epsilon \mathcal{L}(\kappa_a)} z_i \tilde{\zeta}_{ia}(y) = 0 \qquad \text{if} \quad y \notin C,$$
$$= - \sum_w \kappa_a^w(y_2) \qquad \text{if} \quad y \epsilon C - H,$$
$$= (m - 1) \sum_w \kappa_a^w(y_2) \quad \text{if} \quad y = y_2 \epsilon H,$$

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[5] Alternatively, this can be verified by applying the "permutation lemma" [1] to the character table of M.

where $y = y_1 y_2$ *with* $y_1 \epsilon M$, $y_2 \epsilon H$, *for* $y \epsilon C$, *and where* w *ranges over a full system* R_a *of representatives of the right cosets* $F(\kappa_a)w$ *of* $F(\kappa_a)$ *in* L.

Proof. This follows immediately from (3.1) if $y \notin C$. If $y \epsilon C$ then, according to the remarks preceding Lemma 3.2,

$$
(3.3) \quad
\begin{aligned}
f_a \sum_{\tilde{\zeta}_{ia} \epsilon \mathcal{L}(\kappa_a)} z_i \, \tilde{\zeta}_{ia}(y) &= \sum_{i=1}^{n-1} z_i \, \tilde{\zeta}_{ia}(y) \\
&= \sum_{i=1}^{n-1} z_i \sum_{x \epsilon R} \zeta_i^x(y_1) \kappa_a^x(y_2).
\end{aligned}
$$

But $\sum_{i=0}^{n-1} z_i \zeta_i^x$ is the character of the regular representation of M. Hence the sum (3.3) has the value

$$
f_a(\delta m - 1) \sum_{w \epsilon R_a} \kappa_a^w(y_2),
$$

where $\delta = 1$ or 0 according as y_1 is 1 or not. This proves the lemma.

Now we prepare to apply Theorem 2.1 and its corollaries, assuming G satisfies Hypothesis II. The theorem and the first corollary obtained are generalizations of similar results of Feit [4]. It is clear that if G satisfies Hypothesis II then it satisfies Hypothesis I. Equation (3.1) implies that

$$
(3.4) \quad\quad\quad z_j \, \tilde{\zeta}_{ia}(x) = z_i \, \tilde{\zeta}_{ja}(x) \quad\quad\quad (x \, \epsilon \, L - \hat{L}).
$$

Thus $\mathcal{L}(\kappa_a)$ satisfies condition (2.3).

Make the further assumptions that $q \neq 1$ and M is not a non-abelian p-group with $(M : M') < 4q^2$. Then by Lemma 3.1, M is nilpotent, so $M' \neq M$. Hence M has non-principal characters of degree 1, and, in the notation of Theorem 2.1, $l_1 = 1$. If ζ is an irreducible character of M of degree $z > 1$, then $\sum z_i^2 > 2qz$, where z_i ranges over the degrees of all irreducible characters ζ_i of M for which $i \neq 0$ and $z_i < z$ [4, Lemma 2.3]. Therefore

$$
(3.5) \quad\quad\quad (1/f_a) \sum z_i^2 > 2qz/f_a \geq 2z.
$$

But for each a, each character $\tilde{\zeta}_{ia}$ with $i \neq 0$ is obtained from the f_a different characters ζ_i^x for which $x \epsilon F(\kappa_a)$. Hence (3.5) shows that $\mathcal{L}(\kappa_a)$ satisfies (2.4), and thus all the hypotheses of Theorem 2.1. Consequently we have

THEOREM 3.1. *Suppose* G *is a group satisfying Hypothesis II, that* $q \neq 1$, *and that* M *is not a non-abelian* p-*group with* $(M : M') < 4q^2$. *Then for every* a, *there exist irreducible characters* χ_{ia} *of* G *with* $1 \leq i \leq n - 1$, *and a sign* $\varepsilon_a = \pm 1$ *such that*

$$
(3.6) \quad\quad\quad (z_i \, \tilde{\zeta}_{ja} - z_j \, \tilde{\zeta}_{ia})^* = \varepsilon_a (z_i \, \chi_{ja} - z_j \, \chi_{ia}).
$$

We shall refer to the character χ_{ia} as the exceptional character for $\tilde{\zeta}_{ia}$ except when $\mathcal{L}(\kappa_a)$ has only one member, and in this case we shall regard all the characters of G as non-exceptional for $\mathcal{L}(\kappa_a)$. An irreducible character of G which is non-exceptional for all the families $\mathcal{L}(\kappa_a)$ will be called non-exceptional.

If $\mathcal{L}(\kappa_a)$ has only one member, then the remarks preceding Lemma 3.2 imply

that all the non-principal irreducible characters of M are associated in $F(\kappa_a)$ and that $L = F(\kappa_a)$. In particular, M has no proper subgroup which is normal in L. Since $q \neq 1$, M is nilpotent and hence must be an elementary abelian p-group, and $q = m - 1$.

COROLLARY 3.1. *Suppose G satisfies the hypotheses of Theorem 3.1.*

(a) *If χ_{ia} and χ_{ja} are the exceptional characters corresponding to $\tilde{\zeta}_{ia}$ and $\tilde{\zeta}_{ja}$, respectively, then $z_j \chi_{ia} - z_i \chi_{ja}$ vanishes on all elements of G which are not conjugate to members of $C - H$. In particular, $z_j \chi_{ia}(1) = z_i \chi_{ja}(1)$.*

(b)[6] *If κ_a and κ_b are distinct members of $\mathcal{S}$, then no character of G can be the exceptional character both for a member of $\mathfrak{L}(\kappa_a)$ and for a member of $\mathfrak{L}(\kappa_b)$.*

(c) *There exist integers d_{ab} such that if χ_{jb} is exceptional for $\tilde{\zeta}_{jb}$ then*

$$\chi_{jb} \mid L = \varepsilon_b \tilde{\zeta}_{jb} + z_j \{ \sum_{\kappa_a \in \mathcal{S}} d_{ab} \sum_{\tilde{\zeta}_{ia} \in \mathfrak{L}(\kappa_a)} z_i \tilde{\zeta}_{ia} + \mu_b \}.$$

Here d_{ab} depends only on a and b, and μ_b is a (reducible) character of L whose kernel contains M and which depends only on b.

(d) *For every non-exceptional character χ_j of G, there are integers e_{ja} and a (reducible) character ν_j of L whose kernel contains M such that*

$$\chi_j \mid L = \sum_{\kappa_a \in \mathcal{S}} e_{ja} \sum_{\tilde{\zeta}_{ia} \in \mathfrak{L}(\kappa_a)} z_i \tilde{\zeta}_{ia} + \nu_j.$$

Proof. Part (a) is obtained by applying (2.7), and part (b) by applying Corollary 2.2. According to Lemma 3.1, no element of $L - \hat{L}$ is conjugate to any element of $\hat{L}$. Therefore, in the notation of Corollary 2.1, $d_{jt} = l_j d_{11}$. According to part (b), a character of G can be exceptional for at most one character of L. If χ_{jb} is exceptional for $\tilde{\zeta}_{jb}$ then (2.8) can be applied to all irreducible characters α of L not in $\mathfrak{L}(\kappa_b)$. Then part (c) is obtained from (2.6) and (2.9). Part (d) follows directly from (2.9).

Equation (3.1) and Lemma 3.3 can be applied to express (c) and (d) in another way. They also lead to expressions for the degrees of the characters. We state the latter results.

COROLLARY 3.2. *Suppose G satisfies the hypotheses of Theorem 3.1.*

(c$'$) *If χ_{jb} is the exceptional character for $\tilde{\zeta}_{jb}$ then*

$$\chi_{jb}(1) = z_j [\varepsilon_b \, q \kappa_b(1) + (m - 1) \sum_{\kappa_a \in \mathcal{S}} d_{ab}(q/f_a)\kappa_a(1) + \mu_b(1)].$$

(d$'$) *If χ_j is non-exceptional then*

$$\chi_j(1) = (m - 1) \sum_{\kappa_a \in \mathcal{S}} e_{ja}(q/f_a)\kappa_a(1) + \nu_j(1).$$

4. Bounds on the degrees of the characters

We apply the above results to obtain lower bounds for the degrees of the irreducible characters of groups which satisfy Hypothesis II.

THEOREM 4.1. *Suppose G satisfies the hypotheses of Theorem 3.1. If*

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[6] Apparently it is intended in Corollary 2.2 in [4] to assume that λ_1 and λ_2 are not associated in $N(M \times H)$, that is, that $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$.

χ_j is a non-exceptional character of G then either $\chi_j(1) \geq m - 1$ or M is contained in the kernel of χ_j.

Suppose χ_{jb} is exceptional for $\tilde{\zeta}_{jb}$. If $\varepsilon_b = -1$ then $\chi_{jb}(1) \geq (m - 1)/2$. If $\varepsilon_b = 1$ and $\chi_{jb}(1) < m$ then

$$(4.1) \qquad\qquad \chi_{jb} \mid L = \tilde{\zeta}_{jb} + z_j \mu_b ,$$

where μ_b is a (reducible) character of L whose kernel contains M.

Proof. The statement about χ_j follows immediately from Corollaries 3.1(d) and 3.2(d').

Suppose χ_{jb} is exceptional for $\tilde{\zeta}_{jb}$. Then, by the definition following Theorem 3.1, $\mathcal{L}(\kappa_b)$ has more than one member; that is, $(n - 1)/f_b \geq 2$. Therefore $f_b \leq (n - 1)/2 \leq (m - 1)/2$. Equation (3.2) implies that $f_b \leq q$.

If $\varepsilon_b = -1$ then Corollary 3.1(c) implies that $d_{bb} \geq 1$ since

$$(\chi_{jb} \mid L, \tilde{\zeta}_{jb}) \geq 0.$$

It follows from Corollary 3.2(c') and the inequalities above that

$$\chi_{jb}(1) \geq -q\kappa_b(1) + (m - 1) d_{bb}(q/f_b)\kappa_b(1) \geq m - 1 - f_b \geq (m - 1)/2.$$

Now suppose that $\varepsilon_b = 1$ and that $\chi_{jb}(1) < m$. Then Corollaries 3.1(c) and 3.2(c') immediately yield the conclusion stated. This proves the theorem.

This theorem can be applied to obtain lower bounds for the degrees of the faithful (reducible) representations of G. First we derive a lemma.

LEMMA 4.1. *Suppose G satisfies Hypothesis II, and let K be a normal subgroup $\neq \{1\}$ of G. Then $M \lhd G$ or K has non-identity elements which are not conjugate to any elements of $(M \times H) - H$.*

Proof. The product KL is a group, and $(KL:L) = (K:K \cap L)$. Suppose every non-identity element of K is conjugate to some element of $C - H$. Since $K \lhd G$, K consists of the members of all the conjugates of $(C - H) \cap K$ and the identity. Since these conjugates are disjoint,

$$|K| = (G:L) \mid (C - H) \cap K \mid + 1.$$

The left-hand side is divisible by $(K:K \cap L)$ and the first term of the right-hand side by $(KL:L)$. Then $(KL:L) = 1$. Hence $K \subset L$. It follows from Lemma 3.1 that $K \subset \hat{L} \cup \{1\}$. Since $K \lhd G$ and distinct conjugates of $\hat{L}$ are disjoint, it follows that $L = G$ or $K = \{1\}$. This proves the Lemma.

THEOREM 4.2. *Suppose G satisfies Hypothesis II and has a faithful representation X of degree $< (m - 1)/2$. Then one of the following must be true.*
  (a) $N(M \times H) = M \times H$. *That is, $q = 1$.*
  (b) M *is a non-abelian p-group with* $(M:M') < 4q^2$.
  (c) $M \lhd G$. *That is,* $L = G$.

(d) *M is an elementary abelian p-group, and no proper subgroup of M is normal in* $N(M \times H)$.

Proof. Assume that G does not satisfy (a), (b), or (c). Then G satisfies the hypotheses of Theorems 3.1 and 4.1. Furthermore, according to Lemma 3.1, M is nilpotent. Therefore to prove (d) it will be sufficient to prove that M has no proper subgroup M_0 which is normal in $N(M \times H)$, because otherwise the elements in the center of M whose orders divide a fixed prime divisor p of m would form such a subgroup.

Assume that M has a proper subgroup M_0 which is normal in L. According to Theorem 4.1, not all the constituents of X can be non-exceptional. We shall prove that X is not faithful, contradicting our hypotheses.

Clearly M has non-principal irreducible characters ζ_i whose kernels contain M_0. Therefore it follows from Corollary 3.1(c) and (4.1) that if χ_{jb} is exceptional for $\tilde{\zeta}_{jb}$ and has degree $< (m - 1)/2$, then G has a character χ_{ib} which is exceptional for a member of the same family $\mathfrak{L}(\kappa_b)$ and whose kernel contains M_0. For each exceptional constituent χ_{jb} of X, such a character χ_{ib} may be obtained. Let K be the intersection of the kernels of the characters χ_{ib} obtained and of the non-exceptional constituents of X. Then K is a normal subgroup of G and contains M_0. According to Lemma 4.1, K contains non-identity elements which are not conjugate to any elements of $C - H$. According to Corollary 3.1(a), these elements belong to the kernel of X. This is a contradiction, and the theorem is proved.

Finally, we can prove a result which supplements cases (a) and (d) of Theorem 4.2.

THEOREM 4.3. *Suppose G satisfies Hypothesis II and that G has a faithful representation X of degree $\leq m^{1/2} - 1$. Then either M is a non-abelian p-group with $(M:M') < 4q^2$ or $M \lhd G$.*

Proof. Suppose the theorem is false, and that G is a counter example of minimal order. If K is a proper subgroup of G containing M, then $M \lhd K$, because $(L \cap K : C \cap K) \leq (L:C) = q$. Now we show that

$$(4.2) \qquad\qquad (M \times H) \cap x(M \times H)x^{-1} = Z,$$

the center of G, if $x \notin N(M \times H)$. If y belongs to the intersection but not to Z, then, by Hypothesis II(ii), $y \in H \cap xHx^{-1}$. Then M and xMx^{-1} are contained in $C(y)$. Since $C(y) \neq G$, we have $M \lhd C(y)$, and hence $C(y) \subset L$. According to Lemma 3.1, $(q, m) = 1$, and therefore

$$xMx^{-1} \subset M \times H.$$

Hence by Hypothesis II(ii), $xMx^{-1} \subset xHx^{-1}$. Then $M \subset H$, which is impossible. Thus $(M \times H) \cap x(M \times H)x^{-1} \subset Z$. On the other hand, Hypothesis II(i) implies that $Z \subset H$, and hence that Z is contained in the intersection (4.2). This proves (4.2).

Suppose G has a proper normal subgroup K with $M \subset K$. The minimality of G implies that $K \subset N(M)$, and hence all the conjugates of M are in $N(M)$. According to Lemma 3.1, $(q, m) = 1$. Hence all the conjugates of M are contained in $M \times H$. Since M is not normal in G, (4.2) implies that $M \subset Z$, which is impossible. This shows that G has no proper normal subgroup which contains M.

Theorem 4.2 shows that either $q = 1$ or M is abelian. In the latter case we can apply a lemma due to Feit [6, Lemma 4.2]. Our Hypothesis II, (4.2), the fact that M is not normal in G, and the fact proved in the preceding paragraph are special cases of Feit's hypotheses. The conclusion is that every non-linear irreducible character of G has degree $> m^{1/2} - 1$. Therefore the constituents of X are linear, and G is abelian, an impossibility. Hence $q = 1$.

Each element x of G produces a permutation of the conjugates of $M \times H$ by conjugation, and by (4.2) if x leaves two different conjugates fixed, then $x \in Z$. Hence G/Z may be regarded as a transitive group of permutations such that no non-identity element leaves two symbols fixed. By a well-known theorem of Frobenius, it follows that the elements of G which leave no conjugate of $M \times H$ fixed together with the elements of Z form a normal subgroup K of G of order $(G:M \times H)\,|\,Z\,|$. That is, G/Z is a Frobenius group with regular subgroup K/Z.

We have $K \neq Z$ since $G \neq M \times H$. Therefore X has an irreducible constituent with character, say, χ, which does not represent all the elements of K by multiples of the identity matrix. Then $\chi\bar{\chi}$ contains Z in its kernel but not K. The degree of every irreducible representation of a Frobenius group whose kernel does not contain the regular subgroup is a multiple of the index of the regular subgroup. (Cf. (3.1) and the remarks following it.) Hence $\chi\bar{\chi}(1) \geq m$, and $\chi(1) \geq m^{1/2}$. Again we have a contradiction, and the theorem is proved.

Remark. The new simple groups recently discovered by Suzuki satisfy the hypotheses of our Theorem 4.2 and fall under case (b) (see [7], especially §§13–17). Thus case (b) cannot be dropped from this theorem. However, it is not known to the author whether the corresponding case in the conclusion of Theorem 4.3 can be dropped.

REFERENCES

1. R. BRAUER, *On the connection between the ordinary and the modular characters of groups of finite order*, Ann. of Math. (2), vol. 42 (1941), pp. 926–935.
2. R. BRAUER AND H. S. LEONARD JR., *On finite groups with an abelian Sylow group*, Canad. J. Math., vol. 14 (1962), pp. 436–450.
3. C. W. CURTIS AND I. REINER, *Representation theory of finite groups and associative algebras*, New York, Interscience Publishers, 1962.
4. W. FEIT, *On a class of doubly transitive permutation groups*, Illinois J. Math., vol. 4 (1960), pp. 170–186.

5. ———, *Group characters. Exceptional characters*, Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc., vol. 6 (1962), pp. 67–70.

6. ———, *Groups which have a faithful representation of degree less than $p - 1$*, Trans. Amer. Math. Soc., vol. 112 (1964), pp. 287–303.

7. M. SUZUKI, *On a class of doubly transitive groups*, Ann. of Math. (2), vol. 75 (1962), pp. 105–145.

8. J. G. THOMPSON, *Finite groups with fixed-point-free automorphisms of prime order*, Proc. Nat. Acad. Sci. U.S.A., vol. 45 (1959), pp. 578–581.

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