# A PRIORI ESTIMATES FOR DIFFERENTIAL OPERATORS IN $L_\infty$ NORM[1]

BY

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It is well known that if $A$ and $B$ are constant-coefficient partial differential operators, with $A$ elliptic and order $B \leqq$ order $A$, then

$$\int | B\varphi |^2 \leqq \text{const} \int (| \varphi |^2 + | A\varphi |^2)$$

for all infinitely differentiable functions $\varphi$ of compact support. The proof of this "a priori estimate" uses Fourier transforms and the Plancherel theorem. Similar estimates are known for $p^{\text{th}}$ powers $(1 < p < \infty)$ in place of squares, although the easy proof for $p = 2$ does not generalize. In the present paper we investigate the limiting case $p = \infty$, where supremum norms appear in place of $L_p$ integral norms. This case turns out to be genuinely exceptional. For instance (Proposition 2) if $A$ and $B$ have the same order and $B \neq cA$, then no a priori estimate

$$\sup | B\varphi | \leqq \text{const} \sup (| \varphi | + | A\varphi |)$$

is possible. But (Proposition 5) if $B$ has strictly lower order, and $A$ is elliptic, then the estimate is reinstated. In fact (converse half of Proposition 5) in dimension $n \geqq 3$ this property is characteristic of elliptic operators, just as the $L_2$ a priori estimate is characteristic of elliptic operators for the case of equal orders. Before proving these last assertions we must establish (Propositions 3 and 4) some basic facts about the $n$-dimensional Fourier transform that do not seem to be in the literature. The connection between a priori estimates and Fourier transforms is explained in Proposition 1.

The other limiting case $p = 1$ has recently been treated by Ornstein [4]. The results for $L_1$ are essentially the same as those for $L_\infty$, but seem to be much harder to prove.

## Operator domination

If

$$A = \sum a_e \left(\frac{\partial}{\partial x}\right)^e = \sum a_{e_1 \cdots e_n} \left(\frac{\partial}{\partial x_1}\right)^{e_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{e_n}$$

is a partial differential operator with constant coefficients, then its *full characteristic polynomial* is

$$P = \sum a_e (ix)^e = \sum a_{e_1 \cdots e_n} (ix_1)^{e_1} \cdots (ix_n)^{e_n}.$$

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Throughout this paper, the notation $\| f \|$ means

$$\sup \{ | f(x) | : x = (x_1, \cdots, x_n) \in R^n \}.$$

PROPOSITION 1. *Let $B, A_1, \cdots, A_m$ be constant-coefficient differential operators in $n$ variables. Let $Q, P_1, \cdots, P_m$ be their full characteristic polynomials. Then*

$$\| B\varphi \| \leqq \text{const} \sum \| A_k \varphi \| \tag{1.1}$$

*for all infinitely differentiable $\varphi$ of compact support, if and only if*

$$Q = \sum M_k P_k \tag{1.2}$$

*for suitable Fourier-Stieltjes transforms $M_1, \cdots, M_m$.*

*Proof.* Write $D$ for the infinitely differentiable functions of compact support, and $C_0$ for the continuous functions vanishing at infinity. Take $D$ as domain for each of the $A_k$, and by the mapping $\varphi \rightarrow (A_1 \varphi, \cdots, A_m \varphi)$ embed their joint range in the direct sum $\oplus^m C_0$ of $m$ copies of $C_0$. Assuming the a priori estimate (1.1), the functional $(A_1 \varphi, \cdots, A_m \varphi) \rightarrow B\varphi(0)$ on the embedded joint range is continuous with respect to the natural topology of $\oplus^m C_0$. By the Hahn-Banach theorem this functional extends to the whole space $\oplus^m C_0$. And by the Riesz representation theorem we can write

$$B\varphi(0) = \sum \int (A_k \varphi) \mu_k \tag{1.3}$$

for some integrable (i.e., finite total mass) measures $\mu_1, \cdots, \mu_m$ and all $\varphi \in D$. Taking Fourier transforms we have

$$\int \hat{\varphi} Q = \sum \int \hat{\varphi} M_k P_k . \tag{1.4}$$

And since the Fourier transforms $\hat{\varphi}$ of $\varphi \in D$ are sufficiently numerous, we conclude that formula (1.2) of Proposition 1 holds, having assumed the a priori estimate (1.1).

On the other hand, assuming formula (1.2) we can work backwards through the previous steps to obtain formula (1.3). Then taking const $= \int \sum | \mu_k |$, we have

$$| B\varphi(0) | \leqq \text{const} \sum \| A_k \varphi \|.$$

And since the right-hand side is unaffected by translation, we obtain the estimate (1.1). This completes the proof of Proposition 1.

In the sequel, whenever the conditions of Proposition 1 obtain, we shall say that *the $A_k$ jointly dominate $B$* and *the $P_k$ jointly dominate $Q$.*

PROPOSITION 2. *Let $P_1, \cdots, P_m$ and $Q$ be polynomials of degree $\leqq d$, and let $P_1, \cdots, P_m$ jointly dominate $Q$. Then (writing $P_k^d$ and $Q^d$ for the homogene-*

*ous term of degree d) there exist numbers $c_k$ such that*

$$Q^d = \sum c_k P_k^d. \tag{2.1}$$

*Proof.* Let $A_k$ and $B$ be the differential operators having full characteristic polynomials $P_k$ and $Q$, as in the statement of Proposition 1. Write $Q$ as the sum of its homogeneous terms $Q^e$, and similarly $P_k$ as the sum of $P_k^e$. Substituting these sums into formula (1.3) of Proposition 1, we have

$$(\textstyle\sum_e B^e \phi)(0) = \sum_e \sum_k \int (A_k^e \varphi)\mu_k$$

for each infinitely differentiable $\varphi$ of compact support. If we define $\varphi_r$ by $\varphi_r(x) = \varphi(rx) = \varphi(rx_1, \cdots, rx_n)$, then for homogeneous $Q^e$ we have the formula

$$B^e(\varphi_r) = r^e(B^e\varphi)_r. \tag{2.2}$$

Thus replacing $\varphi$ by $\varphi_r$ in (2.2) we have

$$\textstyle\sum_e r^e (B^e \varphi)(0) = \sum_k \sum_e \int r^e(A_k^e \varphi)_r \mu_k.$$

Dividing both sides by $r^d$ and letting $r \to \infty$, we have

$$(B^d\varphi)(0) = \textstyle\sum_k c_k (A_k^d \varphi)(0),$$

where $c_k$ is the measure assigned to the origin by $\mu_k$. Thus, since $\varphi$ is arbitrary enough, we can conclude (2.1), and Proposition 2 is proved.

COROLLARY. *Let $P_1, \cdots, P_m$ and $Q$ all be homogeneous of the same degree. Then in order that the $P_k$ jointly dominate $Q$ it is necessary and sufficient that $Q$ belong to the vector space of polynomials spanned by the $P_k$.*

The proof is immediate.

*Remark.* The proof of Proposition 2 did not actually use the full hypothesis, but only that $\| Q\varphi \| \leqq \text{const} \sum \| P_k \varphi \|$ for all $\varphi$ of small support.

## Fourier transforms

This section is devoted to proving certain facts about Fourier transforms that will be needed in the next section. We begin with a further comment on Proposition 2. According to the proof of that proposition, each numerical coefficient $c_k$ in the expression $Q^d = \sum c_k P_k^d$ can be chosen (and if the $P_k^d$ are linearly independent, must be chosen) as the mass at the origin of the measure $\mu_k$ whose Fourier transform is $M_k$ in the expression $Q = \sum M_k P_k$ of Proposition 1. The following lemma interprets $c_k$ directly in terms of the function $M_k$.

LEMMA. *Let $\mu$ be an integrable measure, let $c$ be the signed mass of $\mu$ at the origin, and let $M$ be the Fourier transform of $\mu$. Then the constant function $c$ can be approximated uniformly by $\pi * M$, with $\pi$ a probability measure.*

*Proof.* This fact is due to Eberlein [3]. His proof first shows that $M$ is weakly-almost-periodic, then applies the Markov-Kakutani fixed-point theorem [1] to approximate $c$ by some convex combination of translates of $M$, or equivalently by $\pi * M$ with $\pi$ some probability measure of finite support.

Here is an elementary proof of the lemma. In fact, taking Fourier transforms, we can get a stronger result. It is possible to approximate the point-mass $c\delta$ in *measure norm* by $\hat{\pi}\mu$, with $\hat{\pi}$ now some positive-definite function such that $\hat{\pi}(0) = 1$. To accomplish the approximation we need only find a sequence of such $\hat{\pi}_j$ converging pointwise to zero everywhere except at the origin, for then with $\sup | \hat{\pi}_j | = \hat{\pi}_j(0) = 1$ we can apply the Lebesgue bounded convergence theorem to conclude that

$$\| \hat{\pi}_j \mu - c\delta \|_1 = \| \hat{\pi}_j(\mu - c\delta) \|_1 = \int | \hat{\pi}_j | \, | \mu - c\delta |$$

converges to zero. So let $V_j$ be a fundamental sequence of neighborhoods of the origin, and let $f_j$ be a positive continuous function vanishing outside $V_j$ such that $\int | f_j |^2 = f_j * \tilde{f}_j(0) = 1$. Then $\hat{\pi}_j = f_j * \tilde{f}_j$ is our sequence, and the lemma is proved.

In the application of this lemma to the converse half of Proposition 5, we would not actually need the full Eberlein conclusion concerning approximation by finite convex combinations. But this conclusion can be obtained by first discarding some small part of the mass of $\pi$ that lies near infinity, then chopping up the remaining mass into small pieces and averaging over each piece. We note also that the proof of the lemma will work for an arbitrary locally compact abelian group; in the case where the group is not metrisable, and hence there exists no fundamental sequence of neighborhoods $V_j$, it is enough to pick a sequence such that $| \mu - c\delta | (V_j)$ converges to zero.

In Proposition 3 below, the phrase *rapid decay* pertains to L. Schwartz's space $S$. Recall that $g \in S$ if $g$ is infinitely differentiable and if for each $m$ the function

$$r^{2m}\Delta^m g = (x_1^2 + \cdots + x_n^2)^m \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^m g$$

is bounded. We shall regard the function $f$ below as an element of the dual space $S'$. As such, it has a well defined Fourier transform $\hat{f}$, also in $S'$, given by $\int g\hat{f} = \int \hat{g} f$ for all $g \in S$.

PROPOSITION 3. *Let $a > 0$, $b > 0$, $a + b = n$. For any function $f$ on Euclidean n-space, the following conditions are equivalent:*

(3.1) *$f$ is homogeneous of degree $-a$, and infinitely differentiable away from the origin.*

(3.2) $f(x) = \int_0^\infty t^{a-1} g(tx) \, dt$, *for some infinitely differentiable $g$ of rapid decay on Euclidean n-space.*

(3.3) *The Fourier transform* $\hat{f}$ *is homogeneous of degree* $-b$, *and infinitely differentiable away from the origin.*

(3.4) $\hat{f}(y) = \int_0^\infty t^{b-1}\hat{g}(ty)\,dt$, *for some* $g$ *of rapid decay.*

*The* $g$ *in condition* (3.2) *and the* $\hat{g}$ *in condition* (3.4) *can be taken as Fourier transforms of each other, and one of them can be chosen to vanish identically near the origin and near infinity. For the equivalence of* (3.1) *with* (3.2) *one need only assume* $a > 0$, *and similarly for the equivalence of* (3.3) *with* (3.4) *only* $b > 0$.

*Proof.* By appealing to the theory of spherical harmonics one can swiftly show the equivalence of (3.1) with (3.3), which is the part of the lemma actually needed for the proof of Proposition 4. In fact, taking a complete normalized set of spherical harmonics $\{P_k\}$ with degree $P_k \leqq$ degree $P_l$ when $k \leqq l$, then $f$ on the sphere is infinitely differentiable if and only if $f(\theta) = \sum c_k P_k(\theta)$ with $c_k$ decaying more rapidly than any fixed power of $1/k$. Also, if the harmonic $P$ has degree $d$, then $r^{-a}P(\theta)$ has Fourier transform $r^{-b}P(\theta)$ up to complex constant $\lambda_d$ depending on $d$ in such a way that the rapid decay of the sequence $c_k$ is not disturbed.

We shall present, however, a more elementary proof, inspired partly by Calderón-Zygmund [2].

(3.1) $\Rightarrow$ (3.2). Take any infinitely differentiable function $\lambda(t)$ on the positive real axis identically zero near 0 and near $\infty$ and with

$$\int_0^\infty t^{-1}\lambda(t)\,dt = 1.$$

Define $g$ by $g(x) = \lambda(|x|)f(x)$. Then for $x \neq 0$, we have

$$\int_0^\infty t^{a-1}g(tx)\,dt = \int_0^\infty t^{a-1}\lambda(t\,|\,x\,|)f(tx)\,dt = f(x)\int_0^\infty t^{-1}\lambda(t\,|\,x\,|)\,dt$$

$$= f(x)\int_0^\infty t^{-1}\lambda(t)\,dt = f(x).$$

(3.2) $\Rightarrow$ (3.1). Given $g \in S$, it is clear that the integral in condition (3.2) defines at least a continuous $f$ homogeneous of degree $-a$. For the purposes of Proposition 4 we do not actually need more than this. But for completeness we present also a proof of differentiability. Represent $f$ as the limit of $f_\varepsilon$, where

$$f_\varepsilon(x) = \int_\varepsilon^{1/\varepsilon} t^{a-1}g(tx)\,dt.$$

We claim that this limit is uniform on any compact subset of the complement of the origin in Euclidean $n$-space. Indeed suppose $0 < r \leqq |\,x\,| \leqq 1/r < \infty$. Then

$$|f(x) - f_\varepsilon(x)| \leqq \left|\int_0^\varepsilon\right| + \left|\int_{1/\varepsilon}^\infty\right|.$$

There is some $c > 0$ such that

$$| g(x) | \leqq c \qquad \text{for} \quad | x | \leqq 1,$$
$$| g(x) | < c | x |^{-a-1} \quad \text{for} \quad | x | \geqq 1.$$

Hence

$$\left| \int_0^{\varepsilon} \right| = \left| \int_0^{\varepsilon} t^{a-1} g(tx)\, dt \right| = \left| \int_0^{\varepsilon |x|} t^{a-1} | x |^{-a} g\left( t \frac{x}{| x |} \right) dt \right| \leqq r^{-a} c \int_0^{\varepsilon / r} t^{a-1}\, dt.$$

On the other hand

$$\left| \int_{1/\varepsilon}^{\infty} \right| = \left| \int_{1/\varepsilon}^{\infty} t^{a-1} g(tx)\, dt \right| = | x |^{-a} \left| \int_{|x|/\varepsilon}^{\infty} t^{a-1} g\left( t \frac{x}{| x |} \right) dt \right| \leqq r^{-a} c \int_{r/\varepsilon}^{\infty} t^{-2}\, dt.$$

From the inequalities above, it is clear that $| \int_0^{\varepsilon} |$ and $| \int_{1/\varepsilon}^{\infty} |$ go to zero with $\varepsilon$; hence $f_\varepsilon$ converges to $f$ uniformly on each compact set not containing the origin. Moreover, let $D^m$ be any homogeneous differentiation operator,

$$D^m = \left( \frac{\partial}{\partial x_1} \right)^{m_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{m_n}.$$

Since differentiation across the integral sign is legitimate here, we have

$$(D^m f_\varepsilon)(x) = \int_{\varepsilon}^{1/\varepsilon} t^{m_1 + \cdots + m_n + a - 1} (D^m g)(tx)\, dt.$$

As $\varepsilon$ goes to 0, the above converges uniformly for $0 < r \leqq x < 1/r < \infty$. The argument is the same as for $f_\varepsilon$, taking $a + m_1 + \cdots + m_n$ as the new $a$. Thus $f$ is infinitely differentiable away from the origin.

(3.2) $\Rightarrow$ (3.4). We argue by duality. Take any $\varphi \in S$. We want

$$\int \varphi(y) \hat{f}(y)\, dy = \int \varphi(y)\, dy \int_0^{\infty} t^{b-1} \hat{g}(ty)\, dt.$$

We have

$$\int \varphi(y) \hat{f}(y)\, dy = \int \hat{\varphi}(x) f(x)\, dx = \int \hat{\varphi}(x)\, dx \int_0^{\infty} t^{a-1} g(tx)\, dt.$$

On the other hand

$$\begin{aligned} \int \varphi(y)\, dy \int_0^{\infty} t^{b-1} \hat{g}(ty)\, dt &= \int_0^{\infty} t^{b-1}\, dt \int \varphi(y) \hat{g}(ty)\, dy \\ &= \int_0^{\infty} t^{b-1}\, dt \int \hat{\varphi}(x)\, t^{-n} g(x/t)\, dx = \int_0^{\infty} t^{-a-1}\, dt \int \hat{\varphi}(x) g(x/t)\, dx \\ &= \int \hat{\varphi}(x)\, dx \int_0^{\infty} t^{-a-1} g(x/t)\, dt = \int \hat{\varphi}(x)\, dx \int_0^{\infty} t^{a-1} g(tx)\, dt. \end{aligned}$$

We have established the implications (3.1) $\Leftrightarrow$ (3.2) $\Rightarrow$ (3.4), from which all other implications can be got by symmetry. Thus Proposition 3 is completely established.

PROPOSITION 4. *If $f$ satisfies the conditions of Proposition 3, then $f$ is a Fourier transform near infinity. In other words there is some integrable function $\hat{g}$ on Euclidean n-space whose Fourier transform $g$ agrees with $f$ identically outside some compact subset.*

*Proof.* The theorem is not obvious even for the function $f(x) = 1/r$. But this special case can be handled by means of the theorem of Young-Kolmogoroff-Rudin [5], which gives sufficient conditions that a radial function $\varphi(r)$ be a Fourier transform of an $L^1$ function, namely that the derivatives of $\varphi$ be alternately positive and negative up to a certain order depending on the dimension $n$. And this single case $f = 1/r$, together with the facts about the Fourier transform of $r^{-a}P(\theta)$, for spherical harmonic $P$, can be used to establish all of Proposition 4.

We present, however, an elementary proof, which we owe to a conversation with Hörmander.

Let $\rho$ be infinitely differentiable, identically 1 near the origin, identically 0 near infinity. Define $g = f - \rho f$. We want to show that the Fourier transform $\hat{g} = \hat{f} - (\rho f)^{\wedge}$ belongs to $L^1$. (As usual, the Fourier transform $\hat{g}$ is defined by duality with the space $S$.) From Proposition 3 we know that $\hat{f}$ is locally $L^1$. And $(\rho f)^{\wedge}$ is an ordinary Fourier transform, and hence continuous. Combining these functions we see that $\hat{g}$ is locally $L^1$.

To show that $\hat{g}$ is also $L^1$ near infinity, we shall demonstrate that for large $m$ the function $r^{2m}\hat{g}$ is bounded, indeed is a classical Fourier transform. Or since multiplication by $r^2$ is the transform of the Laplacean operator $\Delta_x$, we shall demonstrate that $\Delta_x^m g = \Delta_x^m (f - \rho f)$ belongs to $L^1$. We need consider the behavior of $f - \rho f$ only near infinity, because it is differentiable at all finite points. And near infinity we may as well consider $f$ itself. Let us use polar coordinates $x = r\theta, r = |x|, \theta = x/|x|$. We can write

$$\Delta_x = \left(\frac{\partial}{\partial r}\right)^2 + \frac{n-1}{r}\left(\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_\theta$$

where $\Delta_\theta$ stands for the Laplace-Beltrami operator on the unit sphere. The condition (3.1) of Proposition 3 characterizing $f$ can obviously be stated in the equivalent form

$$f(r\theta) = r^{-a}h(\theta),$$

where $h$ is some infinitely differentiable function on the unit sphere. Hence $\Delta_x f = \Delta_x(r^{-a}h(\theta)) = r^{-a-2}[a(a+1)h(\theta) - a(n-1)h(\theta) + \Delta_\theta h(\theta)] = r^{-a-2}h_1(\theta)$. Similarly

$$\Delta_x^m f = r^{-a-2m}h_m(\theta)$$

for some infinitely differentiable function $h_m$ on the unit sphere. Hence when $a + 2m > n$, the function $\Delta_x^m f$ is integrable near infinity, $\Delta_x^m g$ is globally integrable, and Proposition 4 is proved.

## Elliptic operators

The constant-coefficient differential operator $A$ of order $d$ is called *elliptic* if the principal term $P^d$ of its full characteristic polynomial vanishes nowhere except at the origin. In this case we also call the full characteristic polynomial $P$ itself *elliptic.*

PROPOSITION 5. *Let $P$ be a polynomial in $n \geqq 3$ variables, of degree $d \geqq 2$. Then a necessary and sufficient condition that $P$ be elliptic is that $P$ and 1 together dominate all polynomials of degree $\leqq d - 1$. If $n = 2$, the condition is only necessary.*

*Proof.* We first prove necessity. Suppose $P$ elliptic and $Q$ of degree $\leqq d - 1$. Using Proposition 1, we want integrable measures $\mu$ and $\nu$ whose Fourier-Stieltjes transforms $M$ and $N$ satisfy

$$Q = MP + N. \tag{5.1}$$

Our construction will in fact make $\mu$ and $\nu$ integrable functions. First choose $N = Q$ near 0, $= 0$ near $\infty$, and infinitely differentiable everywhere. Such a Fourier transform $N$ exists because

(i) every differentiable function is locally a Fourier transform, and

(ii) every function locally a Fourier transform (near each finite point, and near $\infty$) is globally a Fourier transform.

Once $N$ is constructed, the continuous function $M$ is completely determined, since

$$M(x) = \frac{Q(x) - N(x)}{P(x)}, \qquad \text{for every } x \neq 0.$$

In particular, $M = 0$ near 0, $M = Q/P$ near $\infty$, and $M$ is infinitely differentiable at every finite point. We must decide whether the displayed formula for $M(x)$ defines a Fourier transform. Remarks (i) and (ii) in the construction of $N$ apply here also. Clearly there is no problem near finite points. Hence to finish this half of the proof of Proposition 5 we must show that there is a Fourier transform $M = Q/P$ near $\infty$.

We shall proceed by reducing all cases to the case $P$ homogeneous of degree $d$ and $Q$ homogeneous of degree $d - 1$. This "main case" is a direct consequence of Proposition 4, since $Q/P$ is infinitely differentiable away from the origin and homogeneous of degree $-1$.

*Case* 1. $P$ homogeneous of degree $d$ and $Q$ homogeneous of degree $= d - 1 - k$. We have $Q/P = (1/r^k) \cdot (r^k Q/P)$. Since the pointwise product of Fourier transforms is again a Fourier transform, it is enough that $1/r$ and $r^k Q/P$ each be Fourier transforms near $\infty$. The latter is the main case. And both the main case and the function $1/r$ are instances of Proposition 4.

*Case* 2. $P$ homogeneous of degree $d$, and $Q$ arbitrary. Split $Q$ into its homogeneous terms $Q^e$, apply the preceding case to each of these, and add.

*Case* 3. $P$ arbitrary, $Q$ arbitrary. Split $P$ into its homogeneous terms. $P = P^d + \sum_{e=0}^{d-1} P^e = P^d + \Sigma^{d-1}$. Then

$$\frac{Q}{P} = \frac{Q}{P^d}\left(\frac{1}{1 + \Sigma^{d-1}/P^d}\right).$$

By the preceding case, both $Q/P^d$ and $\Sigma^{d-1}/P^d$ are Fourier transforms near $\infty$. Choose a neighborhood $V$ of $\infty$ such that

$$\sup_{x \epsilon V} | \Sigma^{d-1}(x)/P^d(x) | < 1.$$

Let $A$ be the algebra of Fourier transforms with its natural Banach algebra norm. For each $f \epsilon A$ let $f_V$ be the restriction of $f$ to $V$, and let $A_V$ be the algebra of all such, with the natural quotient norm. If $c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots$ is a power series with radius of convergence 1, and $f \epsilon A$ has $\sup_{x \epsilon V} f(x) < 1$, then from the theory of Banach algebras we know that $c_0 + c_1 f_V + c_2 f_V^2 + \cdots$ converges in the Banach algebra $A_V$. In particular, substituting $f = \Sigma^{d-1}/P^d$ into the Maclaurin expansion of $1/(1 + \lambda)$, we see that $1/(1 + \Sigma^{d-1}/P^d)$ belongs to $A_V$. Hence the arbitrary case finally reduces to the main case, and the direct half of Proposition 5 is proved.

We now undertake the converse, assuming $n \geqq 3$. We must demonstrate the incompatibility of the following two assumptions:

$$Q = MP + N, \tag{5.1}$$

$$P^d(x_0) = 0. \tag{5.2}$$

In (5.1) the polynomial $P$ has degree $d \geqq 2$, the polynomial $Q$ degree $\leqq d - 1$, the functions $M$, $N$ are Fourier-Stieltjes transforms whose choice depends on $Q$. In (5.2) of course $x_0$ is some point other than the origin.

From now on we shall write $P = P^d + P^{d-1} + \Sigma^{d-2}$, with $\Sigma^{d-2}$ representing all the terms of degree $\leqq d - 2$. If we choose $Q$ homogeneous of degree $d - 1$ such that $Q(x_0) \neq 0$, then on the ray determined by $x_0$ we have

$$Q(tx_0) = t^{d-1}Q(x_0) = M(t^{d-1}P^{d-1}(x_0) + \Sigma^{d-2}(tx_0)) + N.$$

Since $M$ and $N$ are bounded, and since $\Sigma^{d-2}(tx_0) \leqq \text{const} \cdot t^{d-2}$, we must have $P^{d-1}(x_0) \neq 0$, and without loss of generality we can assume $P^{d-1}(x_0) = 1$.

Now consider the sphere $\{x : | x | = | x_0 |\}$. We claim that there will exist at $x_0$ some (pure first order) derivation $T$ tangential to the sphere and such that $TP^d(x_0)$ is real (possibly zero). Indeed this assertion about a homogeneous polynomial in $n$ real variables at a point on the real $(n - 1)$-sphere becomes an assertion about an arbitrary polynomial at the origin in real $(n - 1)$-space: Given a complex polynomial $F$ in $\nu \geqq 2$ real variables, we want to find real numbers $c_k$ not all zero such that $\sum c_k \dfrac{\partial F}{\partial x_k}(0)$ is real. Equivalently we want the differential $dF$ (which coincides with the pure linear term $F^1$ of $F$, and can be regarded as a linear map from real $\nu$-space into real 2-space) either to include the real axis in its range or else to send some nonzero $\nu$-vector into zero. But when $\nu \geqq 2$, this always happens because $\nu$ = dimension (range $F^1$) + dimension (nullspace $F^1$).

Returning now to $x_0$ on the sphere, we claim that $P^d$ cannot vanish identically on the great circle determined by $x_0$ and the direction $T$, or indeed on any other great circle. For if the homogeneous polynomial $P^d$ vanishes on a great circle, then $P^d$ vanishes on the whole 2-dimensional subspace $Z$ spanned by that great circle. Let $Q_Z$ be an arbitrary polynomial of degree $d - 1$ in 2 real variables, regarded as a function on $Z$. Let $Q$ be the same polynomial regarded as a function on $n$-space, or for that matter let $Q$ be any polynomial of degree $d - 1$ in $n$ real variables whose restriction to $Z$ is $Q_Z$. Writing subscript $Z$ systematically for this restriction map, the equation (5.1) gives us $Q_Z = M_Z P_Z + N_Z$. The functions $M_Z$ and $N_Z$ are Fourier-Stieltjes transforms on the plane. (Proof: Write

$$M = M^1 - M^2 + i(M^3 - M^4)$$

with the $M^j$ positive definite. Then $M_Z = M_Z^1 - M_Z^2 + i(M_Z^3 - M_Z^4)$. But the $M_Z^j$ are positive definite on the plane by direct appeal to the matrix definition.) Hence $P_Z$, which has degree at most $d - 1$, dominates an arbitrary polynomial $Q_Z$ of degree $d - 1$, contradicting Proposition 2.

Hence choose $y_0$ on the great circle determined by $x_0$ and $T$ so that $P^d(y_0) \neq 0$, and for the rest of the proof restrict everything to the plane $Z$ spanned by $x_0$ and $y_0$. We will use classical notation $(x, y)$ for rectangular coordinates in $Z$, and will assume that $x_0$ and $y_0$ have coordinates (1, 0) and (0, 1) respectively. This is legitimate because the argument from now on is purely affine. In this notation, our assumptions so far amount to

$$P^d(1, 0) = 0, \tag{5.2*}$$

$$P^{d-1}(1, 0) = 1, \tag{5.3}$$

$$\frac{\partial P^d}{\partial y}(1,0) \quad \text{real}. \tag{5.5*}$$

From now on we shall refer to (5.1) only in the following instance:

$$P^{d-1} = MP + N. \tag{5.1*}$$

For the sake of concreteness, we choose to derive most of our subsequent formulas from the factorization

$$P^d = \textstyle\prod_{k=1}^d (a_k x - b_k y). \tag{5.4}$$

Such factorization is possible because there are only 2 variables. The $(a_k : b_k)$ are the *homogeneous roots* of $P^d$. Ellipticity of $P$ would mean all $a_k$ and $b_k$ nonzero and all $a_k/b_k$ nonreal. But we are now assuming $P$ nonelliptic, and in fact (5.2*) can be restated in the form

(5.2**) $$P^d = y\prod^{d-1}(a_k x - b_k y).$$

Hence

$$\frac{\partial P^d}{\partial y} = \prod^{d-1}(a_k x - b_k y) + y\frac{\partial}{\partial y}\prod^{d-1}(a_k x - b_k y).$$

And

$$\frac{\partial P^d}{\partial y}(x, 0) = ax^{d-1}, \qquad \text{with } a = \prod^{d-1} a_k.$$

We shall find that in the presence of assumptions (5.1*) and (5.2*) either one of the alternatives

(5.5) $$\frac{\partial P^d}{\partial y}(x, 0) \neq 0 \quad \text{or} \quad \frac{\partial P^d}{\partial y}(x, 0) = 0$$

leads to a contradiction.

First suppose $\frac{\partial P^d}{\partial y}(x, 0) = ax^{d-1} \neq 0$, so that we may write

$$P^d = ay\prod^{d-1}(x - (b_k/a_k)y).$$

Then along the horizontal line $y = -1/a$ we have

$$P^d(x, -1/a) = -\prod^{d-1}(x + b_k/a_k a).$$

And letting $x \to +\infty$ along $y = -1/a$ we have

(5.6) $$P^d(x, -1/a) \sim -x^{d-1}.$$

On the other hand, $y/x \to 0$ along this same horizontal line. Since

$$P^{d-1}(x, y) = x^{d-1}P^{d-1}(1, y/x),$$

from (5.3) we have

(5.7) $$P^{d-1}(x, -1/a) \sim x^{d-1}.$$

Let us write $o^k$ for any function such that $o^k/x^k \to 0$ as $x \to \infty$, and $O^k$ for any function such that $O^k/x^k$ is bounded as $x \to \infty$. Then the asymptotic expressions (5.6) and (5.7) can be written equivalently:

(5.6*) $$P^d(x, -1/a) = -x^{d-1} + o^{d-1},$$

(5.7*) $$P^{d-1}(x, -1/a) = x^{d-1} + o^{d-1}.$$

We also have

(5.8) $$M = O^1, \qquad N = O^1.$$

And substituting (5.6*), (5.7*), (5.8) into (5.1*) we obtain on $y = -1/a$,

$$x^{d-1} = o^{d-1} + O^1(-x^{d-1} + x^{d-1} + o^{d-1}) + O^1.$$

This asymptotic absurdity is what results from assuming $\frac{\partial P}{\partial y}(x, 0) \neq 0$.

We now must arrive at a contradiction on the other horn of (5.5). As a preliminary step we shall demonstrate that $M \to 1$ for $x \to \infty$, $y = \text{const.}$ On the $x$-axis, the formula (5.1*) becomes

$$x^{d-1} = N + M(x^{d-1} + o^{d-1}).$$

And from this formula, without any information about $\partial P^d/\partial y$, we at once deduce that $M \to 1$ along the $x$-axis. But since we are now assuming $\frac{\partial P^d}{\partial y}(1, 0) = 0$, or equivalently

$$P^d = y^2 \prod^{d-2} (a_k x - b_k y), \tag{5.9}$$

we are able to show that $M$ approaches 1 along every horizontal line.

In fact, setting $y = b$ in (5.9),

$$P^d(x, b) = b^2 \prod^{d-2} (a_k x - b_k b) \sim b^2 \left(\prod^{d-2} a_k\right) x^{d-2}.$$

Hence

$$P^d(x, b) = o^{d-1}. \tag{5.10}$$

And, independent of (5.5), we still have

$$P^{d-1}(x, b) \sim x^{d-1}, \tag{5.11}$$

$$N(x, b) = O^1, \tag{5.12}$$

arguing as in the special case $b = -1/a$. Substituting (5.10), (5.11), (5.12) into (5.1*), we have $x^{d-1} = M(x^{d-1} + o^{d-1}) + o^{d-1}$ along each horizontal line $y = b$. In particular

$$M(x, b) \to 1 \quad \text{as} \quad x \to \infty. \tag{5.13}$$

We now use the Eberlein lemma to get our ultimate contradiction. In fact, if $\alpha$ and $\beta$ are the convex invariant means of $M$ and $N$, from formula (5.1*) and Proposition 2 we have

$$0 = \alpha P^d + \beta,$$

which forces $\alpha = 0$. Then 0 can be approximated uniformly by convex combinations of translates of $M$. But this is impossible, since by (5.13) every translate of $M$ has limit 1 along the $x$-axis. This completes the proof of Proposition 5 in dimension $n \geqq 3$.

Finally let us remark that the converse half of Proposition 5 is actually false in dimension $n = 2$. (We owe this observation, and the counterexample justifying it, to Malgrange.) Consider the nonelliptic operator

$$A = \left(\frac{\partial}{\partial x} + 1\right)\left(\frac{\partial}{\partial y} + 1\right).$$

If $\theta$ is the Heaviside function ($= 0$ on the negative real axis, and $= 1$ on the positive real axis), and if $e(x, y) = \theta(x)\theta(y) \exp(-x - y)$, then $Ae = \delta$

(Dirac measure). We have $\dfrac{\partial}{\partial x}\delta = A\dfrac{\partial e}{\partial x}$. But

$$\frac{\partial e}{\partial x} = [\delta_x - \theta(x) \exp(-x)]\theta(y) \exp(-y),$$

which is an integrable measure. Hence by Proposition 1, $A$ dominates $\partial/\partial x$. By symmetry, $A$ also dominates $\partial/\partial y$.

## Bibliography


1. N. Bourbaki, *Espaces vectoriels topologiques*, Chapters I and II, Appendix, pp. 114–115, Éléments de Mathematique, XV, Book V, Paris, 1953.
2. A. P. Calderón and A. Zygmund, *Algebras of certain singular operators*, Amer. J. Math., vol. 78 (1956), pp. 310–320.
3. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc., vol. 67 (1949), pp. 217–240.
4. D. Ornstein, *A non-inequality for differential operators in the $L_1$ norm*, Arch. Rational Mech. Anal., vol. 11 (1962), pp. 40–49.
5. W. Rudin, *Representation of functions by convolutions*, J. Math. Mech., vol. 7 (1958), pp. 103–115.



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