T-GROUPS AND THEIR GEOMETRY

BY

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Introduction

This paper began with the question: How can one describe the abstract Pappus configuration within its automorphism group $\mathfrak{G}$ of order 108? (For a discussion of this group, see e.g. Levi [2, pp. 108 and 109].) The answer is simple, and not surprising. To every point of the configuration corresponds an involution in $\mathfrak{G}$ that leaves this point and no other point fixed; to every line corresponds an involution in $\mathfrak{G}$ that leaves this line and no other line fixed. A point and a line are incident if and only if the corresponding involutions commute. These $9 + 9 = 18$ involutions generate $\mathfrak{G}$; the inner automorphisms of $\mathfrak{G}$ induce the automorphisms of the configuration.

$\mathfrak{G}$ contains a subgroup $\mathfrak{B}_9$ of order 9 (notation as in [2]) that acts as translation group on the 9 points, and a subgroup $\mathfrak{B}_8$ of order 9 that acts, dually, as translation group on the 9 lines. $\mathfrak{B}_9$ and $\mathfrak{B}_8$ generate a group $\mathfrak{B}_5$ of order 27, and are normal in $\mathfrak{B}_5$. The group $\mathfrak{B}_3 = \mathfrak{B}_9 \cap \mathfrak{B}_8$ is a direct factor of $\mathfrak{B}_9$ and of $\mathfrak{B}_8$. It is also possible to describe the Pappus configuration in $\mathfrak{B}_5$ by identifying the elements of $\mathfrak{B}_9$ with the 9 points and the elements of $\mathfrak{B}_8$ with the 9 lines.

Analytically, the Pappus configuration can be described by deleting the vertical lines of an affine plane over the field GF(3) with 3 elements; $\mathfrak{G}$ is then represented by linear transformations over GF(3).

To generalize this situation, we call a group G a *T-group*, if $G = A \cdot B$ is a product of two abelian normal subgroups A and B, and if $C = A \cap B$ is a direct factor of A and of B. In §1 we associate with each T-group $G = A \cdot B$ a *P-system* (i.e. an incidence system with parallelism) $\langle A, B\rangle$. In §8 we construct for T-groups without elements of order 2, a semidirect product $\Omega = V \cdot G$, with a four-group V and G normal in Ω. In Ω one can define a P-system $\langle P_0, L_0\rangle$ in terms of the involutions of Ω, and $\langle P_0, L_0\rangle$ is isomorphic to $\langle A, B\rangle$. The inner automorphisms of Ω induce collineations in $\langle P_0, L_0\rangle$; under certain mild conditions, the group of all collineations and dualities of $\langle P_0, L_0\rangle$ is canonically isomorphic with the automorphism group of Ω.

Let $\mathfrak{E}$ be a projective plane that is (Y, Y)- and (ω, ω)-transitive for some $Y \mid \omega$ (i.e. a plane over a distributive quasi-field). Let A be the group of all translations with axis ω and B the group of all translations with center Y. Then $G = A \cdot B$ is a T-group. (The T indicates that A and B are translation groups). The group Ω is now the group generated by all point-reflections with axis ω and all line-reflections with center Y. If $\mathfrak{E}$ is the projective plane

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over the field GF(3), the groups Ω, G, A, B, C are linear representations of the groups $\mathfrak{S}$, $\mathfrak{V}_5$, $\mathfrak{V}_9$, $\mathfrak{V}_8$, $\mathfrak{V}_3$ resp.

Instead of considering planes over distributive quasi-fields, we consider in §4 P-systems over arbitrary rings, and their associated T-groups. In §5 *transitive P-systems* are characterized by configurations and transitivity properties. A transitive P-system possesses a *regular T-group* of collineations, and in §6 a coordinate ring R is defined in terms of multiplication and commutation in this group. In this way, one gets a one-to-one correspondence between *regular T-groups*, *transitive P-systems*, and *rings R with* 1 (where R is determined up to isotopisms). In §2 we consider homomorphisms of a T-group $G = A \cdot B$, and induced homomorphisms of the associated P-system $\langle A, B\rangle$.

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1. T-groups and P-systems

A system $\langle \mathfrak{A}, \mathfrak{B}\rangle$ consisting of a set $\mathfrak{A}$, a set $\mathfrak{B}$, and an incidence relation, denoted by $\mathfrak{a} \mid \mathfrak{b}$ for $\mathfrak{a} \in \mathfrak{A}$ and $\mathfrak{b} \in \mathfrak{B}$, is an incidence system. The elements of $\mathfrak{A}$ are called points and the elements of $\mathfrak{B}$ lines. The system $\langle \mathfrak{A}, \mathfrak{B}\rangle^{du} = \langle \mathfrak{B}, \mathfrak{A}\rangle$, with $\mathfrak{b} \mid \mathfrak{a}$ if and only if $\mathfrak{a} \mid \mathfrak{b}$, is the dual of $\langle \mathfrak{A}, \mathfrak{B}\rangle$.

Suppose there exists an equivalence relation on $\mathfrak{A}$, denoted by $\mathfrak{a}_1 \parallel \mathfrak{a}_2$, and an equivalence relation on $\mathfrak{B}$, denoted by $\mathfrak{b}_1 \parallel \mathfrak{b}_2$.

We call $\langle \mathfrak{A}, \mathfrak{B}\rangle$ an *incidence system with parallelism* or *P-system*, if the following holds:

Given a point $\mathfrak{a}_1$ and a line $\mathfrak{b}_1$, then there is

(i) exactly one point $\mathfrak{a}_2$ such that $\mathfrak{a}_2 \mid \mathfrak{b}_1$ and $\mathfrak{a}_2 \parallel \mathfrak{a}_1$,

and

(ii) exactly one line $\mathfrak{b}_2$ such that $\mathfrak{b}_2 \mid \mathfrak{a}_1$ and $\mathfrak{b}_2 \parallel \mathfrak{b}_1$.

(For a similar concept, see Sperner [4].) The dual of a P-system is a P-system.

A typical example of a P-system is the affine plane over a field, minus its vertical lines. Hence if we say that two parallel points have the same "abscissa" and that two parallel lines have the same "slope", we can restate (i) and (ii):

(i) Through every point there is exactly one line with given slope.

(ii) On every line there is exactly one point with given abscissa.

A homomorphism of a P-system $\langle \mathfrak{A}, \mathfrak{B}\rangle$ onto a P-system $\langle \mathfrak{A}', \mathfrak{B}'\rangle$ is a pair of mappings of $\mathfrak{A}$ onto $\mathfrak{A}'$ and of $\mathfrak{B}$ onto $\mathfrak{B}'$ that preserve incidence and parallelism. (Corresponding definitions hold for isomorphism, collineation and duality.) $\langle \mathfrak{A}, \mathfrak{B}\rangle$ is self-dual if there exists an isomorphism of $\langle \mathfrak{A}, \mathfrak{B}\rangle$ onto the dual $\langle \mathfrak{B}, \mathfrak{A}\rangle$.

Remark. A homomorphism φ that is one-to-one, is an isomorphism.

Proof. We have to show that φ^{-1} preserves incidence and parallelism.

Suppose at first that $\mathfrak{a}\varphi \mid \mathfrak{b}\varphi$. There exists exactly one $\mathfrak{a}_1$ such that $\mathfrak{a}_1 \mid \mathfrak{b}$ and $\mathfrak{a}_1 \parallel \mathfrak{a}$. Then $\mathfrak{a}_1\varphi \mid \mathfrak{b}\varphi$, $\mathfrak{a}_1\varphi \parallel \mathfrak{a}\varphi$, together with $\mathfrak{a}\varphi \mid \mathfrak{b}\varphi$, imply that $\mathfrak{a}_1\varphi = \mathfrak{a}\varphi$; hence $\mathfrak{a}_1 = \mathfrak{a} \mid \mathfrak{b}$. Suppose next that $\mathfrak{b}_1\varphi \parallel \mathfrak{b}_2\varphi$. Take $\mathfrak{a} \mid \mathfrak{b}_1$. There exists exactly one $\mathfrak{b}_3$ such that $\mathfrak{b}_3 \mid \mathfrak{a}$ and $\mathfrak{b}_3 \parallel \mathfrak{b}_2$. Then $\mathfrak{a}\varphi \mid \mathfrak{b}_1\varphi$, $\mathfrak{a}\varphi \mid \mathfrak{b}_3\varphi$, and $\mathfrak{b}_1\varphi \parallel \mathfrak{b}_2\varphi \parallel \mathfrak{b}_3\varphi$ imply that $\mathfrak{b}_1\varphi = \mathfrak{b}_3\varphi$; hence $\mathfrak{b}_1 = \mathfrak{b}_3 \parallel \mathfrak{b}_2$.

Notation. $\mathfrak{L}(\mathfrak{a})$ is the line-pencil of all lines $\mathfrak{b} \mid \mathfrak{a}$; $\mathfrak{P}(\mathfrak{b})$ is the point-row of all points $\mathfrak{a} \mid \mathfrak{b}$.

A group $G = A \cdot B$ that is the product of two abelian normal subgroups A and B, has the following well-known properties:

$C = A \cap B$ is contained in the center $Z(G)$ of G.

The derived group $D(G)$ of G is contained in C.

Notation. a, b, c, g shall always denote elements in A, B, C, G resp. $(g, h) = g^{-1}h^{-1}gh$ is the commutator of g and h in G. If H and K are complexes in G, then (H, K) is the set of all commutators (h, k) with $h \in H$ and $k \in K$.

Since $D(G) \subseteq Z(G)$, we have

$$(g_1 g_2, g_0) = (g_1, g_0)(g_2, g_0)$$

and

$$(g_0, g_1 g_2) = (g_0, g_1)(g_0, g_2);$$

i.e. the mappings $g \to (g, g_0)$ and $g \to (g_0, g)$ are homomorphisms of G into C.

PROPOSITION 1. *Let $G = A \cdot B$ be the product of two subgroups A and B of G, $B \lhd G$. (Hence $(A, B) \subseteq B$.) If α and β are two homomorphisms of A and B into a group H such that $\alpha = \beta$ on $A \cap B$, and $(b\beta, a\alpha) = (b, a)\beta$ for all $a \in A$ and $b \in B$, then α and β can be extended to a (unique) homomorphism η of G into H.*

Proof. If $a_1 b_1 = a_2 b_2$, then $a_2^{-1}a_1 = b_2 b_1^{-1}$ in $A \cap B$; hence

$$(a_2\alpha)^{-1}a_1\alpha = b_2\beta(b_1\beta)^{-1} \quad \text{or} \quad a_1\alpha b_1\beta = a_2\alpha b_2\beta.$$

Therefore $(ab)\eta = a\alpha\, b\beta$ is well defined on G.

$$\begin{aligned}(a_1 b_1 a_2 b_2)\eta &= (a_1 a_2 b_1(b_1, a_2)b_2)\eta \\ &= a_1\alpha a_2\alpha b_1\beta(b_1\beta, a_2\alpha)b_2\beta = a_1\alpha b_1\beta a_2\alpha b_2\beta \\ &= (a_1 b_1)\eta(a_2 b_2)\eta.\end{aligned}$$

Suppose that a group $G = A \cdot B$ is product of two abelian normal subgroups A and B and that $C = A \cap B$ is a direct factor of A and of B, say $A = A_0 \times C$ and $B = B_0 \times C$. We call such a group G, together with the system of subgroups A_0, B_0, C, a *T-group*.

Since $A_0 \cap B = e$, $G = A_0 B$ is a semidirect product, and every $g \in G$ has a unique representation $g = abc$ with $a \in A_0$, $b \in B_0$, $c \in C$.

With a T-group $G = A \cdot B$ we associate an incidence system $\langle A, B\rangle$ with points A and lines B by defining

$$a_0 c_1 \mid b_0 c_2 \quad \Leftrightarrow \quad (a_0, b_0) = c_1 c_2^{-1} \qquad (a_0 \in A_0, b_0 \in B_0, c_i \in C).$$

Furthermore we define

$$a \parallel a' \quad \Leftrightarrow \quad a \equiv a' \bmod C,$$

$$b \parallel b' \quad \Leftrightarrow \quad b \equiv b' \bmod C.$$

If $G = A \cdot B$ is a T-group, the "dual" group $G^{du} = B \cdot A$ is a T-group with the associated incidence system $\langle B, A \rangle$, the dual of $\langle A, B \rangle$. Hence the duality principle is valid: we have to prove only one of two dual statements.

PROPOSITION 2. *If $G = A \cdot B$ is a T-group, then $\langle A, B \rangle$ is a P-system, and*

$$\mathfrak{L}(a_0 c) = a_0^{-1} B_0 a_0 c, \qquad \mathfrak{P}(b_0 c) = b_0^{-1} A_0 b_0 c.$$

Proof. a_0 and b_0 denote elements in A_0 and B_0 resp. We have $a_0 c' \mid b_0 c$ if and only if $c' = (a_0, b_0)c$. Hence

$$\mathfrak{P}(b_0 c) = \text{set of all } a_0(a_0, b_0)c = b_0^{-1} A_0 b_0 c.$$

Similarly $\mathfrak{L}(a_0 c) = a_0^{-1} B_0 a_0 c$. $\mathfrak{L}(a_0 c)$ contains exactly one line $b \parallel b_0$, namely $b = b_0(b_0, a_0)c$. $\mathfrak{P}(b_0 c)$ contains exactly one point $a \parallel a_0$, namely $a = a_0(a_0, b_0)c$.

For T-groups, Proposition 1 can be stated as follows:

PROPOSITION 1′. *Let $G = A \cdot B$ be a T-group with $A = A_0 \times C$ and $B = B_0 \times C$. Suppose that α, β, γ are homomorphisms of A_0, B_0, C resp. into a group H such that*
(i) *$a\alpha$ and $b\beta$ commute with $c\gamma$, and*
(ii) *$(a\alpha, b\beta) = (a, b)\gamma$ (for all $a \in A_0$, $b \in B_0$, $c \in C$).*
Then α, β, γ can be extended to a homomorphism of G into H.

THEOREM 1. *If $G = A \cdot B$ is a T-group, then G has a faithful representation G^* as collineation group of $\langle A, B \rangle$. A^* is sharply transitive on the points, B^* is sharply transitive on the lines, and G^* is sharply transitive on the incident point-line-pairs, of $\langle A, B \rangle$.*

Proof. Let $A = A_0 \times C$ and $B = B_0 \times C$. For $a_0 \in A_0$ define the map $a_0 \alpha = [a_0 \rho, a_0 \sigma]$ by

$$a(a_0 \rho) = a a_0 \qquad \text{on } A,$$

$$b(a_0 \sigma) = a_0^{-1} b a_0 \quad \text{on } B.$$

(Note that $c(a_0 \rho) \neq c(a_0 \sigma)$.) $a_0 \rho$ is a permutation of A. Since $B \lhd G$, $a_0 \sigma$ is a permutation of B. Put $a = a_1 c$ with $a_1 \in A_0$. Then

$$\mathfrak{L}(a)(a_0 \sigma) = (a_1^{-1} B_0 a_1 c)(a_0 \sigma) = a_0^{-1} a_1^{-1} B_0 a_1 a_0 c = \mathfrak{L}(a_1 a_0 c) = \mathfrak{L}(a(a_0 \rho));$$

hence $a_0 \alpha$ preserves incidence. $a_0 \rho$ clearly maps parallel points onto parallel points; and $b(a_0 \sigma) = b(b, a_0) \parallel b$. Hence $a_0 \alpha$ is a collineation of $\langle A, B \rangle$. It follows now from the definition of α, that α is an isomorphism of A_0 into the collineation group K of $\langle A, B\rangle$.

Similarly define $b_0\beta = [b_0\rho, b_0\sigma]$ for $b_0 \in B_0$ by

$$a(b_0\rho) = b_0^{-1}ab_0 \quad \text{on } A,$$

$$b(b_0\sigma) = bb_0 \quad \text{on } B.$$

β is an isomorphism of B_0 into K.

Define $c\gamma = [c\rho, c\sigma]$ for $c \in C$ by

$$a(c\rho) = ac \quad \text{on } A,$$

$$b(c\sigma) = bc \quad \text{on } B.$$

It is easy to check that γ is an isomorphism of C into K.

$a_0\rho$ and $b_0\rho$ commute with $c\rho$, and $a_0\sigma$ and $b_0\sigma$ commute with $c\sigma$; i.e. $a_0\alpha$ and $b_0\beta$ commute with $c\gamma$.

We have

$$a(a_0\rho, b_0\rho) = b_0^{-1}b_0aa_0^{-1}b_0^{-1}a_0b_0 = a\cdot(a_0, b_0)\rho;$$

and similarly

$$b(a_0\sigma, b_0\sigma) = b\cdot(a_0, b_0)\sigma.$$

Hence $(a_0\alpha, b_0\beta) = (a_0, b_0)\gamma$, and α, β, γ can be extended to a homomorphism $*$ of G into K by Prop. 1′.

Let $g = a_0b_0c$ and suppose that point $e(g\rho) = e$ and line $e(g\sigma) = e$. Then $e(g\sigma) = b_0c = e$ implies that

$$b_0 = c = e, \quad \text{and} \quad e(g\rho) = e(a_0\rho) = a_0 = e;$$

hence $g = e$. We have proved that if $e(g\rho) = e$ and $eg(\sigma) = e$, then $g = e$. In particular if g^* is the identity on $\langle A, B\rangle$, then $g = e$; hence $*$ is an isomorphism of G into K.

As a consequence of the definition of $*$, A^* is sharply transitive on A and B^* is sharply transitive on B. With $g = a_0b_0c$, we have $e(g\rho) = a_0(a_0, b_0)c$ and $e(g\sigma) = b_0c$. Hence G^* is transitive on the incident point-line-pairs of $\langle A, B\rangle$. Since only e^* leaves point and line e fixed, G^* is actually sharply transitive.

PROPOSITION 3. *Suppose that a T-group $G = A\cdot B$ with $A = A_0 \times C$ and $B = B_0 \times C$ acts as collineation group on a P-system $\langle \mathfrak{A}, \mathfrak{B}\rangle$ such that*

(i) *A and B are sharply transitive on $\mathfrak{A}$ and $\mathfrak{B}$ resp.;*

(ii) *there exist $\mathfrak{a}_0 \mid \mathfrak{b}_0$ such that A_0 leaves $\mathfrak{b}_0$ and B_0 leaves $\mathfrak{a}_0$ fixed;*

(iii) *$\mathfrak{a}c \parallel \mathfrak{a}$ and $\mathfrak{b}c \parallel \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b}, c$.*

Then the map θ given by

$$\mathfrak{a}_0a \to a, \qquad \mathfrak{b}_0b \to b$$

is an isomorphism of $\langle \mathfrak{A}, \mathfrak{B}\rangle$ onto $\langle A, B\rangle$.

$$g\theta = \theta g^* \qquad \textit{for all } g \in G.$$

Proof. θ is one-to-one because of the sharp transitivity. $a_0 c_1 \mid b_0 c_2$ implies that $c_1 c_2^{-1} = a_0^{-1} b_0^{-1} a_0 b_0$, or $b_0 a_0 c_1 = a_0 b_0 c_2$. Hence $\mathfrak{a}_0 b_0 a_0 c_1 \mid \mathfrak{b}_0 a_0 b_0 c_2$, i.e. $\mathfrak{a}_0 a_0 c_1 \mid \mathfrak{b}_0 b_0 c_2$. Hence θ^{-1} preserves incidence. $\mathfrak{a}_0 ac \parallel \mathfrak{a}_0 a$ and $\mathfrak{b}_0 bc \parallel \mathfrak{b}_0 b$ imply that θ^{-1} preserves parallelism. Let $g = a_0 b_0 c$. Then

$$\mathfrak{a}_0 a\theta g^* \theta^{-1} = ag^* \theta^{-1} = b_0^{-1} a a_0 b_0 c \theta^{-1} = \mathfrak{a}_0 ag.$$

$$\mathfrak{b}_0 b\theta g^* \theta^{-1} = bg^* \theta^{-1} = a_0^{-1} b a_0 b_0 c \theta^{-1} = \mathfrak{b}_0 bg.$$

Hence $\theta g^* \theta^{-1} = g$.

$$\begin{tikzcd} \langle \mathfrak{A}, \mathfrak{B} \rangle \arrow[r, "\theta"] \arrow[d, "g"'] & \langle A, B \rangle \arrow[d, "g^*"] \\ \langle \mathfrak{A}, \mathfrak{B} \rangle \arrow[r, "\theta"] & \langle A, B \rangle \end{tikzcd}$$

Let $G = A \cdot B$ be a T-group, and put $Z = Z(G)$, $A_1 = A_0 \cap Z$ and $B_1 = B_0 \cap Z$.

PROPOSITION 4.

$$Z = A_1 \times B_1 \times C;$$

$$\mathfrak{L}(a_1 c_1) \subseteq \mathfrak{L}(a_1 c_2) \iff a_1 \equiv a_2 \bmod A_1 \quad \textit{and} \quad c_1 = c_2;$$

$$\mathfrak{P}(b_1 c_1) \subseteq \mathfrak{P}(b_2 c_2) \iff b_1 \equiv b_2 \bmod B_1 \quad \textit{and} \quad c_1 = c_2$$

$$(\textit{for } a_i \in A_0,\ b_i \in B_0,\ c_i \in C).$$

Proof. If $g = abc$ is in Z, then $b'g = gb'$ implies that $b'a = ab'$; hence $a \in Z$, and similarly $b \in Z$, which proves that $Z = A_1 \times B_1 \times C$. Note that $a \in Z$ if and only $(a, B_0) = e$, and $b \in Z$ if and only if $(A_0, b) = e$.

We have

$$\mathfrak{L}(a_i c_i) = a_i^{-1} B_0 a_i c_i.$$

$a_1^{-1} B_0 a_1 c_1 \subseteq a_2^{-1} B_0 a_2 c_2$ implies that $a^{-1} B_0 ac \subseteq B_0$, where $a = a_1 a_2^{-1}$ and $c = c_1 c_2^{-1}$. $a^{-1} eac = c \in B_0$ implies that $c = e$. But then for every $b \in B_0$, $(a, b) = a^{-1} b^{-1} ab \in B_0 b = B_0$; hence $(a, B_0) = e$, and $a \in Z$.

The converse follows from $a_1^{-1} B_0 a_1 = a_2^{-1} B_0 a_2$. Note that $\mathfrak{L}(a) \subseteq \mathfrak{L}(a')$ implies $\mathfrak{L}(a) = \mathfrak{L}(a')$.

COROLLARY 4. *The mappings $a \to \mathfrak{L}(a)$ and $b \to \mathfrak{P}(b)$ are one-to-one if and only if $Z(G) = C$.*

PROPOSITION 5. *The following two conditions are equivalent:*

(i) $(a, b) = e$ *implies* $a \in C$ or $b \in C$ (*for all* $a \in A$ *and* $b \in B$);

(ii) $\langle A, B \rangle$ *is a partial plane.*

Proof. "Partial plane" means as usual that two points have at most one line in common, and two lines have at most one point in common. If $a = a_0 c_1$ and $b = b_0 c_2$ with $a_0 \in A_0$ and $b_0 \in B_0$, then $(a, b) = (a_0, b_0)$. Hence suppose at first that $(a_0, b_0) = e$ with $a_0 \neq e$ in A_0 and $b_0 \neq e$ in B_0. Then $e, a_0 \mid e, b_0$ and $\langle A, B\rangle$ is not a partial plane. Conversely if $\langle A, B\rangle$ is not a partial plane, then there are two distinct points incident with two distinct lines. Because of the transitivity of G^*, we may assume that point and line e are two of the elements. We have $e, a \mid e, b$. $a \mid e$ implies that a $(\neq e)$ in A_0, and $e \mid b$ implies that b $(\neq e)$ in B_0 ; $a \mid b$ implies that $(a, b) = e$.

Note that condition (i) is stronger than $C = Z(G)$.

Put $K = A_1 \times B_1$ with A_1 and B_1 as above in Prop. 4. Let $\mathfrak{A}$ be the set of all line-pencils $\mathfrak{L}(a)$ and $\mathfrak{B}$ the set of all point-rows $\mathfrak{P}(b)$. Define incidence and parallelism on $\langle \mathfrak{A}, \mathfrak{B}\rangle$ by

$$\mathfrak{L}(a) \mid \mathfrak{P}(b) \quad \text{if} \quad a \mid b,$$

$$\mathfrak{L}(a) \parallel \mathfrak{L}(a') \quad \text{if} \quad a \equiv a' \bmod A_1 C,$$

$$\mathfrak{P}(b) \parallel \mathfrak{P}(b') \quad \text{if} \quad b \equiv b' \bmod B_1 C.$$

PROPOSITION 6. *$G/K = AK/K \cdot BK/K$ is a T-group. The maps $a \to \mathfrak{L}(a)$ and $b \to \mathfrak{P}(b)$ induce an isomorphism of $\langle AK/K, BK/K\rangle$ onto $\langle \mathfrak{A}, \mathfrak{B}\rangle$.*

Proof. A_1 and B_1 are subgroups of Z, hence normal in G; hence K is normal in G, and AK/K and BK/K are normal in G/K. Since

$$AK = A_0 \times B_1 \times C \quad \text{and} \quad BK = A_1 \times B_0 \times C,$$

we have

$$AK \cap BK = K \times C, \quad AK/K \simeq (A_0/A_1) \times C, \quad \text{and} \quad BK/K \simeq (B_0/B_1) \times C.$$

Note that $(a_0, B) \subseteq K$ implies that $a_0 \in A_1$, and $(A, b_0) \subseteq K$ implies that $b_0 \in B_1$. Hence $Z(G/K) = CK/K \simeq C$. (See also Prop 10.)

Call two elements of an incidence system $\langle \mathfrak{A}, \mathfrak{B}\rangle$ *connected* if they are connected by an incidence chain, as e.g. $\mathfrak{a}_1$ and $\mathfrak{b}_4$ in $\mathfrak{a}_1 \mid \mathfrak{b}_2 \mid \mathfrak{a}_3 \mid \mathfrak{b}_4$. Connected is clearly an equivalence relation, and every incidence system is the union of pairwise disconnected components (equivalence classes).

Let $G = A \cdot B$ be a T-group with $A = A_0 \times C$ and $B = B_0 \times C$, and $D = D(G)$ the derived group of G. Put $A_1 = A_0 \times D$ and $B_1 = B_0 \times D$.

PROPOSITION 7. *$G_1 = A_1 \cdot B_1$ is a T-group. The components of $\langle A, B\rangle$ are the $[C{:}D]$ translates $\langle A_1, B_1\rangle(c^*)$ of $\langle A_1, B_1\rangle$.*

Proof. $a_0 c_1 \mid b_0 c_2$ implies that $(a_0, b_0) = c_1 c_2^{-1}$; hence $c_1 \equiv c_2 \bmod D$. Hence if $a_0 c_1$ and $b_0 c_2$ are connected, then $c_1 \equiv c_2 \bmod D$. To prove the converse, observe that the relation "connected" is preserved under collineations of $\langle A, B\rangle$. Hence if e, c_1 and c_2 are connected, then $c_1(c_2^*)^{-1} = c_1 c_2^{-1}$ and $c_2(c_2^*)^{-1} = e$ are connected; i.e. the c's that are connected with e, form a subgroup of C. This subgroup contains D, since $c = (a_0, b_0)$ implies that $e \mid b_0 \mid a_0 c \mid c$, and—as proved above—is contained in D. Hence e and c are connected if and only if c in D, or c_1 and c_2 are connected if and only if

$c_1 \equiv c_2 \bmod D$. Clearly $a_0 c_1$ and $b_0 c_2$ connected if and only if c_1 and c_2 connected.

COROLLARY 7. *$C = D$ if and only if $\langle A, B\rangle$ is connected* (i.e. *has only one component*).

PROPOSITION 8. *The following three statements are equivalent.*
(i) *Every $c \in C$ is a commutator $c = (a, b)$.*
(ii) *Every line b intersects (in a point) at least one line of every pencil $\mathfrak{L}(a)$.*
(iii) *Every point a is joined (by a line) to at least one point of every row $\mathfrak{P}(b)$.*

Proof. Suppose that (ii) holds, and (line) c is given. There is a point $a_0 c \mid c$ and a line $b_0 \mid e$, such that $a_0 c \mid b_0$, i.e. $(a_0, b_0) = c$. Hence (ii) implies (i), and similarly (iii) implies (i). Conversely consider line b and pencil $\mathfrak{L}(a)$. We have $a(a\rho)^{-1} = e$ and $b(a\sigma)^{-1} = b_0 c$, say. Then $e(b_0 \rho)^{-1} = e$ and $b_0 c(b_0 \sigma)^{-1} = c$. Hence we may assume that $b = c$ and $a = e$. $c = (a_0, b_0)$ for some $a_0 \in A_0$ and $b_0 \in B_0$ means that $a_0 c \mid b_0$ with $a_0 c \mid c$ and $b_0 \in \mathfrak{L}(e)$.

Note that condition (i) is stronger than $C = D$.

In every T-group $G = A \cdot B$, we have $D \subseteq C \subseteq Z$. $G = A \times B$ (with $C = e$) shows that we can have $D = C \neq Z$. $G = A = B = C$ shows that we can have $D \neq C = Z$.

2. Homomorphisms

Let $G = A \cdot B$ and $G' = A' \cdot B'$ be two T-groups with $A = A_0 \times C$, $B = B_0 \times C$ and $A' = A_0' \times C'$, $B' = B_0' \times C'$. We call a homomorphism φ of G onto G' that maps A_0 onto A_0', B_0 onto B_0' and C onto C', a *T-homomorphism* of G onto G'.

PROPOSITION 9. *If φ is a T-homomorphism of the T-group $G = A \cdot B$ onto $G' = A' \cdot B'$, then φ induces a homomorphism φ^* of $\langle A, B\rangle$ onto $\langle A', B'\rangle$.*

$$g^* \varphi^* = \varphi^*(g\varphi)^* \qquad \textit{for all } g \in G.$$

Conversely suppose that every $c \in C$ is a commutator $c = (a, b)$ and that $C' = Z(G')$. Then every homomorphism of $\langle A, B\rangle$ onto $\langle A', B'\rangle$ is a product $\varphi^* g'^*$, with uniquely determined homomorphism φ of G onto G', and $g' \in G'$.

Proof. Let φ be a given T-homomorphism. Since $C\varphi = C'$, φ preserves parallelism. Suppose that $ac_1 \mid bc_2$ ($a \in A_0$, $b \in B_0$). Then $(a, b) = c_1 c_2^{-1}$; hence

$$(a\varphi, b\varphi) = c_1 \varphi(c_2 \varphi)^{-1},$$

i.e. $(ac_1)\varphi \mid (bc_2)\varphi$. Hence φ preserves incidence.

$$(a(g\rho))\varphi = (a\varphi)(g\varphi\rho)$$

and

$$(b(g\sigma))\varphi = (b\varphi)(g\varphi\sigma)$$

prove that $g^* \varphi^* = \varphi^*(g\varphi)^*$.

$$\begin{tikzcd} \langle A, B\rangle \arrow[r, "\varphi^*"] \arrow[d, "g^*"] & \langle A', B'\rangle \arrow[d, "(g\varphi)^*"] \\ \langle A, B\rangle \arrow[r, "\varphi^*"] & \langle A', B'\rangle \end{tikzcd}$$

Conversely let κ_1 be a homomorphism of $\langle A, B\rangle$ onto $\langle A', B'\rangle$. Then there exists a unique $g' \in G'$ such that point $e\kappa_1 = e'g'\rho$ and line $e\kappa_1 = e'g'\sigma$. Then $\kappa = \kappa_1(g'^*)^{-1}$ is a homomorphism of $\langle A, B\rangle$ onto $\langle A', B'\rangle$ that maps point and line e onto e'.

$e \parallel c$ implies that $e' \parallel c\kappa$; hence $C\kappa \subseteq C'$ (for points and lines). Suppose that κ maps point c onto c' and line c onto c''. Then $c \mid c$ implies that $c' \mid c''$, i.e. $c' = c''$, and we are justified in writing $c\kappa$ for the image of point and line c. $a \mid e$ implies that $a\kappa \mid e'$; i.e. $A_0\,\kappa \subseteq A_0'$ and similarly $B_0\,\kappa \subseteq B_0'$.

Let $a \in A_0$. $ac \mid c$ and $ac \parallel a$ imply that $(ac)\kappa \mid c\kappa$ and $(ac)\kappa \parallel a\kappa$; hence $(ac)\kappa = a\kappa \cdot c\kappa$. Similarly for $b \in B_0$, $(bc)\kappa = b\kappa \cdot c\kappa$. Therefore $A' = A\kappa = A_0\,\kappa \cdot C\kappa \subseteq A_0'\,C'$ implies that $A_0\,\kappa = A_0'$ and $C\kappa = C'$, and similarly $B_0\,\kappa = B_0'$.

Given c_1 and c_2 in C, there are $a \in A_0$ and $b \in B_0$ such that $(a, b) = c_1\,c_2^{-1}$, i.e. $ac_1 \mid bc_2$ as well as $ac_1\,c_2^{-1} \mid b$. These two incidences imply that

$$a\kappa \cdot c_1\,\kappa \mid b\kappa \cdot c_2\,\kappa \quad \text{and} \quad a\kappa(c_1\,c_2^{-1})\kappa \mid b\kappa,$$

i.e. that

$$(a\kappa, b\kappa) = c_1\,\kappa(c_2\,\kappa)^{-1} \quad \text{and} \quad (a\kappa, b\kappa) = (c_1\,c_2^{-1})\kappa.$$

Hence $c_1\,\kappa(c_2\,\kappa)^{-1} = (c_1\,c_2^{-1})\kappa$; i.e. κ induces a homomorphism on C.

We have $a(a, b) \mid b$ $(a \in A_0,\ b \in B_0)$. Hence $a\kappa(a, b)\kappa \mid b\kappa$, i.e. $(a\kappa, b\kappa) = (a, b)\kappa$. Then

$$\begin{aligned}(a\kappa, (b_1\,b_2)\kappa) &= (a, b_1\,b_2)\kappa = ((a, b_1)(a, b_2))\kappa \\ &= (a, b_1)\kappa(a, b_2)\kappa = (a\kappa, b_1\,\kappa)(a\kappa, b_2\,\kappa) = (a\kappa, b_1\,\kappa \cdot b_2\,\kappa),\end{aligned}$$

or

$$(a\kappa, (b_1\,b_2)\kappa(b_1\,\kappa)^{-1}(b_2\,\kappa)^{-1}) = e'.$$

If a runs through A_0, then $a\kappa$ runs through A_0', and $C' = Z(G')$ implies that

$$(b_1\,b_2)\kappa(b_1\,\kappa)^{-1}(b_2\,\kappa)^{-1} = e';$$

i.e. κ induces a homomorphism on B_0, and similarly on A_0. Together with $(a\kappa, b\kappa) = (a, b)\kappa$, κ can be extended by Prop. 1′ to a T-homomorphism φ of G onto G'; i.e. $\kappa = \varphi^*$ and $\kappa_1 = \varphi^* g'^*$.

Denote by Φ_0 the group of all T-automorphisms of the T-group $G = A \cdot B$. Then we have the following as a consequence of Prop. 9.

THEOREM 2. *If in a T-group $G = A \cdot B$, every $c \in C$ is a commutator $c = (a,b)$ and if $C = Z(G)$, then the group* K *of all collineations of $\langle A, B\rangle$ is equal to the semidirect product* $\mathrm{K} = \Phi_0^* G^*$.

Remark. We can interpret $\Phi_0 \cdot G$ as subgroup of the holomorph of G. Elements g and φ are switched according to the rule

$$g \cdot \varphi = \varphi(g^{\varphi}).$$

($g \cdot \varphi$ is a product in the holomorph; g^{φ} is the image of g under φ.) Then the formula $g^* \varphi^* = \varphi^*(g\varphi)^*$ implies that the map $\varphi g \rightarrow \varphi^* g^*$ is an isomorphism of the subgroup $\Phi_0 \cdot G$ of the holomorph onto the collineation group $\Phi_0^* \cdot G^*$ of $\langle A, B \rangle$.

If $C \neq Z$ or if $D \neq C$, there are "in general" collineations κ of $\langle A, B \rangle$ (leaving point and line e fixed) that are not induced by automorphisms of G.

If $C \neq Z$, suppose e.g. that $A_1 = A_0 \cap Z \neq e$. Let α be a permutation of A_0 that leaves e and the cosets modulo A_1 invariant. If A_0 is not too small, there are such permutations α that are not automorphisms of A_0. Define κ by

$$(ac)\kappa = a\alpha c, \qquad (bc)\kappa = bc \qquad (a \in A_0,\ b \in B_0).$$

Then κ is a collineation of $\langle A, B \rangle$ that is not induced by an automorphism of G.

If $e \neq D \neq C$, take c not in D and $d \neq e$ in D. Put

$$\begin{aligned} \kappa &= d^* && \text{on } \langle A_0 Dc, B_0 Dc \rangle \\ &= \text{identity} && \text{otherwise.} \end{aligned}$$

Then κ is a collineation of $\langle A, B \rangle$. If $[C:D] \geqq 3$, there are c_1 and c_2, both $\neq c \bmod D$, such that $c = c_1 c_2$. Then $c_1 \kappa = c_1$, $c_2 \kappa = c_2$, but $(c_1 c_2)\kappa = c\kappa = c\,d \neq c_1 c_2$. If $[C:D] = 2$ and in addition $d^2 \neq e$, then $c\kappa = cd$, but $c^2\kappa = c^2 \neq (cd)^2$. In both cases, κ is not induced by an automorphism of G.

PROPOSITION 10. *Let $G = A \cdot B$ be a T-group with $A = A_0 \times C$ and $B = B_0 \times C$. Then the subgroup K of G is kernel of a T-homomorphism of G if and only if $K = A_1 B_1 C_1$ with A_1, B_1, C_1 subgroups of A_0, B_0, C resp. and $(A_1, B_0) \subseteq C_1$ and $(A_0, B_1) \subseteq C_1$.*

$K = A_1 C_1 \cdot B_1 C_1$ is a T-group.

Proof. If φ is a T-homomorphism of G with kernel K, put

$$A_1 = A_0 \cap K, \qquad B_1 = B_0 \cap K, \qquad C_1 = C \cap K.$$

Then

$$(A_1, B_0) \subseteq C \cap K = C_1, \quad \text{and} \quad (A_0, B_1) \subseteq C_1.$$

Hence $K = A_1 C_1 \cdot B_1 C_1$ is a T-group.

Conversely suppose that A_1, B_1, C_1 are subgroups of A_0, B_0, C resp. and that $(A_1, B_0) \subseteq C_1$ and $(A_0, B_1) \subseteq C_1$. B_0 normalizes $A_1 C_1$ and A_0 normalizes $B_1 C_1$. Hence $K = A_1 B_1 C_1 \lhd G$.

$$AK \cap BK = A_0 B_1 C \cap A_1 B_0 C = A_1 B_1 C = KC; \qquad AK = A_0 K \cdot CK.$$

$$A_0 K \cap CK = A_0 B_1 C_1 \cap A_1 B_1 C = A_1 B_1 C_1 = K.$$

$$A_0 K/K \simeq A_0/A_0 \cap K = A_0/A_1 ; \qquad CK/K \simeq C/C \cap K = C/C_1 .$$

Hence

$$G/K \simeq (A_0/A_1 \times C/C_1)\cdot(B_0/B_1 \times C/C_1).$$

Compute commutators according to the rule

$$(a_0 A_1 , b_0 B_1) = (a_0 , b_0)(a_0 , B_1)(A_1 , b_0)(A_1 , B_1) \equiv (a_0 , b_0) \mod C_1 .$$

Let H be the group of all automorphisms η of a T-group $G = A \cdot B$ such that for all $a \in A$, $b \in B$, $c \in C$, $a\eta \parallel a$, $b\eta \parallel b$, and $c\eta = c$, and suppose that

$$A = A_0 \times C = A_1 \times C, \qquad B = B_0 \times C = B_1 \times C.$$

PROPOSITION 11. *There exists exactly one automorphism $\eta \in \mathrm{H}$ that maps A_0 onto A_1 and B_0 onto B_1. η^* is an isomorphism of $\langle A, B\rangle_0$ onto $\langle A, B\rangle_1$.*

Proof. Define a mapping η as follows. η is the identity on C. If $a_0 \parallel a_1$ (with $a_i \in A_i$), define $a_0 \eta = a_1$; and if $b_0 \parallel b_1$ (with $b_i \in B_i$), define $b_0 \eta = b_1$. Then

$$(a_0 \eta, b_0 \eta) = (a_1 , b_1) = (a_0 , b_0) = (a_0 , b_0)\eta.$$

Hence by Prop. 1′, η can be extended to an automorphism of G. An automorphism η that maps A_0 onto A_0 and B_0 onto B_0, is clearly the identity. Hence η is uniquely determined. By Prop. 9, η^* is the desired isomorphism.

Remark. If $G = A \cdot B$ is a T-group, the structure of the associated P-system $\langle A, B\rangle$ does not depend on the choice of the direct factors A_0 and B_0.

Let $G = A \cdot B$ be a T-group with $A = A_0 \times C$ and $B = B_0 \times C$. Denote by Φ the group of all automorphisms φ of G that map A onto A and B onto B, (hence also C onto C), and by Φ_0 the subgroup of all T-automorphisms of G. Let $\varphi \in \Phi$ and $\eta \in \mathrm{H}$. Then

$$a\varphi^{-1}\eta \parallel a\varphi^{-1};$$

hence $a\varphi^{-1}\eta\varphi \parallel a$; similarly $b\varphi^{-1}\eta\varphi \parallel b$ and $c\varphi^{-1}\eta\varphi = c$. Hence $\varphi^{-1}\eta\varphi \in \mathrm{H}$, i.e. $\mathrm{H} \lhd \Phi$. Given φ, there exists exactly one η such that $A_0 \varphi = A_0 \eta$ and $B_0 \varphi = B_0 \eta$; hence $\varphi\eta^{-1} \in \Phi_0$. Since $\Phi_0 \cap \mathrm{H} = 1$; $\Phi = \Phi_0 \cdot \mathrm{H}$ is the semidirect product of Φ_0 and H.

3. Finite T-groups

PROPOSITION 12. *The direct product of two T-groups is a T-group.*

Proof. Let $G_i = A_i \times B_i$ be two T-groups with

$$A_i = A_{i0} \times C_i \quad \text{and} \quad B_i = B_{i0} \times C_i \qquad (i = 1, 2).$$

Then

$$G = G_1 \times G_2 = (A_1 \times A_2)\cdot(B_1 \times B_2)$$

with

$$C = A_1 \times A_2 \cap B_1 \times B_2 = C_1 \times C_2,$$

$$A_1 \times A_2 = A_{10} \times A_{20} \times C_1 \times C_2 \quad \text{and} \quad B_1 \times B_2 = B_{10} \times B_{20} \times C_1 \times C_2 .$$

Remark.

$$\text{center } Z(G_1 \times G_2) = Z(G_1) \times Z(G_2).$$

$$\text{commutator } (a_1 \times a_2, b_1 \times b_2) = (a_1, b_1) \times (a_2, b_2) \quad \text{with } a_i \in A_i, b_i \in B_i ;$$

hence

$$\text{derived group } D(G_1 \times G_2) = D(G_1) \times D(G_2).$$

Also every $c_1 \times c_2$ is a commutator if and only if every c_1 and every c_2 is a commutator ($c_i \in C_i$).

This suggests that we define the direct product of two P-systems as follows:

$$\langle \mathfrak{A}_1, \mathfrak{B}_1 \rangle \times \langle \mathfrak{A}_2, \mathfrak{B}_2 \rangle = \langle \mathfrak{A}_1 \times \mathfrak{A}_2, \mathfrak{B}_1 \times \mathfrak{B}_2 \rangle,$$

with incidence and parallelism defined by

$$\begin{aligned} \mathfrak{a}_1 \times \mathfrak{a}_2 \mid \mathfrak{b}_1 \times \mathfrak{b}_2 &\Leftrightarrow \mathfrak{a}_1 \mid \mathfrak{b}_1 \text{ and } \mathfrak{a}_2 \mid \mathfrak{b}_2 \\ \mathfrak{a}_1 \times \mathfrak{a}_2 \parallel \mathfrak{a}_1' \times \mathfrak{a}_2' &\Leftrightarrow \mathfrak{a}_1 \parallel \mathfrak{a}_1' \text{ and } \mathfrak{a}_2 \parallel \mathfrak{a}_2' \\ \mathfrak{b}_1 \times \mathfrak{b}_2 \parallel \mathfrak{b}_1' \times \mathfrak{b}_2' &\Leftrightarrow \mathfrak{b}_1 \parallel \mathfrak{b}_1' \text{ and } \mathfrak{b}_2 \parallel \mathfrak{b}_2' . \end{aligned}$$

This makes the direct product into a P-system; and especially for the direct product $G_1 \times G_2$ of two T-groups, we have

$$\langle A_1 \times A_2, B_1 \times B_2 \rangle \simeq \langle A_1, B_1 \rangle \times \langle A_2, B_2 \rangle.$$

If $a^m = e$, then $(a, b)^m = (a^m, b) = e$. Hence we have the following:
1. If $a^m = b^n = e$ and $(m, n) = 1$, then $(a, b) = e$, i.e. $ab = ba$;
2. if $a^{p^\alpha} = b^{p^\beta} = e$ and $\gamma = \min(\alpha, \beta)$, then $(a, b)^{p^\gamma} = e$.

Let $G = A \cdot B$ be a T-group with $A = A_0 \times C$ and $B = B_0 \times C$, and D the derived group of G. If A_0 has exponent p^α and B_0 has exponent p^β, then D has exponent at most p^γ, where $\gamma = \min(\alpha, \beta)$. (The exponent of a group G is the smallest positive integer k such that $g^k = e$ for all $g \in G$, provided k exists).

Proposition 13. *If in a T-group $G = A \cdot B$, A and B are p-groups, then G is a p-group. If p^m is the maximum of the exponents of A and B, then G has exponent p^m, except possibly in the case that exponent of A = exponent of B = 2^m, where G can also have exponent 2^{m+1}.*

Proof. We have

$$(ab)^{p^m} = a^{p^m} b^{p^m} (b, a)^{1+2+\cdots+(p^m-1)} = (b, a)^{(1/2)p^m(p^m-1)}.$$

If p is odd, then $(a, b)^{p^m} = e$. If $p = 2$ and if A and B have distinct exponents, then $(a, b)^{2^{m-1}} = e$. If $\exp(A) = \exp(B) = 2^m$, then

$$(ab)^{2^{m+1}} = (b, a)^{2^{m}(2^m-1)} = e.$$

An example for the exceptional case is the dihedral group $G = \{a, b\}$ of order 8 generated by the permutations $a = (14)(23)$ and $b = (24)$. Then $ab = (1234)$ and $c = (a, b) = (ab)^2 = (13)(24)$. $G = A \cdot B$ with $A = \{a, c\}$ and $B = \{b, c\}$.

A T-group $G = A \cdot B$ is nilpotent. Hence a finite T-group is direct product of its Sylow subgroups.

PROPOSITION 14. *The Sylow subgroups of a finite T-group $G = A \cdot B$ are T-groups, and G is their direct product.*

Proof. Let A_i and B_i be the p_i-Sylow subgroups of A and of B. Then $C_i = A_i \cap B_i$ is the p_i-Sylow subgroup of $C = A \cap B$. $A_i \lhd G$ since A_i is characteristic in A; similarly $B_i \lhd G$. Hence $G_i = A_i B_i \lhd G$. Since G_i is a p_i-group, $G_i \cap G_j = e$ for $i \neq j$; furthermore $(G_i, G_j) = e$, i.e. G_i and G_j commute elementwise. Hence $G = \text{dir} \prod G_i$, and the G_i's are the Sylow subgroups of G. $A = A_0 \times C$ implies that $A_i = (A_i \cap A_0) \times C_i$, and similarly $B_i = (B_i \cap B_0) \times C_i$.

PROPOSITION 15. *Let $G = A \cdot B$ be a finite T-group. If $\langle A, B\rangle$ is a connected partial plane, then G is a p-group and A and B are elementary abelian. If moreover $p \neq 2$, then G has exponent p.*

Proof. Let $A = A_0 \times C$ and $B = B_0 \times C$. Suppose $a_0 \in A_0$ has order p and $b_0 \in B_0$ has order q, $p \neq q$ primes. Then $(a_0, b_0) = e$, contradicting that $\langle A, B\rangle$ is a partial plane, (see Prop. 5). Therefore A_0 and B_0 are both p-groups for the same prime p. Then the derived group D is also a p-group, but $C = D$ since $\langle A, B\rangle$ is connected. Thus A and B are p-groups, hence $G = A \cdot B$ is a p-group. Suppose that A_0 has an element of order p^2, say a_0. Pick $b_0 \in B_0$ of order p. Then $(a_0^p, b_0) = (a_0, b_0^p) = e$ with both a_0^p and b_0 not in C, contradicting that $\langle A, B\rangle$ is a partial plane. Therefore A_0 and similarly B_0 are both elementary abelian, and so are A and B. If $p \neq 2$, this implies that G has exponent p.

4. T-groups associated with rings

Let R be a ring (associative or not). (x, y, u, v, s, t will denote elements of R.) Let $\langle \mathfrak{A}(R), \mathfrak{B}(R)\rangle$ be the following incidence system: $\mathfrak{A}(R)$ is the set of ordered pairs $\langle x, y\rangle$; $\mathfrak{B}(R)$ is the set of ordered pairs $\langle u, v\rangle$; incidence is defined by

$$\langle x, y\rangle \mid \langle u, v\rangle \quad \Leftrightarrow \quad x \cdot u = y + v.$$

If we call two points with same abscissa x parallel, and two lines with same slope u parallel, then $\langle \mathfrak{A}(R), \mathfrak{B}(R)\rangle$ is a P-system.

One sees easily that $t\mathbf{a}$, $t\mathbf{b}$, $t\mathbf{c}$, defined as follows, are collineations of $\langle \mathfrak{A}(R), \mathfrak{B}(R)\rangle$:

$$\begin{aligned} \langle x, y\rangle t\mathbf{a} &= \langle x + t, y\rangle, & \langle u, v\rangle t\mathbf{a} &= \langle u, v + tu\rangle, \\ \langle x, y\rangle t\mathbf{b} &= \langle x, y + xt\rangle, & \langle u, v\rangle t\mathbf{b} &= \langle u + t, v\rangle, \\ \langle x, y\rangle t\mathbf{c} &= \langle x, y + t\rangle, & \langle u, v\rangle t\mathbf{c} &= \langle u, v - t\rangle. \end{aligned}$$

Let $A_0(R)$ be the group of all $t\mathbf{a}$, $B_0(R)$ the group of all $t\mathbf{b}$, and $C(R)$ the group of all $t\mathbf{c}$. Then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are isomorphisms of the additive group R^+ onto $A_0(R)$, $B_0(R)$, $C(R)$ resp.

Put $A(R) = A_0(R) \times C(R)$ and $B(R) = B_0(R) \times C(R)$. One sees easily that $A(R) \cap B(R) = C(R)$. Furthermore

$$\langle x, y\rangle (s\mathbf{a})^{-1}(t\mathbf{b})^{-1} s\mathbf{a} t\mathbf{b} = \langle x, y + st\rangle$$

and

$$\langle u, v\rangle (s\mathbf{a})^{-1}(t\mathbf{b})^{-1} s\mathbf{a} t\mathbf{b} = \langle u, v - st\rangle;$$

hence

$$(s\mathbf{a}, t\mathbf{b}) = (s\cdot t)\mathbf{c}.$$

Therefore $A(R)$ and $B(R)$ are normal in $G(R)$, and $G(R) = A(R)\cdot B(R)$ is a T-group. $G(R)$ is abelian if and only if $R\cdot R = 0$ (i.e. R is a zero-ring).

The conditions of Prop. 3 are satisfied with $\mathfrak{a}_0 = \langle 0, 0\rangle$ and $\mathfrak{b}_0 = \langle 0, 0\rangle$. Hence the canonical map θ given by

$$\text{point } \langle x, y\rangle \to x\mathbf{a} y\mathbf{c}$$

$$\text{line } \langle u, v\rangle \to u\mathbf{b}(-v)\mathbf{c}$$

is an isomorphism of $\langle \mathfrak{A}(R), \mathfrak{B}(R)\rangle$ onto $\langle A(R), B(R)\rangle$ such that

$$g\theta = \theta g^* \qquad \text{for all } g \in G(R).$$

The triple (α, β, γ) is a homotopism of a ring R_1 onto a ring R_2, if α, β, γ are three homomorphisms of R_1^+ onto R_2^+ that satisfy

$$s\alpha\cdot t\beta = (s\cdot t)\gamma \qquad \text{for all } s, t \in R_1 .$$

A homotopism (α, β, γ) of R_1 onto R_2 induces a homorphism $(\alpha, \beta, \gamma)^*$ of $\langle \mathfrak{A}(R_1), \mathfrak{B}(R_1)\rangle$ onto $\langle \mathfrak{A}(R_2), \mathfrak{B}(R_2)\rangle$ given by

$$\text{point } \langle x, y\rangle \to \langle x\alpha, y\gamma\rangle,$$

$$\text{line } \langle u, v\rangle \to \langle u\beta, v\gamma\rangle.$$

PROPOSITION 16. *There is a one-to-one correspondence between homotopisms (α, β, γ) of R_1 onto R_2 and T-homomorphisms φ of $G(R_1)$ onto $G(R_2)$. This correspondence is determined by*

$$\mathbf{a}_1\varphi = \alpha\mathbf{a}_2, \qquad \mathbf{b}_1\varphi = \beta\mathbf{b}_2, \qquad \mathbf{c}_1\varphi = \gamma\mathbf{c}_2 .$$

Proof. In the diagram

$$\begin{tikzcd} R_1^+ \arrow[r, "\alpha"] \arrow[d, "\mathbf{a}_1"'] & R_2^+ \arrow[d, "\mathbf{a}_2"] \\ A_0(R_1) \arrow[r, "\varphi"] & A_0(R_2) \end{tikzcd}$$

$\mathbf{a}_1$ and $\mathbf{a}_2$ are isomorphisms. Hence in the equation $\mathbf{a}_1 \varphi = \alpha\mathbf{a}_2$, one of the homomorphisms α and φ determines the other one. The same holds for $\mathbf{b}_1 \varphi = \beta\mathbf{b}_2$ and $\mathbf{c}_1 \varphi = \gamma\mathbf{c}_2$.

If (α, β, γ) is a given homotopism, then

$$(s\mathbf{a}_1, t\mathbf{b}_1)\varphi = (s\cdot t)\mathbf{c}_1 \varphi = (s\cdot t)\gamma\mathbf{c}_2 = (s\alpha\cdot t\beta)\mathbf{c}_2 = (s\alpha\mathbf{a}_2, t\beta\mathbf{b}_2) = (s\mathbf{a}_1 \varphi, t\mathbf{b}_1 \varphi)$$

implies that φ can be extended by Prop. 1′ to a homomorphism of $G(R_1)$ onto $G(R_2)$.

Conversely if φ is a given homomorphism, then

$$(s\alpha\cdot t\beta)\mathbf{c}_2 = (s\alpha\mathbf{a}_2, t\beta\mathbf{b}_2) = (s\mathbf{a}_1 \varphi, t\mathbf{b}_1 \varphi) = (s\mathbf{a}_1, t\mathbf{b}_1)\varphi = (s\cdot t)\mathbf{c}_1 \varphi = (s\cdot t)\gamma\mathbf{c}_2$$

implies that (α, β, γ) is a homotopism.

COROLLARY 16. *The group* A *of all autotopisms of a ring* R *and the group* Φ_0 *of all* T*-automorphisms of* $G(R)$ *are isomorphic.*

Put $\mathfrak{T}_i = \langle \mathfrak{A}(R_i), \mathfrak{B}(R_i) \rangle$ and $T_i = \langle A(R_i), B(R_i) \rangle$.

PROPOSITION 17. *The following diagram is commutative.*

$$\begin{tikzcd} & \mathfrak{T}_1 \arrow[r, "(\alpha,\beta,\gamma)^*"] \arrow[d, "\theta_1"'] & \mathfrak{T}_2 \arrow[d, "\theta_2"] & \\ \mathfrak{T}_1 \arrow[r, "\theta_1"] \arrow[d, "g_1"] & T_1 \arrow[r, "\varphi^*"] \arrow[d, "g_1^*"] & T_2 \arrow[d, "(g_1\varphi)^*"] & \mathfrak{T}_2 \arrow[l, "\theta_2"'] \arrow[d, "g_1\varphi"] \\ \mathfrak{T}_1 \arrow[r, "\theta_1"'] & T_1 \arrow[r, "\varphi^*"'] & T_2 & \mathfrak{T}_2 \arrow[l, "\theta_2"] \\ & \mathfrak{T}_1 \arrow[r, "(\alpha,\beta,\gamma)^*"'] \arrow[u, "\theta_1"] & \mathfrak{T}_2 \arrow[u, "\theta_2"'] & \end{tikzcd}$$

$(\alpha, \beta, \gamma) \leftrightarrow \varphi$ *according to Prop.* 16.

Proof. We only have to check that

$$\theta_1 \varphi^* = (\alpha, \beta, \gamma)^* \theta_2 .$$

We have

$$\langle x, y\rangle\theta_1 \varphi^* = (x\mathbf{a}_1\, y\mathbf{c}_1)\varphi = x\mathbf{a}_1\, \varphi y\mathbf{c}_2\, \varphi = x\alpha\mathbf{a}_2\, y\gamma\mathbf{c}_2 = \langle x\alpha, y\gamma\rangle\theta_2 = \langle x, y\rangle(\alpha, \beta, \gamma)^*\theta_2$$

and

$$\langle u, -v\rangle\theta_1 \varphi^* = (u\mathbf{b}_1\, v\mathbf{c}_1)\varphi = u\mathbf{b}_1\, \varphi v\mathbf{c}_1\, \varphi$$

$$= u\beta\mathbf{b}_2\, v\gamma\mathbf{c}_2 = \langle u\beta, -v\gamma\rangle\theta_2 = \langle u, -v\rangle(\alpha, \beta, \gamma)^*\theta_2 .$$

We may look at the five squares of the diagram as five faces of a cube, and get in this way the remaining relation

$$g_1\,(\alpha, \beta, \gamma)^* = (\alpha, \beta, \gamma)^*\,(g_1\,\varphi).$$

In other words the canonical map θ induces a complete isomorphism between the structure of the $\mathfrak{T}$-level and of the T-level.

A product $(\alpha, \beta, \gamma)^* g_2$ $(g_2 \in G(R_2))$ is of the form

$$\langle x_1, y_1\rangle \rightarrow \langle x_1\alpha + r_2,\ y_1\gamma + x_1\alpha\cdot s_2 + t_2\rangle \quad (r_2, s_2, t_2 \text{ arbitrary in } R_2).$$

Hence let us call every homomorphism $(\alpha, \beta, \gamma)^* g_2$ a *semilinear transformation.*

We have $(s\mathbf{a}, t\mathbf{b}) = (s\cdot t)\mathbf{c}$ (for all $s, t \in R$). Hence $r\mathbf{c} \in C(R)$ is a commutator if and only if r is a product in R, $r = st$. $s\mathbf{a}$ is in $Z(G(R))$ if and only if $s\cdot R = 0$; $t\mathbf{b}$ is in $Z\,(G(R))$ if and only if $R\cdot t = 0$. Hence $Z(G(R)) = C(R)$ if and only if there are no annihilators in R ($r \neq 0$ is an annihilator means that $r\cdot R = 0$ or $R\cdot r = 0$.) Therefore the next proposition follows from Prop. 9.

PROPOSITION 18. *If every element in R_1 is a product and if R_2 does not possess any annihilators, then every homomorphism of $\langle\mathfrak{A}(R_1), \mathfrak{B}(R_1)\rangle$ onto $\langle\mathfrak{A}(R_2), \mathfrak{B}(R_2)\rangle$ is a semilinear transformation $(\alpha, \beta, \gamma)^* g_2$ with uniquely determined factors.*

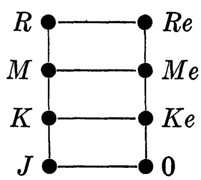
We have as a corollary

THEOREM 3. *If R is a ring with 1 and A its autotopism group, then the collineation group of $\langle\mathfrak{A}(R), \mathfrak{B}(R)\rangle$ is equal to the group $\mathrm{A}^*\cdot G(R)$ of semilinear transformations.*

Remark. Let (α, β, γ) be a homotopism of a ring R with 1 onto a ring R' with $1'$. Then (α, β, γ) is a homomorphism (i.e. $\alpha = \beta = \gamma$) if and only if $(\alpha, \beta, \gamma)^*$ maps point and line $\langle 1, 0\rangle$ onto $\langle 1', 0\rangle$ (since then $s\gamma = s\alpha\cdot 1\beta = s\alpha$ and $t\gamma = 1\alpha\cdot t\beta = t\beta$.) Hence if R is a ring with 1, then the automorphism group of R is isomorphic to the group of all collineations of $\langle\mathfrak{A}(R), \mathfrak{B}(R)\rangle$ that leave point and line $\langle 0, 0\rangle$ and point and line $\langle 1, 0\rangle$ fixed.

PROPOSITION 19. *Let R be a ring with 1, R' a ring that is finite or has a 1, and (α, β, γ) a homotopism of R onto R'. Then α, β, γ have the same kernel M, M is an ideal in R, and (α, β, γ) is the product of the canonical homomorphism of R onto R/M and the induced isotopism of R/M onto R'.*

Proof. Let K, L, M be the kernels of α, β, γ respectively. Clearly $K \subseteq K\cdot R$. $(K\cdot R)\gamma = K\alpha\cdot R\beta = 0$ implies that $K\cdot R \subseteq M$; hence $K \subseteq K\cdot R \subseteq M$, and similarly $L \subseteq R\cdot L \subseteq M$. R^+/K, R^+/L, and R^+/M are all isomorphic to R'^+; hence if R' is finite, then clearly $K = L = M$. Now suppose that R' has a 1 and that $e\beta = 1$. We have $(re)\gamma = r\alpha$; hence $re \in M$ if and only if $r \in K$, i.e. $Re \cap M = Ke$. Consider the homomorphism $r \to re$ of R^+ onto R^+e, and let J be its kernel. $Je = 0$ implies that $J \subseteq K$.



$Me = M \cap Re = Ke$ implies that $M = K$; similarly $M = L$. Since $M\cdot R \subseteq M$ and $R\cdot M \subseteq M$, M is an ideal in R. Let φ be the canonical homomorphism of R onto R/M and define $(\alpha', \beta', \gamma')$ by

$$(r + M)\alpha' = r\alpha, \qquad (r + M)\beta' = r\beta, \qquad (r + M)\gamma' = r\gamma.$$

Then $(\alpha, \beta, \gamma) = \varphi(\alpha', \beta', \gamma')$.

COROLLARY 19. *If R and R' are rings with* 1 *and κ a homomorphism of $\langle \mathfrak{A}(R), \mathfrak{B}(R)\rangle$ onto $\langle \mathfrak{A}(R'), \mathfrak{B}(R')\rangle$, then there exists an ideal M of R such that $\kappa = \kappa_1\kappa_2$ is the product of the canonical homomorphism κ_1* of *$\langle \mathfrak{A}(R), \mathfrak{B}(R)\rangle$ onto $\langle \mathfrak{A}(R/M), \mathfrak{B}(R/M)\rangle$ and the induced isomorphism κ_2 of $\langle \mathfrak{A}(R/M), \mathfrak{B}(R/M)\rangle$ onto $\langle \mathfrak{A}(R'), \mathfrak{B}(R')\rangle$.*

Proof. Use Prop. 18 and Prop. 19.

5. Regular T-groups and transitive P-systems

The mapping $a \to (a, b)$ is a homomorphism of A into C; the mapping $b \to (a, b)$ is a homomorphism of B into C. We call $a \in A$ *regular* if and only if $b \to (a, b)$ is onto C with kernel C, and $b \in B$ *regular* if and only if $a \to (a, b)$ is onto C with kernel C.

If $A = A_0 \times C$ and $B = B_0 \times C$, then a is regular if and only if $b_0 \to (a, b_0)$ is an isomorphism of B_0 onto C, and b is regular if and only if $a_0 \to (a_0, b)$ is an isomorphism of A_0 onto C.

If a is regular, then every ac, i.e. every $a' \parallel a$, is regular (similarly for b). We say that a T-group $G = A\cdot B$ is *regular*, if there exist a regular $a \in A$ and a regular $b \in B$. If $G = A\cdot B$ is regular, then $D(G) = C = Z(G)$, $A_0 \simeq B_0 \simeq C$, and Thm. 2 applies.

Given are two lines b_1c_1 and b_2c_2 $(b_i \in B_0)$. The two lines have a point in common if and only if there exist $a \in A_0$ and $c \in C$ such that

$$(\dagger) \qquad (a, b_1) = cc_1^{-1} \quad \text{and} \quad (a, b_2) = cc_2^{-1}.$$

But a solution a, c of $(\dagger)$ corresponds to a solution a of $(a, b_1b_2^{-1}) = c_1^{-1}c_2$. Hence we have the following:

The number of points common to b_1c_1 and b_2c_2 $(b_i \in B_0)$ is equal to the number of solutions $a \in A_0$ of $(a, b_1b_2^{-1}) = c_1^{-1}c_2$;

The number of lines common to a_1c_1 and a_2c_2 $(a_i \in A_0)$ is equal to the number of solutions $b \in B_0$ of $(a_1a_2^{-1}, b) = c_1c_2^{-1}$.

In particular, b is regular if and only if the lines bc_1 and c_2 intersect in exactly one point for every c_1 and c_2; a is regular if and only if the points ac_1 and c_2 are joined by exactly one line for every c_1 and c_2.

Let $\langle \mathfrak{A}, \mathfrak{B} \rangle$ be a P-system. If we call every equivalence class $[\mathfrak{a}]$ of points $\mathfrak{a}' \parallel \mathfrak{a}$ an *improper line* or *vertical line*, and every equivalence class $[\mathfrak{b}]$ of lines $\mathfrak{b}' \parallel \mathfrak{b}$ an *improper point* or *point at infinity*, then a line $\mathfrak{b}$ and a vertical line $[\mathfrak{a}]$ intersect in exactly one point; a point $\mathfrak{a}$ and a point at infinity $[\mathfrak{b}]$ are joined by exactly one line. We adjoin formally a point $\mathfrak{a}_\infty$ that is incident with all vertical lines, and a line $\mathfrak{b}_\infty$ that is incident with all points at infinity.

We say that a pair of lines $\mathfrak{b}_0$ and $\mathfrak{b}_1$ is a *regular pair* if and only if the system $\mathfrak{N}(\mathfrak{b}_0, \mathfrak{b}_1)$ consisting of the points in $\mathfrak{A}$, the lines in $[\mathfrak{b}_0]$, the vertical lines, and the lines in $[\mathfrak{b}_1]$, is a $[\mathfrak{b}_0]$-$[\mathfrak{a}_\infty]$-$[\mathfrak{b}_1]$-net; (terminology as in Pickert [3, p. 42]); we say that a pair of two points $\mathfrak{a}_0$ and $\mathfrak{a}_1$ is a *regular pair* if and only if the system $\mathfrak{N}(\mathfrak{a}_0, \mathfrak{a}_1)$ consisting of the lines in $\mathfrak{B}$, the points in $[\mathfrak{a}_0]$, the points at infinity, and the points in $[\mathfrak{a}_1]$, is a (dual) $[\mathfrak{a}_0]$-$[\mathfrak{b}_\infty]$-$[\mathfrak{a}_1]$-net.

The group generated by the U-, V-, and W- automorphisms (see [3, p. 51]) of a U-V-W-net $\mathfrak{N}$ is abelian and simply transitive on the points of $\mathfrak{N}$ if and only if $\mathfrak{N}$ is a Thomsen-net (see [3, p. 59]). In that case, this group is the direct product of the group of U-automorphisms and the group of V-automorphisms of $\mathfrak{N}$, and is called the translation group of $\mathfrak{N}$.

Now let $G = A \cdot B$ be a T-group.

Proposition 20. *$b \in B$ is regular if and only if the pair e, b is regular in $\langle A, B \rangle$, and then the group A^* is canonically isomorphic to the translation group of $\mathfrak{N}(e, b)$; $a \in A$ is regular if and only if the pair e, a is regular in $\langle A, B \rangle$, and then the group B^* is canonically isomorphic to the translation group of $\mathfrak{N}(e, a)$.*

Let $\langle \mathfrak{A}, \mathfrak{B} \rangle$ be a P-system with the following properties (i), (ii) and (iii).

(i) $\mathfrak{L}(\mathfrak{a}_1) \subseteq \mathfrak{L}(\mathfrak{a}_2)$ implies that $\mathfrak{a}_1 = \mathfrak{a}_2$;
$\mathfrak{P}(\mathfrak{b}_1) \subseteq \mathfrak{P}(\mathfrak{b}_2)$ implies that $\mathfrak{b}_1 = \mathfrak{b}_2$.

(ii) There is a regular pair $\mathfrak{b}_0$, $\mathfrak{b}_1$ such that $\mathfrak{N}(\mathfrak{b}_0, \mathfrak{b}_1)$ is a Thomsen-net; there is a regular pair $\mathfrak{a}_0$, $\mathfrak{a}_1$ such that $\mathfrak{N}(\mathfrak{a}_0, \mathfrak{a}_1)$ is a Thomsen-net.

(It is no restriction to assume that $\mathfrak{a}_0 \mid \mathfrak{b}_0$, $\mathfrak{a}_0 \mid \mathfrak{b}_1$ and $\mathfrak{a}_1 \mid \mathfrak{b}_0$.)

A collineation of $\langle \mathfrak{A}, \mathfrak{B} \rangle$ that maps every line onto a parallel line and induces a translation on $\mathfrak{N}(\mathfrak{b}_0, \mathfrak{b}_1)$ is called a *point-translation*; a collineation that maps every point onto a parallel point and induces a translation on $\mathfrak{N}(\mathfrak{a}_0, \mathfrak{a}_1)$

is called a *line-translation.* Because of (i), the point- and line-translations are uniquely determined by their restrictions to $\mathfrak{N}(\mathfrak{b}_0, \mathfrak{b}_1)$, and to $\mathfrak{N}(\mathfrak{a}_0, \mathfrak{a}_1)$ resp.; hence they form two abelian groups.

(iii) The group A of point-translation of $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is transitive on $\mathfrak{A}$, and the group B of line-translations of $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is transitive on $\mathfrak{B}$.

A P-system that satisfies (i), (ii) and (iii) is called *transitive.*

If $G = A \cdot B$ is a regular T-group, the P-system $\langle A, B \rangle$ is transitive because of Prop. 20.

THEOREM 4. *If $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is a transitive P-system, the group $G = A \cdot B$ generated by the point-translations A and by the line-translations B of $\langle \mathfrak{A}, \mathfrak{B} \rangle$, is a regular T-group. $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is isomorphic to $\langle A, B \rangle$.*

Proof. There is no danger if we identify in this proof the point-translations of $\langle \mathfrak{A}, \mathfrak{B} \rangle$ with the translations of $\mathfrak{N}(\mathfrak{b}_0, \mathfrak{b}_1)$ and the line-translations of $\langle \mathfrak{A}, \mathfrak{B} \rangle$ with the translations of $\mathfrak{N}(\mathfrak{a}_0, \mathfrak{a}_1)$. If κ is a collineation of $\langle \mathfrak{A}, \mathfrak{B} \rangle$, it follows from the definition of A and of B that $\kappa \in A \cap B$ if and only if $\mathfrak{a}\kappa \parallel \mathfrak{a}$ on $\mathfrak{A}$ and $\mathfrak{b}\kappa \parallel \mathfrak{b}$ on $\mathfrak{B}$. Therefore $C = A \cap B$ is equal to the group of $\mathfrak{a}_\infty$-translations of $\mathfrak{N}(\mathfrak{b}_0, \mathfrak{b}_1)$ and equal to the group of $\mathfrak{b}_\infty$-translations of $\mathfrak{N}(\mathfrak{a}_0, \mathfrak{a}_1)$. Let A_0 be the group of $[\mathfrak{b}_0]$-translations of $\mathfrak{N}(\mathfrak{b}_0, \mathfrak{b}_1)$ and B_0 the group of $[\mathfrak{a}_0]$-translations of $\mathfrak{N}(\mathfrak{a}_0, \mathfrak{a}_1)$. Then $A = A_0 \times C$, $B = B_0 \times C$, and $\mathfrak{b}_0 A_0 = \mathfrak{b}_0$, $\mathfrak{a}_0 B_0 = \mathfrak{a}_0$.

We want to show that $(a, b) \in C$ $(a \in A_0, b \in B_0)$. Consider the four points $\mathfrak{a}$, $\mathfrak{a}a^{-1}$, $\mathfrak{a}a^{-1}b^{-1}$, and $\mathfrak{a}a^{-1}b^{-1}a$. $\mathfrak{a}a^{-1} \parallel \mathfrak{a}a^{-1}b^{-1}$ implies that $\mathfrak{a} \parallel \mathfrak{a}a^{-1}b^{-1}a$; hence $\mathfrak{a}a^{-1}b^{-1}ab \parallel \mathfrak{a}a^{-1}b^{-1}a \parallel \mathfrak{a}$; i.e. $\mathfrak{a}(a, b) \parallel \mathfrak{a}$. Dually we get $\mathfrak{b}(a, b) \parallel \mathfrak{b}$. Therefore $(a, b) \in C$. Since $C \subseteq Z(G)$, this proves that $A \lhd G$ and $B \lhd G$. Therefore $G = A \cdot B$ is a T-group. Hence the conditions of Prop. 3 are satisfied, and $\langle \mathfrak{A}, \mathfrak{B} \rangle$ and $\langle A, B \rangle$ are isomorphic. If $\mathfrak{a}_0 a_1 = \mathfrak{a}_1$ and $\mathfrak{b}_0 b_1 = \mathfrak{b}_1$, then a_1 and b_1 are regular in G.

6. Introduction of coordinates

Suppose that $G = A \cdot B$ is a regular T-group with $a_1 \in A_0$ and $b_1 \in B_0$ regular. Then the mapping $a \to (a, b_1)$ is an isomorphism of A_0 onto C, and the mapping $b \to (a_1, b)$ is an isomorphism of B_0 onto C.

Let γ be an isomorphism of an additive group R onto C.

$$(s + t)\gamma = s\gamma \cdot t\gamma \qquad (s, t \in R).$$

Let α be the isomorphism of R onto A_0 given by

$$t\gamma = (t\alpha, b_1);$$

let β be the isomorphism of R onto B_0 given by

$$t\gamma = (a_1, t\beta).$$

(α is equal to γ followed by the inverse of $a \to (a, b_1)$.)

Define multiplication on R by

$$(s \cdot t)\gamma = (s\alpha, t\beta).$$

Put $t_1\gamma = (a_1, b_1)$, so that $t_1\alpha = a_1$ and $t_1\beta = b_1$. Then $t_1 = 1$ in R, since

$$(t \cdot t_1)\gamma = (t\alpha, b_1) = t\gamma, \qquad (t_1 \cdot t)\gamma = (a_1, t\beta) = t\gamma.$$

Multiplication is distributive; e.g.

$$(s \cdot (t + t'))\gamma = (s\alpha, t\beta \cdot t'\beta) = (s\alpha, t\beta)(s\alpha, t'\beta)$$
$$= (s \cdot t)\gamma(s \cdot t')\gamma = (s \cdot t + s \cdot t')\gamma.$$

Hence R is a ring with 1.

Now suppose that $a_i \in A_0$ and $b_i \in B_0$ $(i = 1, 2)$ are regular in G. Denote the corresponding isomorphisms by α_i, β_i, and γ_i, and the two rings by R_i. Then $(\alpha_1\alpha_2^{-1}, \beta_1\beta_2^{-1}, \gamma_1\gamma_2^{-1})$ is an isotopism of R_1 onto R_2, since

$$(s\alpha_1\alpha_2^{-1} \cdot t\beta_1\beta_2^{-1})\gamma_2 = (s\alpha_1, t\beta_1) = (s \cdot t)\gamma_1.$$

Hence the ring $R = R(G)$, constructed as above, is uniquely determined up to isotopisms.

Proposition 21. *If $G = A \cdot B$ is a regular T-group and $R = R(G)$, then G is T-isomorphic to $G(R)$.*

Proof. Suppose that $A = A_0 \times C$ and $B = B_0 \times C$. Then

$$t\alpha \to t\mathfrak{a}, \qquad t\beta \to t\mathfrak{b}, \qquad t\gamma \to t\mathfrak{c}$$

are isomorphisms of A_0, B_0, C onto $A_0(R)$, $B_0(R)$, $C(R)$ resp.

Since

$$(s\alpha, t\beta) = (s \cdot t)\gamma \to (s \cdot t)\mathfrak{c} = (s\mathfrak{a}, t\mathfrak{b}),$$

these isomorphisms can be extended by Prop. 1′ to an isomorphism of G onto $G(R)$.

Remark. $t\mathfrak{a}$ is regular if and only if $t \cdot u = v$ has a unique solution u for every v (i.e. t is left-nonsingular); $t\mathfrak{b}$ is regular if and only if $x \cdot t = y$ has a unique solution x for every y (i.e. t is right-nonsingular).

Theorem 5. *If $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is a transitive P-system, then there exists a ring R with 1, uniquely determined up to isotopisms, such that $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is isomorphic to $\langle \mathfrak{A}(R), \mathfrak{B}(R) \rangle$.*

Proof. Use Thm. 4 and Prop. 21.

Remark. The definition of point- and line-translation in a P-system $\langle \mathfrak{A}, \mathfrak{B} \rangle$ was dependent on the choice of two particular nets. Thm. 5 shows that the point- and line-translations of a transitive P-system $\langle \mathfrak{A}, \mathfrak{B} \rangle$ induce translations on every net $\mathfrak{N}(\mathfrak{b}_0, \mathfrak{b}_1)$ and $\mathfrak{N}(\mathfrak{a}_0, \mathfrak{a}_1)$ (for regular pairs $\mathfrak{b}_0$, $\mathfrak{b}_1$ and $\mathfrak{a}_0$, $\mathfrak{a}_1$).

Hence in a transitive P-system, the concept of point- and line-translation is independent of the choice of particular nets.

7. Duality

The dual of an incidence system $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is defined by $\langle \mathfrak{A}, \mathfrak{B} \rangle^{\mathrm{du}} = \langle \mathfrak{B}, \mathfrak{A} \rangle$, and the dual of a T-group $G = A \cdot B$ by $G^{\mathrm{du}} = B \cdot A$. Note that as abstract groups $G = G^{\mathrm{du}}$.

Let $G = A \cdot B$ be a T-group with $A = A_0 \times C$ and $B = B_0 \times C$, and let ψ be an automorphism of G that switches A and B, hence leaves C invariant. There exists exactly one $\eta \in \mathrm{H}$ (see Prop. 11) such that $A_0\psi = B_0\eta$ and $B_0\psi = A_0\eta$. Then the automorphism $\psi\eta^{-1}$ maps A_0, B_0, C onto B_0, A_0, C resp., hence induces an isomorphism of $\langle A, B \rangle$ onto $\langle B, A \rangle = \langle A, B \rangle^{\mathrm{du}}$. Conversely suppose that G satisfies the conditions of Thm. 2 and that there exists a duality of $\langle A, B \rangle$, i.e. an isomorphism κ_1 of $\langle A, B \rangle$ onto $\langle B, A \rangle$. Then κ_1 has a unique product representation $\kappa_1 = \kappa g^*$ where κ is induced by an automorphism of G that maps A_0, B_0, C onto B_0, A_0, C resp. Hence in this case, the group Δ of all collineations and dualities of $\langle A, B \rangle$ is equal to the semidirect product $\Delta = \Psi_0^* G^*$, where Ψ_0 is the group of all automorphisms of G that map either A_0, B_0, C onto A_0, B_0, C resp., or A_0, B_0, C onto B_0, A_0, C resp.

R^{op} denotes the "opposite" ring of R with multiplication $\circ$ defined by $x \circ y = yx$ (product in R). $xu = y + v \Leftrightarrow u \circ x = v + y$ shows that the anti-isomorphism $t \to t$ of R onto R^{op} induces an isomorphism of $\langle \mathfrak{B}(R), \mathfrak{A}(R) \rangle = \langle \mathfrak{A}(R), \mathfrak{B}(R) \rangle^{\mathrm{du}}$ onto $\langle \mathfrak{A}(R^{\mathrm{op}}), \mathfrak{B}(R^{\mathrm{op}}) \rangle$. The corresponding isomorphism δ of $G(R)^{\mathrm{du}}$ onto $G(R^{\mathrm{op}})$ is given by $r\mathbf{b}\delta = r\mathbf{a}$, $s\mathbf{a}\delta = s\mathbf{b}$, $t\mathbf{c}\delta = (-t)\mathbf{c}$.

Suppose that a ring R possesses an anti-autotopism, i.e. an isotopism (α, β, γ) of R onto R^{op}. Then (α, β, γ) induces an isomorphism $(\alpha, \beta, \gamma)^*$ of $\langle \mathfrak{A}(R), \mathfrak{B}(R) \rangle$ onto $\langle \mathfrak{A}(R^{\mathrm{op}}), \mathfrak{B}(R^{\mathrm{op}}) \rangle$, hence onto the dual $\langle \mathfrak{B}(R), \mathfrak{A}(R) \rangle$ of $\langle \mathfrak{A}(R), \mathfrak{B}(R) \rangle$. Conversely suppose that R is a ring with 1 and that $\langle \mathfrak{A}(R), \mathfrak{B}(R) \rangle$ is self-dual. Then there exists a semilinear transformation of $\langle \mathfrak{A}(R), \mathfrak{B}(R) \rangle$ onto $\langle \mathfrak{A}(R^{\mathrm{op}}), \mathfrak{B}(R^{\mathrm{op}}) \rangle$, and R possesses an anti-autotopism. We have proved

PROPOSITION 22. *If R is a ring with 1, then $\langle \mathfrak{A}(R), \mathfrak{B}(R) \rangle$ is self-dual if and only if R possesses an anti-autotopism.*

PROPOSITION 23. *If a regular T-group $G = A \cdot B$ possesses an automorphism ψ of order 2 that switches A and B, then there exists a ring R with involutorial anti-automorphism γ such that $G \simeq G(R)$. $\gamma = 1$ if and only if $c\psi = c^{-1}$ on C. (In that case R is commutative.)*

Proof. Let $A = A_0 \times C$ and a_1 regular in A_0. Put $B_0 = A_0\psi$ and $b_1 = a_1\psi$. Then $B = B_0 \times C$ and b_1 regular in B_0. Construct $R = R(G)$ as above. Then $G \simeq G(R)$. Identify ψ with the corresponding automorphism of $G(R)$. Then $\psi\delta$ (with $\delta : G(R)^{\mathrm{du}} \to G(R^{\mathrm{op}})$, as above) is an isomorphism of $G(R)$

onto $G(R^{\mathrm{op}})$, induced by an isotopism (α, β, γ) of R onto R^{op}. Since $\psi\delta$ maps $\langle 1, 0\rangle$ onto $\langle 1, 0\rangle$, (α, β, γ) is an isomorphism of R onto R^{op}.

Now $t\mathrm{c}\psi\delta = t\gamma\mathrm{c}$, hence

$$t\mathrm{c}\psi = t\gamma\mathrm{c}\delta = (-t\gamma)\mathrm{c} = (t\gamma\mathrm{c})^{-1}.$$

Therefore $t\mathrm{c}\psi = (t\mathrm{c})^{-1}$ for all $t \in R$ if and only if $\gamma = 1$; i.e. $c\psi = c^{-1}$ on C if and only if $\gamma = 1$.

8. The V-extension of a T-group

In §8 we assume that every T-group is nonabelian and has no elements of order 2.

Let $G = A \cdot B$ be a T-group with $A = A_0 \times C$ and $B = B_0 \times C$. Define the map π_0 as follows:

$$a\pi_0 = a^{-1} \text{ on } A \quad \text{and} \quad b\pi_0 = b \text{ on } B_0 .$$

Then

$$(a\pi_0 , b\pi_0) = (a^{-1}, b) = (a, b)^{-1} = (a, b)\pi_0$$

implies by Prop. 1′ that π_0 can be extended to a T-automorphism π_0 of G. Similarly there exists a T-automorphism λ_0 of G determined by

$$a\lambda_0 = a \text{ on } A_0 \quad \text{and} \quad b\lambda_0 = b^{-1} \text{ on } B.$$

Then A_0 is the subgroup of A that is centralized by λ_0, and B_0 is the subgroup of B that is centralized by π_0.

Since $D = \{(A_0 , B_0)\} \neq e$, we have $A_0 \neq e$ and $B_0 \neq e$. Hence π_0 and λ_0 are two distinct, and commuting, elements of order 2; i.e. $V = \{\pi_0 , \lambda_0\}$ is a four-group. We call the semidirect product $\Omega = V \cdot G$, contained in the holomorph of G, the *V-extension* of the T-group $G = A \cdot B$, (V for Vierergruppe).

We have $\pi_0 a_1 \cdot \pi_0 a_2 = a_1^{-1} a_2$ in Ω. Hence the subset $P = \pi_0 A$ of Ω consists of involutions (elements of order 2), and $P \cdot P = A$. Similarly $L = \lambda_0 B$ consists of involutions, and $L \cdot L = B$. Hence $\Omega = \{P, L\}$ is generated by the involutions in P and L. Note that the subgroups $\{P\}$ and $\{L\}$ of Ω are proper.

We define the incidence system $\langle P, L\rangle$ by

$$\pi \mid \lambda \iff \pi\lambda = \lambda\pi.$$

Furthermore

$$\pi \parallel \pi' \iff \pi\pi' \in C,$$

$$\lambda \parallel \lambda' \iff \lambda\lambda' \in C,$$

define equivalence relations on P and on L.

Let P_0 be the class of all conjugates of π_0 in Ω, and L_0 the class of all conjugates of λ_0. Since π_0 and λ_0 commute, we only have to form conjugates with

$g \in G$. Every $g \in G$ has a representation $g = b_0 a$ with $b_0 \in B_0$ and $a \in A$. Then $g^{-1}\pi_0 g = a^{-1}\pi_0 a = \pi_0 a^2$. Hence $P_0 = \pi_0 A^2 \subseteq P$; similarly $L_0 = \lambda_0 B^2 \subseteq L$.

Because of the definition of π_0 and λ_0, we have A and B normal in Ω, hence also $C \lhd \Omega$. Furthermore

$$\omega^{-1}P\omega = \omega^{-1}\pi_0 A\omega = \pi_0 A = P,$$

and similarly

$$\omega^{-1}L\omega = L, \qquad \text{for all } \omega \in \Omega.$$

Remark. If Ω and G are finite groups (or torsion groups), then always $A^2 = A$ and $B^2 = B$; hence $P_0 = P$ and $L_0 = L$.

PROPOSITION 24. *$\langle P_0, L_0\rangle$ is a P-system. The map χ:*

$$a \longrightarrow a^{-1}\pi_0 a = \pi_0 a^2$$

$$b \longrightarrow b^{-1}\lambda_0 b = \lambda_0 b^2$$

is an isomorphism of $\langle A, B\rangle$ onto $\langle P_0, L_0\rangle$.

Proof. The map χ is clearly one-to-one and onto. Let $\pi = \pi_0 a^2 c_1^2$ and $\lambda = \lambda_0 b^2 c_2^2$ (with $a \in A_0$, $b \in B_0$). Then

$$\begin{aligned} \pi\lambda = \lambda\pi \quad & \text{if and only if} \quad \pi_0\lambda_0 a^2 b^2 c_1^{-2} c_2^2 = \lambda_0\pi_0 b^2 a^2 c_2^{-2} c_1^2 \\ & \text{if and only if} \quad (a^2, b^2) = (a, b)^4 = c_1^4 c_2^{-4} \\ & \text{if and only if} \quad (a, b) = c_1 c_2^{-1} \\ & \text{if and only if} \quad ac_1 \mid bc_2 . \end{aligned}$$

For $a_0 \in A_0$, $(a_0 c)^2 \in C$ implies that $a_0^2 \in A_0 \cap C$, i.e. $a_0^2 = e$ and $a_0 = e$. Hence for $a \in A$, $a^2 \in C$ if and only if $a \in C$. Therefore

$$\begin{aligned} a_1 \parallel a_2 \quad & \text{if and only if} \quad (a_1^{-1}a_2)^2 \in C \\ & \text{if and only if} \quad \pi_0 a_1^2 \cdot \pi_0 a_2^2 \in C \\ & \text{if and only if} \quad \pi_0 a_1^2 \parallel \pi_0 a_2^2. \end{aligned}$$

This proves that $\langle P_0, L_0\rangle$ is a P-system, and that χ is an isomorphism.

(*Remark.* $\langle P, L\rangle$ is not always a P-system.)

P_0 and L_0 are classes of conjugate involutions, and C is normal in Ω. Hence the inner automorphisms of Ω induce collineations in $\langle P_0, L_0\rangle$.

Notation. Denote the inner automorphism of a group G, induced by $g \in G$, by $i(g)$. If a homomorphism φ of a T-extension $\Omega = V \cdot G$ onto $\Omega' = V' \cdot G'$ induces a homomorphism of $\langle P_0, L_0\rangle$ onto $\langle P_0', L_0'\rangle$ (or onto the dual $\langle L_0', P_0'\rangle$), denote the induced homomorphism by $\varphi^{\#}$.

PROPOSITION 25. *The map $\omega \longrightarrow i(\omega)^{\#}$ is an isomorphism of Ω onto the induced group $i(\Omega)^{\#}$ of collineations of $\langle P_0, L_0\rangle$.*

$$\omega^*\chi = \chi i(\omega)^{\#} \qquad \text{for all } \omega \in \Omega.$$

$$Z(\Omega) = e.$$

Proof. To prove that $\omega^*\chi = \chi i(\omega)^{\#}$, note the following. If $g = a_0b_0c$, then

$$ag^* = b_0^{-1}aa_0b_0c = b_0^{-1}ag;$$

and, for $v \in V$,

$$av^* = v^{-1}av \quad (\text{in } \Omega).$$

Now let $\omega = vg$ ($v \in V$, $g \in G$). Then

$$a\chi i(\omega) = \omega^{-1}a^{-1}\pi_0 a\omega;$$

$$a\omega^*\chi = av^*g^*\chi = (v^{-1}av)g^*\chi = b_0^{-1}(v^{-1}av)g\chi = (b_0^{-1}v^{-1}a\omega)\chi = \omega^{-1}a^{-1}\pi_0 a\omega.$$

Similarly $b\chi i(\omega) = b\omega^*\chi$.

Next we prove that $\omega^* = 1$ implies that $\omega = e$. Let $\omega = vg$. Then for point and line e in $\langle A, B\rangle$, $e = e\omega^* = ev^*g^* = eg^*$ implies that $g = e$ (by Thm. 1). Now $v^{-1}av = a$ for all $a \in A$ and $v^{-1}bv$ for all $b \in B$; hence $v^{-1}gv = g$ on G, i.e. $v = e$. Hence all groups Ω, $i(\Omega)$, $i(\Omega)^{\#}$ and Ω^* are isomorphic; in particular ω in $Z(\Omega)$ implies $\omega = e$.

Remark. If $\varphi^{\#}$ is a homomorphism of $\langle P_0, L_0\rangle$ onto $\langle P_0', L_0'\rangle$, then there exists a unique $g' \in G'$ such that

$$\pi_0\varphi = \pi_0'i(g') \quad \text{and} \quad \lambda_0\varphi = \lambda_0'i(g').$$

Hence $\varphi = \varphi_0 i(g')$, where φ_0 is a homomorphism of Ω onto Ω' that maps π_0 onto π_0' and λ_0 onto λ_0'.

We determine the involutions in Ω. We have $\Omega = G \cup \pi_0 G \cup \lambda_0 G \cup \pi_0\lambda_0 G$. Let $g = ab$ with $a \in A$, $b \in B_0$. Then

$$\pi_0 ab \cdot \pi_0 ab = a^{-1}bab = b(a, b)b = b^2(a, b) = e$$

if and only if $b = e$. Hence $\pi_0 A = P$ is the set of all involutions in $\pi_0 G$. Similarly $\lambda_0 B = L$ is the set of all involutions in $\lambda_0 G$. If π and λ commute, then $\pi\lambda \in \pi_0\lambda_0 G$ has order 2. Let $g = ab$ with $a \in A_0$, $b \in B$, and suppose that $\pi_0\lambda_0 g = \pi_0 a \cdot \lambda_0 b$ has order 2. But a product of two distinct involutions ($\pi_0 a$ and $\lambda_0 b$) has order 2 if and only if they commute. Hence the involutions in $\pi_0\lambda_0 G$ are the products $\pi\lambda$ of commuting π and λ.

Let ω be an involution in Ω. Denote by $J(\omega)$ the group generated by all products $\omega_1\omega_2$ of conjugates ω_1 and ω_2 of ω in Ω.

PROPOSITION 26. *Let ω be an involution in $\Omega = V \cdot G$. Then ω is in P or in L if and only if $J(\omega)$ is abelian. If ω_1 and ω_2 are involutions in P or in L, then both are in P, or both are in L, if and only if $(J(\omega_1), J(\omega_2)) = e$.*

Proof. Let π be in P. Then $J(\pi) \subseteq P \cdot P = A$. Furthermore $\pi \cdot a^{-1}\pi a = a^2 \in J(\pi)$. Hence $A^2 \subseteq J(\pi) \subseteq A$. Similarly $B^2 \subseteq J(\lambda) \subseteq B$.

Now let $\omega = \pi_0\lambda_0 g$ be an involution in $\pi_0\lambda_0 G$. Then for $a \in A_0$,

$$a^{-1}\omega a = \pi_0\lambda_0 aga = \omega a^2(a, g);$$

hence

$$\omega \cdot a^{-1}\omega a = a^2(a, g) \in J(\omega).$$

Similarly for $b \in B_0$, $b^2(b, g) \in J(\omega)$. But

$$(a^2(a, g), b^2(b, g)) = (a^2, b^2) = (a, b)^4 = e$$

if and only if $(a, b) = e$. Since $D(G) \neq e, J(\omega)$ is not abelian.

Clearly $(J(\pi_1), J(\pi_2)) = e$ and $(J(\lambda_1), J(\lambda_2)) = e$. $A^2 \subseteq J(\pi)$ and $B^2 \subseteq J(\lambda)$ imply that $(J(\pi), J(\lambda)) \supseteq (A^2, B^2) = (A, B)^4 \neq e$.

PROPOSITION 27. *Let φ be a homomorphism of $\Omega = V \cdot G$ onto $\Omega' = V'G'$. Then φ maps either P onto P' and L onto L', or P onto L' and L onto P'.*

If in addition $A'^2 = A'$ and $B'^2 = B'$ in G', then φ induces a collineation or a duality $\varphi^{\#}$ of $\langle P_0, L_0\rangle$ onto $\langle P', L'\rangle$.

Proof. If for some $\pi \in P$, $\pi\varphi = e'$, then $P\varphi = (\pi A)\varphi = A\varphi = e'$, since there are no involutions in $A\varphi$. Hence either $P\varphi = e'$, or all $\pi\varphi \in P\varphi$ are involutions. In the second case, $J(\pi\varphi) = J(\pi)\varphi$ abelian implies that $\pi\varphi \in P'$ or $\pi\varphi \in L'$;

$$(J(\pi_1\varphi), J(\pi_2\varphi)) = (J(\pi_1), J(\pi_2))\varphi = e'$$

implies that either $P\varphi \subseteq P'$ or $P\varphi \subseteq L'$. Similarly either $L\varphi = e'$, or $L\varphi \subseteq P'$, or $L\varphi \subseteq L'$. We must have either $P\varphi \subseteq P'$ and $L\varphi \subseteq L'$, or $P\varphi \subseteq L'$ and $L\varphi \subseteq P'$; for in every other case

$$\Omega' = \Omega\varphi = \{P\}\varphi \subseteq \{P'\} \neq \Omega',$$

or

$$\Omega' = \Omega\varphi = \{L\}\varphi \subseteq \{L'\} \neq \Omega',$$

leads to a contradiction.

An involution ω not in P or in L is of type $\omega = \pi\lambda$ (with commuting $\pi \in P$ and $\lambda \in L$). Then $\omega\varphi = \pi\varphi \cdot \lambda\varphi$ is an involution which is neither in P' nor in L'. Now every involution in Ω' is image of some involution in Ω. Hence we must have equality everywhere: $P\varphi = P'$ and $L\varphi = L'$, or $P\varphi = L'$ and $L\varphi = P'$.

To prove the last statement, note that a homomorphism onto maps conjugate classes onto conjugate classes, and that $P' = P_0'$ and $L' = L_0'$.

PROPOSITION 28. *If φ is a homomorphism of $\Omega = V \cdot G$ such that $D\varphi \neq e$, then $G\varphi = A\varphi \cdot B\varphi$ is a T-group, and $\Omega\varphi$ its V-extension.*

Proof. Suppose that $\pi\varphi = e$ for some $\pi \in P$. Then $P\varphi = (\pi A)\varphi = A\varphi = e$, since $A\varphi$ does not contain any involutions. But then

$$D\varphi = \{(A, B)\}\varphi = \{(A\varphi, B\varphi)\} = e$$

would contradict $D\varphi \neq e$. Hence all $\pi\varphi \in P\varphi$ and all $\lambda\varphi \in L\varphi$ are involutions. Clearly $A\varphi$ and $B\varphi$ normal in $G\varphi$, and $G\varphi = A\varphi \cdot B\varphi$.

All elements of $A\varphi$ anti-commute with $\pi_0\varphi$;
all elements of $B\varphi$ anti-commute with $\lambda_0\varphi$;
all elements of $A_0\varphi$ commute with $\lambda_0\varphi$;
all elements of $B_0\varphi$ commute with $\pi_0\varphi$.

Hence
$$A_0\varphi \cap C\varphi = B_0\varphi \cap C\varphi = e \quad \text{and} \quad A\varphi \cap B\varphi = C\varphi.$$
Therefore
$$A\varphi = A_0\varphi \times C\varphi \quad \text{and} \quad B\varphi = B_0\varphi \times C\varphi,$$
and $G\varphi$ is a T-group. Since $D(G\varphi) = D(G)\varphi \neq e$, $\Omega\varphi$ is the V-extension of $G\varphi$.

Prop. 28 can be stated as follows: If $K \lhd \Omega$ and $K \not\supseteq D$, then GK/K is a T-group and Ω/K its V-extension. (Compare with Prop. 10.)

Let A be a group of automorphisms of a group G. Then the holomorph of G contains the semidirect product $\mathrm{A}\cdot G$. Let φ be a homomorphism of G with kernel K. If for all automorphisms $\alpha \in \mathrm{A}$, we have $K^\alpha \subseteq K$, then
$$\varphi\alpha^\varphi = \alpha\varphi$$
defines an automorphism α^φ of G^φ. The map $\alpha \to \alpha^\varphi$ is a homomorphism of A onto A^φ. We have
$$(g^\varphi, \alpha^\varphi) = (g^\varphi)^{-1}(\alpha^\varphi)^{-1}(g^\varphi)(\alpha^\varphi) = (g^\varphi)^{-1}g^{\varphi\alpha^\varphi}$$
$$= (g^\varphi)^{-1}g^{\alpha\varphi} = (g^{-1}\alpha^{-1}g\alpha)^\varphi = (g, \alpha)^\varphi;$$
hence by Prop. 1, the map $\alpha g \to \alpha^\varphi g^\varphi$ is a homomorphism of $\mathrm{A}\cdot G$ onto $\mathrm{A}^\varphi \cdot G^\varphi$.

PROPOSITION 29. *Let Ω and Ω' be V-extensions of the T-groups G and G'. Then there exists a natural one-to-one correspondence between the T-homomorphisms of G onto G' and those homomorphisms of Ω onto Ω' that map π_0 onto π_0' and λ_0 onto λ_0'.*

Such a homomorphism φ of Ω onto Ω' induces a homomorphism of $\langle P_0, L_0\rangle$ onto $\langle P_0', L_0'\rangle$, and
$$\varphi^*\chi' = \chi\varphi^\#.$$

Proof. Let φ be a T-homomorphism of G onto G'. By Prop. 10, the kernel K of φ is of the form $K = A_1B_1C_1$ with $A_1 \subseteq A_0$, $B_1 \subseteq B_0$, $C_1 \subseteq C$; hence $K^v \subseteq K$ for all $v \in V$. By the above remarks, φ can be extended to a homomorphism φ of $\Omega = V\cdot G$ onto $\Omega' = V'\cdot G'$. Since clearly $\pi_0\varphi = \varphi\pi_0'$ and $\lambda_0\varphi = \varphi\lambda_0'$, φ maps π_0 onto π_0' and λ_0 onto λ_0'. φ maps clearly $P = \pi_0 A$ onto $\pi_0' A' = P'$, and L onto L'.

Conversely suppose now that φ is a homomorphism of Ω onto Ω' that maps π_0 onto π_0' and λ_0 onto λ_0'. Then φ maps P onto P' and L onto L' by Prop. 27; hence also A onto A' and B onto B', and C onto C' Since λ_0 and λ_0'

centralize exactly A_0 and A_0, φ must map A_0 into A_0'. But since $A = A_0 \times C$ is mapped onto $A' = A_0' \times C'$, φ necessarily maps A_0 onto A_0'. Similarly φ maps B_0 onto B_0', and induces a T-homomorphism of G onto G'.

Since $\pi_0 \varphi = \pi_0'$, φ maps P_0 onto P_0', and also L_0 onto L_0'. Hence φ induces a homomorphism $\varphi^{\#}$ of $\langle P_0, L_0\rangle$ onto $\langle P_0', L_0'\rangle$. Finally we have

$$a\varphi\chi' = \pi_0'(a\varphi)^2 = (\pi_0 a^2)\varphi = a\chi\varphi$$

and

$$b\varphi\chi' = \lambda_0'(b\varphi)^2 = (\lambda_0 b^2)\varphi = b\chi\varphi.$$

PROPOSITION 30. *Let $G = A \cdot B$ and $G' = A' \cdot B'$ be two (nonabelian) T-groups such that*

(i) *every $c \in C$ is a commutator $c = (a, b)$;*
(ii) $C' = Z(G')$;
(iii) $A'^2 = A'$ *and* $B'^2 = B'$.

Then every collineation and every duality of $\langle P_0, L_0\rangle$ onto $\langle P', L'\rangle$ is induced by a homomorphism of the V-extension $\Omega = V \cdot G$ onto $\Omega' = V' \cdot G'$.

Proof. Denote by ψ_0 a homomorphism of Ω onto Ω' that maps either π_0 and λ_0 onto π_0' and λ_0', or π_0 and λ_0 onto λ_0' and π_0' resp. By Prop. 9, every collineation and every duality of $\langle A, B\rangle$ onto $\langle A', B'\rangle$ is of the type $\psi_0^* g'^*$ $(g' \in G')$. Hence by Prop. 24 and Prop. 29 every collineation and every duality of $\langle P_0, L_0\rangle$ onto $\langle P', L'\rangle$ is of the type $\psi^{\#}$ with $\psi = \psi_0 i(g')$.

We have as a corollary

THEOREM 6. *Let $G = A \cdot B$ be a T-group such that*

(i) *every $c \in C$ is a commutator $c = (a, b)$;*
(ii) $C = Z(G) \neq e$;
(iii) $A^2 = A$ *and* $B^2 = B$.

Then the automorphism group of the V-extension $\Omega = V \cdot G$ of G induces, and is isomorphic to, the group of all collineations and dualities of $\langle P, L\rangle$.

Let R be a ring such that $r + r = 0$ only for $r = 0$ (and $R \cdot R \neq 0$). Define the collineations π_0 and λ_0 of $\langle \mathfrak{A}(R), \mathfrak{B}(R)\rangle$ by

$$\langle x, y\rangle\pi_0 = \langle -x, -y\rangle, \qquad \langle u, v\rangle\pi_0 = \langle u, -v\rangle,$$
$$\langle x, y\rangle\lambda_0 = \langle x, -y\rangle, \qquad \langle u, v\rangle\lambda_0 = \langle -u, -v\rangle.$$

Then $V(R) = \{\pi_0, \lambda_0\}$ is a four-group. Since

$$\pi_0 r\mathbf{a}\pi_0 = (-r)\mathbf{a}, \qquad \lambda_0 r\mathbf{a}\lambda_0 = r\mathbf{a},$$
$$\pi_0 r\mathbf{b}\pi_0 = r\mathbf{b}, \qquad \lambda_0 r\mathbf{b}\lambda_0 = (-r)\mathbf{b},$$
$$\pi_0 r\mathbf{c}\pi_0 = (-r)\mathbf{c}, \qquad \lambda_0 r\mathbf{c}\lambda_0 = (-r)\mathbf{c},$$

$\Omega(R) = V(R) \cdot G(R)$ is the V-extension of $G(R)$. The transformations in $\Omega(R)$ are of the type

$$\langle x, y\rangle \to \langle \pm x + r, \pm y + xs + t\rangle.$$

If R is a ring with 1 and some h such that $h + h = 1$, then $A(R)^2 = A(R)$ and $B(R)^2 = B(R)$; hence Thm. 6 applies. (This holds in every distributive quasi-field (division-ring) of characteristic $\neq 2$.)

9. Characterization of the V-extension in terms of its generating involutions

Let Ω be a group with the following properties:

(i) $\Omega = \{P, L\}$ is generated by two sets P and L of involutions;
(ii) $P \cdot P \cdot P \subseteq P$ and $L \cdot L \cdot L \subseteq L$;
(iii) $\lambda P \lambda \subseteq P$ and $\pi L \pi \subseteq L$ for every $\lambda \in L$ and $\pi \in P$;
(iv) distinct involutions in P do not commute, distinct involutions in L do not commute.

Then (ii) implies that $A = P \cdot P$ and $B = L \cdot L$ are abelian groups. (iii) implies that A and B are normal in Ω, hence normal in $G = A \cdot B \cdot$ (iv) implies that A and B do not have elements of order 2.

Notation. π always denotes elements in P; λ always denotes elements in L. $\pi \mid \lambda$, or $\lambda \mid \pi$, means that π and λ commute. $C = A \cap B$. $\pi \parallel \pi'$ means that $\pi\pi' \in C$, and $\lambda \parallel \lambda'$ means that $\lambda\lambda' \in C$.

Since $\pi_1 \pi_2 = \pi \cdot \pi\pi_1 \pi_2$, we have $A = \pi P$, hence $\pi A = P$; similarly $B = \lambda L$ and $\lambda B = L \cdot$ (ii) implies that

$$\pi \cdot \pi'\pi''\pi = \pi \cdot \pi\pi''\pi' = (\pi'\pi'')^{-1};$$

i.e. $\pi a \pi = a^{-1}$ on A; similarly $\lambda b \lambda = b^{-1}$ on B.

(v) There are π_0 and λ_0 such that $\pi_0 \mid \lambda_0$; to every π there is some π' such that $\pi' \mid \lambda_0$ and $\pi' \parallel \pi$; to every λ there is some λ' such that $\lambda' \mid \pi_0$ and $\lambda' \parallel \lambda$;
(vi) there are π_1, π_2, λ_1, λ_2 such that $\pi_1 \mid \lambda_1$, $\pi_1 \mid \lambda_2$, $\pi_2 \mid \lambda_1$ and $\pi_2 \lambda_2 \neq \lambda_2 \pi_2$.

THEOREM 7. *A group Ω is the V-extension of a nonabelian T-group G without elements of order 2 if and only if Ω satisfies the properties* (i) *to* (vi).

Proof. If Ω is a V-extension, then (i) to (iv) are clear. Every $a = \pi_0\pi \in A$ is a product $a = a_0 c$ with $a_0 \in A_0$, $c \in C$. If $a_0 = \pi_0 \pi'$, then $c = \pi'\pi$ and (v) follows. There are $a = \pi_0 \pi_1 \in A_0$ and $b = \lambda_0 \lambda_1 \in B_0$ such that $(a, b) \neq e$; hence (vi) follows.

Conversely suppose now that Ω satisfies (i) to (vi). We have to show that $G = A \cdot B$ is a T-group. Define A_0 as the subgroup of A that is centralized by λ_0. (v) implies that $a = \pi_0 \pi = \pi_0 \pi' \cdot \pi'\pi$ with $\pi' \mid \lambda_0$, hence $\pi_0 \pi' \in A_0$, and $\pi'\pi \in C$. Since moreover all $a \in A_0$ commute with λ_0 and all $c \in C$ anticommute with λ_0, we have $A = A_0 \times C$; similarly $B = B_0 \times C$, where B_0 is the subgroup of B that is centralized by π_0.

$$\pi_1 \pi_2 \lambda_1 \lambda_2 = \pi_1 \lambda_1 \pi_2 \lambda_2 \neq \pi_1 \lambda_1 \lambda_2 \pi_2 = \lambda_1 \lambda_2 \pi_1 \pi_2$$

implies that G is not abelian.

Suppose that $g = a_0 b$ with $a_0 \in A_0$, $b \in B$, and that $g^2 = e$. Then $(a_0 b)^2 = a_0^2 b^2(b, a_0) = e$ implies that $a_0^2 = e$; hence $a_0 = e$, and $b = e$. G has no elements of order 2. Put $V = \{\pi_0, \lambda_0\}$. Then $\Omega = V \cdot G$ is the V-extension of G.

If Ω satisfies (i), (ii) and (iii), then

$$\pi\lambda\pi\lambda \subseteq P \cdot P \cap L \cdot L = C,$$

i.e. $(P, L) \subseteq C$. If in addition there exist $\pi_0 \mid \lambda_0$, then $C^2 \subseteq (P, L)$. For let $c = \pi_0 \pi = \lambda_0 \lambda$; then $\lambda_0 \pi = \pi_0 \lambda$; hence

$$c^2 = \pi_0 \pi \cdot \lambda_0 \lambda = \pi_0 \cdot \lambda \pi_0 \cdot \lambda = (\pi_0, \lambda).$$

PROPOSITION 31. *If Ω satisfies* (i), (ii) *and* (iii), *then* (vii) *and* (viii) *equivalent*:

(vii) $(P, L) \subseteq C^2$;

(viii) *to every π and λ there exists $c \in C$ such that $\pi c \mid \lambda$*;

(i), (ii), (iii), (vi) *and* (vii) *together imply* (v).

Proof. $\pi\lambda\pi\lambda = c^2$ if and only if $\pi\lambda c^{-1} = \lambda\pi c$ if and only if $\pi c \cdot \lambda = \lambda \cdot \pi c$. Since (viii) implies (v), the last statement follows.

If Ω satisfies (i) to (iv), and if P and L are finite (or if A and B are torsion groups), then $A^2 = A$, $B^2 = B$, and $C^2 = (P, L) = C$. Hence we have

PROPOSITION 32. *A finite group Ω is the V-extension of a nonabelian T-group G without elements of order* 2 *if and only if Ω satisfies the properties* (i) *to* (iv) *and* (vi).

10. Some remarks on projective planes

If $\mathfrak{E}$ is a projective plane and $Y \mid \omega$ an incident point-line-pair, denote by $\mathfrak{E}\,(Y \mid \omega)$ the incidence system one obtains from $\mathfrak{E}$ by deleting all lines through Y and all points on ω. $\mathfrak{E}(Y \mid \omega)$ is a P-system; every pair of nonparallel points, and of nonparallel lines, is regular. If moreover $\mathfrak{E}$ is (Y, Y)- and (ω, ω)-transitive, (hence a translation plane; see e.g. Pickert [3, Chapter 8]), then $\mathfrak{E}(Y \mid \omega)$ is a transitive P-system, and the methods of §6 can be used to introduce coordinates in $\mathfrak{E}$. As is well known (see e.g. Pickert [3, p. 101]), $\mathfrak{E}$ is a plane over a distributive quasifield (division-ring). If A denotes the group of all translations with axis ω and B the group of all translations with center Y, then $G = A \cdot B$ is a T-group, and all a not in C and all b not in C are regular.

Conversely if $G = A \cdot B$ is a T-group in which all a not in C and all b not in C are regular, then there exists a projective plane $\mathfrak{E}$ as above such that $\langle A, B \rangle \simeq \mathfrak{E}(Y \mid \omega)$.

The collineation group of $\mathfrak{E}(Y \mid \omega)$ is equal to the group of all semilinear transformations of $\mathfrak{E}(Y \mid \omega)$ (as defined in §4; see Thm. 2).

If $\mathfrak{E}$ has characteristic $\neq 2$, the V-extension Ω of G is the group generated by all point-reflections with axis ω and all line-reflections with center Y. The group of all collineations and dualities of $\mathfrak{E}(Y \mid \omega)$ is canonically isomorphic with the automorphism group of Ω.

Since $\langle P, L\rangle \simeq \mathfrak{E}(Y \mid \omega)$, Ω satisfies the following:

(vi′) There are π_0, π_1, λ_0, λ_1 such that
λ_1 is the only line incident with π_0 and π_1;
π_1 is the only point incident with λ_0 and λ_1;

(ix) if $\pi\pi' \notin C$, there exists $\lambda \mid \pi, \pi'$;
if $\lambda\lambda' \notin C$, there exists $\pi \mid \lambda, \lambda'$.

The following converse holds:

If Ω is a finite group that satisfies (i) to (iv), (vi′) and (ix), there exists a finite projective plane $\mathfrak{E}$ over a distributive quasi-field such that $\langle P, L\rangle \simeq \mathfrak{E}(Y \mid \omega)$.

Proof. By Prop. 32, Ω is the V-extension of a T-group G, which is regular because of (vi′), hence can be coordinatized by a ring R with 1. (ix) implies that $xa = b$, and $au = b$, have solutions x and u if $a \neq 0$. This together with R finite, implies that $R^{\times}$ is a loop.

As a consequence of Cor. 19, we have the following:

If $\mathfrak{E}$ is a projective plane and $\mathfrak{E}(Y \mid \omega)$ is the homomorphic image of a regular P-system $\langle \mathfrak{A}(R), \mathfrak{B}(R)\rangle$, then there exists a maximal ideal M in R such that $\mathfrak{E}(Y \mid \omega)$ is isomorphic to $\langle \mathfrak{A}(R/M), \mathfrak{B}(R/M)\rangle$.

(See also Klingenberg [1, p. 108, S 28].)

If F is the Galois field with 3 elements, then $\langle \mathfrak{A}(F), \mathfrak{B}(F)\rangle$ is a representation of the abstract Pappus configuration, as was stated in the introduction. Since 0, 1, and -1 are all the elements of F, the group $\Omega(F)$ (isomorphic to the group $\mathfrak{G}$ of the introduction) is now the complete collineation group, and has index 2 in the group of all collineations and dualities of $\langle \mathfrak{A}(F), \mathfrak{B}(F)\rangle$. Therefore $\Omega(F) \cong \mathfrak{G}$ has index 2 in its automorphism group.

Added in proof. Several results of this paper are contained in A. A. Albert, *Finite division algebras and finite planes*, Proceedings of Symposia in Applied Mathematics, Amer. Math. Soc., vol. X(1960), pp. 53–70. T-groups occur there as *elementary collineation groups*. Theorem 3 in § 4 and Theorem 5 in § 6 correspond to Theorem 7 and Theorem 6 resp., in Albert's paper.

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