CONVOLUTIONS, MEANS, AND SPECTRA

BY
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1. Introduction

In two recent papers ([6] and [7] of the bibliography) Kesten studied symmetric probability densities φ on discrete groups G. For each such φ he defined first a matrix and then an operation on $l_2(G)$, an operation easily shown to be equivalent to right convolution, $\circ\varphi$, by φ. The properties found in [6] for the family of groups where $\lambda_2(\varphi)$, the spectral radius of $\circ\varphi$ operating in $l_2(G)$, is 1 suggested to Kesten the result he proved in [7]: $\lambda_2(\varphi) = 1$ *for every symmetric* φ *on* G *if and only if there is an invarant mean on the bounded functions of* G.

In this paper I exploit some results of my earlier work, [1] on invariant means and [2] on uniform rotundity, to give a simpler proof of a more general result. My proof uses uniform rotundity in place of symmetry and strong amenability (see [1, §5, Theorem 1]) to replace Følner's characterization [4] of groups with invariant means. This simpler proof with no dependence on self-adjointness applies to all l_p spaces, $p > 1$, and applies also to some semigroups which are not groups. In this semigroup case, where right and left invariance are independent properties of means, it turns out that right invariance of means is to be used in studying right convolutions.

The relation between these results and those of the paper [3] of Dieudonné, which is concerned with locally compact groups with the property that right convolution by each (Haar measurable) probability density is an operator of norm one in every L_p over G, are discussed but not settled in the final section of this paper.

Theorem 3 isolates from the many results a response to Kesten's hope that some more direct construction of invariant means might be found when 1 is in the spectrum of enough operators $\circ\varphi$.

2. Principal results

It is assumed in this section that each semigroup Σ discussed has right cancellation (rc) and has a nonempty set U of right units (ru). If φ is a probability density on Σ, $P_\varphi = \{\sigma : \varphi(\sigma) > 0\}$. $\circ\varphi$, or right convolution by φ, is defined precisely in the next section, as are $\delta\sigma$ and f^p.

THEOREM 1. *The following conditions on an (rcru) semigroup* Σ *are equivalent:*

(a) Σ *is right amenable.*

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(b) *For each probability density* φ *on* Σ *and each* $p \geqq 1$ *the linear operator* $\circ\varphi$ *has, when considered as a linear operator from* $l_p(\Sigma)$ *into* $l_p(\Sigma)$, *the number* 1 *in its spectrum.*

(c) *For each probability density* φ *on* Σ *and each* $p \geqq 1$ *the spectral radius* $\lambda_p(\varphi)$ *of the operator* $\circ\varphi$ *in* $l_p(\Sigma)$ *is* 1.

(d) *For each probability density* φ *on* Σ *and each* $p \geqq 1$ *the norm of* $\circ\varphi$ *in* $l_p(\Sigma)$, $\| \circ\varphi \|_{p\to p}$, *is* 1.

(e) *For each finite* (*or countable*) *subset* ξ *of* Σ *there are at least one* $p > 1$ *and at least one probability density* φ *on* Σ *such that* $\| \circ\varphi \|_{p\to p} = 1$, $P_\varphi \supseteq \xi$, *and* $P_\varphi \cap U$ *is not empty.*

(f) *Every finitely* (*or countably*) *generated subsemigroup of* Σ *is contained in a strongly right amenable countable subsemigroup of* Σ.

(g) Σ *is strongly right amenable.*

Starting from (b) and using Lemmas 5 and 6 we give a geometric condition.

THEOREM 2. *For* (*rcru*) *semigroups conditions* (a) *to* (g) *are equivalent to*

(h) *For each* $p > 1$, K, *the norm-closed convex hull of the set of right shift operators in* $l_p(\Sigma)$, $\{\circ\,\delta\sigma : \sigma \in \Sigma\}$, *consists of operators* T *each of which is of norm one and has* 1 *in its spectrum.*

THEOREM 3. *For* (*rcru*) *semigroups if for any* $p > 1$, $(f_n, n \in \Delta)$ *is a net of nonnegative elements of norm one in* $l_p(\Sigma)$ *such that for every probability density* φ, $\| f_n \circ \varphi \|_p \to 1$, *then* $(f_n^p, n \in \Delta)$ *is a net of probability densities converging in norm to right invariance.*

For groups some additional results follow.

THEOREM 4. *When* Σ *is a group with unit element* u, *the conditions* (a) *to* (h) *are also equivalent to each of the following:*

(e′) *For each finitely* (*or countably*) *generated subgroup* Σ' *of* Σ *there are at least one* $p > 1$ *and one probability density* φ *such that* $u \in P_\varphi \subseteq \Sigma'$, P_φ *generates the group* Σ', *and* $\| \circ\varphi \|_{p\to p} = 1$.

(f′) *Every finitely* (*or countably*) *generated subgroup* Σ' *of* Σ *is amenable.*

(a_l), (b_l), $\cdots$, (h_l), *the conditions obtained from* (a), $\cdots$, (h) *by replacing right amenability by left amenability and right convolution by left convolution.*

The theorem of Kesten [7, p. 150] is similar to the case of (a) $\Leftrightarrow$ (e‴) of the following

COROLLARY. *If* Σ *is a countable group, then* (a) *is equivalent to*

(e″) *There are at least one* $p > 1$ *and at least one probability density* φ *on* Σ *such that* $\| \circ\varphi \|_{p\to p} = 1$, $u \in P_\varphi$, *and* P_φ *generates* Σ.

(e‴) *Like* (e″) *with "*1 *in the spectrum of* $\circ\varphi$*" replacing "*$\| \circ\varphi \|_{p\to p} = 1$*".*

3. Definitions and some lemmas

Terminology for amenable semigroups and general normed spaces can be found in references [1] and [2], respectively.

A *semigroup* is a system Σ of elements with an associative binary rule of multiplication. A semigroup Σ has *right cancellation* when for a, b, and c in Σ, $ac = bc$ implies $a = b$. u is a *right unit* in Σ if $au = a$ for all a in Σ.

If $p \geqq 1$, $l_p(\Sigma)$ is (see [2, Chapter 2, §2]) the normed linear space of all those complex-valued functions f on Σ for which $\| f \|_p$, defined to mean $(\sum_{s \epsilon \Sigma} | f(s) |^p)^{1/p}$, is finite. $l_\infty(\Sigma)$, on rare occasions, will be used to refer to the space $m(\Sigma)$ of all complex-valued functions x on Σ such that $\| x \|_\infty$, defined to be $\sup \{ | x(s) | : s \epsilon \Sigma \}$, is finite.

An f in any of these function spaces is called *nonnegative*, written $f \geqq 0$, if for every s in Σ the function-value $f(s)$ is a nonnegative number.

A *mean* on Σ is a linear functional μ on $m(\Sigma)$ such that (i) $\| \mu \| = 1$, and (ii) $\mu(x) \geqq 0$ if $x \epsilon m(\Sigma)$ and $x \geqq 0$. If e is the constantly 1 function on Σ, then a mean μ also satisfies (iii) $\mu(e) = 1$; also any two of the conditions (i), (ii), and (iii) imply the third.

The *left* [or *right*] *translation operators* l_s [or r_s] in $m(\Sigma)$ are defined when Σ is a semigroup by the following formulas (see [1, §4]): For each x in $m(\Sigma)$ and each t in Σ,

$$[l_s x](t) = x(st) \qquad [\text{or} \quad [r_s x](t) = x(ts)].$$

A mean μ on Σ is *left* [or *right*] *invariant* whenever

$$\mu(l_s x) = \mu(x) \qquad [\text{or} \quad \mu(r_s x) = \mu(x)]$$

for every x in $m(\Sigma)$ and s in Σ. Σ is called *left* [or *right*] *amenable* if there exists a left [or right] invariant mean on Σ. Σ is *amenable* if there is at least one mean μ on Σ which is both left and right invariant. From [1] we quote §4, (A): *If Σ is both left and right amenable, it is amenable;* and §4, (B): *A left* [or *right*] *amenable group is amenable.*

A *probability density* on Σ is a nonnegative element φ of $l_1(\Sigma)$ such that $\sum_{\sigma \epsilon \Sigma} \varphi(\sigma) = 1$. It is easily seen that if φ is a probability density on Σ, then $Q\varphi$, defined from $l_1(\Sigma)$ to $m(\Sigma)^*$ by

$$[Q\varphi](x) = \sum_{\sigma \epsilon \Sigma} \varphi(\sigma) x(\sigma)$$

for every x in $m(\Sigma)$, is a mean on Σ. For this reason the probability densities on Σ were called, by slight abuse of language, *finite or countable means* in my paper [1].

Define the (Kronecker) δ-mapping of Σ into $l_1(\Sigma)$ by: For each s in Σ, δs is that element of $l_1(\Sigma)$ which is 1 at s and 0 elsewhere.

The operation of convolution is defined among functions on Σ so that δ is an isomorphism of Σ into the multiplicative semigroup of the convolution algebra $l_1(\Sigma)$. Formally if ψ and φ are functions on Σ, convolution is defined by

$$[\psi \circ \varphi](\sigma) = \sum_{st = \sigma} \psi(s) \varphi(t),$$

where the sum is over the unordered set of all those ordered pairs (s, t) of elements of Σ for which $st = \sigma$. Because this set of pairs has no natural order, $[\psi \circ \varphi](\sigma)$ is defined only when the series is unconditionally (that is, absolutely) convergent. Hereafter the defining formula will be abbreviated as

$$(1) \qquad [\psi \circ \varphi](\sigma) = \sum_{st=\sigma} \psi(s)\varphi(t) \qquad \text{for each } \sigma \text{ in } \Sigma.$$

§5 of [1] discusses the semigroup algebra $l_1(\Sigma)$; it is shown there that

$$(2) \qquad \| \psi \circ \varphi \|_1 \leqq \| \psi \|_1 \, \| \varphi \|_1 ,$$

and that the convolution of two probability densities is another probability density.

Even in a semisgroup, for each probability density φ the operator $\circ\,\varphi$ of right convolution by φ can be regarded as a linear, not-necessarily-closed-or-continuous operator defined on some linear, not-necessarily-closed subspace of $l_p(\Sigma)$ which includes $l_1(\Sigma)$. (See §5, C1 and C2, for examples.) However, in the special case of a right-cancellation semigroup Σ, $\circ\,\varphi$ is defined everywhere in each $l_p(\Sigma)$, $p \geqq 1$, and carries $l_p(\Sigma)$ into $l_p(\Sigma)$ without increase of any norm. (See Lemma 4, below.)

If f is a complex-valued function on Σ and if d is a positive number, define the function f^d on Σ by

$$[f^d](s) = | f(s) |^d \operatorname{sign} f(s) \qquad \text{for each } s \text{ in } \Sigma.$$

Here, for a complex number z, sign $z = z/| z |$ if $z \neq 0$, sign $0 = 0$.

LEMMA 1. *Let Σ be a set, let p and d be positive numbers, and let f be an element of $l_p(\Sigma)$; then*

$$\text{(i)} \quad f^d \in l_{p/d}(\Sigma) \quad \textit{and} \quad \text{(ii)} \quad \| f^d \|_{p/d} = [\| f \|_p]^d.$$

The proof follows at once from the definitions.

LEMMA 2. *Let φ and ρ be probability densities on a set Σ, let p exceed* 1, *and define $f = \varphi^{1/p}$, $r = \rho^{1/p}$. Then*

(i) $\| f \|_p = 1 = \| r \|_p$, *so f and r are on the unit sphere of $l_p(\Sigma)$,*
(ii) $\| f - r \|_p \leqq [\| \varphi - \rho \|_1]^{1/p}$, *and*
(iii) $\| \varphi - \rho \|_1 \leqq p2^{p-1} \| f - r \|_p$.

Proof. (i) follows from Lemma 1.

(ii). If $0 < a < b$, concavity of the p$^{\text{th}}$-power function implies that $(b - a)^p < b^p - a^p$. Hence

$$[\| f - r \|_p]^p = \sum_{s \in \Sigma} | f(s) - r(s) |^p \leqq \sum_s | [f(s)]^p - [r(s)]^p |$$
$$= \sum_s | \varphi(s) - \rho(s) | = \| \varphi - \rho \|_1 .$$

(iii). The theorem of the mean applied to the p^{th}-power function in $0 \leqq a \leqq t \leqq b \leqq 1$ yields a number t between a and b such that

$$b^p - a^p = (b - a)pt^{p-1} \leqq (b - a)p(a + b)^{p-1}.$$

Then

$$\| \varphi - \rho \|_1 = \sum_{s \epsilon \Sigma} | \varphi(s) - \rho(s) | = \sum_s | f^p(s) - r^p(s) |$$
$$\leqq \sum_s p | f(s) - r(s) | | f(s) + r(s) |^{p-1}.$$

If q is chosen in the usual way so that $1/p + 1/q = 1$, then $p/q = p - 1$, and $| f + r |^{p-1} \epsilon l_q(\Sigma)$; by Hölder's inequality

$$\| \varphi - \rho \|_1 \leqq p [\sum_s | f(s) - r(s) |^p]^{1/p} [\sum_s | f(s) + r(s) |^p]^{1/q}$$
$$\leqq p \| f - r \|_p [\| f + r \|_p]^{p/q} \leqq p 2^{p-1} \| f - r \|_p .$$

The homeomorphism of l_1 with l_p was proved by Mazur [9]; this is his proof, and it is used to get *uniform* continuity between the positive parts of the unit spheres.

Lemma 3. *Let Σ be a right-cancellation semigroup, and let p and d be positive numbers. Then*

(i) *for each s in Σ and each f defined on Σ*

$$[f^d] \circ \delta s = [f \circ \delta s]^d.$$

(ii) *For each s in Σ and each $p \geqq 1$ the right shift operation $\circ \delta s$ is an isometry of $l_p(\Sigma)$ into $l_p(\Sigma)$. The range of $\circ \delta s$ is all of $l_p(\Sigma)$ if and only if right multiplication by s is a permutation of Σ.*

(iii) *If φ is a probability density on Σ and $p \geqq 1$, then*

(largest real number in spectrum of $\circ \varphi$ in $l_p(\Sigma)$)

$$\leqq \lambda_p(\varphi) \leqq \| \circ \varphi \|_{p \to p} \leqq \| \varphi \|_1 = 1.$$

Proof of (i). For each σ, τ in Σ,

$$[f^d \circ \delta\sigma](\tau) = \sum_{st=\tau} f^d(s) \delta\sigma(t) = \sum_{s\sigma=\tau} f^d(s).$$

By the right-cancellation property this last sum has no more than one term. By interpreting an empty sum as zero, the original sum equals

$$[\sum_{s\sigma=\tau} f(s)]^d = [[f \circ \delta\sigma](\tau)]^d = [f \circ \delta\sigma]^d(\tau).$$

Since this holds for each τ in Σ, $f^d \circ \delta\sigma = [f \circ \delta\sigma]^d$.

Proof of (ii). If $\tau \epsilon \Sigma$, then $[f \circ \delta\sigma](\tau) = \sum_{s\sigma=\tau} f(s)$, and by right cancellation the sum has only one term. Hence

$$\| f \circ \delta\sigma \|_p = (\sum_{\tau\epsilon\Sigma} | \sum_{s\sigma=\tau} f(s) |^p)^{1/p} = (\sum_{s\epsilon\Sigma} | f(s) |^p)^{1/p} = \| f \|_p .$$

If right multiplication by σ is a permutation of Σ, then for given g in $l_p(\Sigma)$, $f \circ \delta\sigma = g$ can be solved for f by setting $f(s) = g(s\sigma)$, so $\circ \delta\sigma$ is a mapping of $l_p(\Sigma)$ *onto* $l_p(\Sigma)$. If right multiplication by σ is not a permutation of Σ, it must fail to map Σ onto Σ, that is, there is a τ in Σ which is not in $\Sigma\sigma$. Hence, no matter how f may be chosen, $\delta\tau$ is not $f \circ \delta\sigma$. But $\| \delta\tau \|_p = 1$, so $\delta\tau \epsilon l_p(\Sigma)$ and is not in the range of $\circ \delta\sigma$.

Proof of (iii). If $f \in l_p(\Sigma)$, then

$$f \circ \varphi = \sum_{\sigma \in \Sigma} \varphi(\sigma)[f \circ \delta\sigma].$$

Hence

$$\| f \circ \varphi \|_p = \| \sum_{\sigma \in \Sigma} \varphi(\sigma) f \circ \delta\sigma \|_p \leqq \sum_{\sigma \in \Sigma} \varphi(\sigma) \| f \circ \delta\sigma \|_p$$
$$= \sum_{\sigma \in \Sigma} \varphi(\sigma) \| f \|_p = \| f \|_p \| \varphi \|_1 = \| f \|_p .$$

Hence

$$\| \circ \varphi \|_{p \to p} = \sup \{ \| f \circ \varphi \|_p : \| f \|_p \leqq 1 \} \leqq 1.$$

The other inequalities of (iii) are standard properties of elements of a normed algebra; see [8].

LEMMA 4. *Let Σ be a semigroup with right cancellation and a right unit, let p be > 1, and let φ be a probability density such that P_φ contains a right unit u of Σ. Then the following conditions on φ are equivalent:*

(i) $\| \circ \varphi \|_{p \to p} = 1$.

(ii) *There exist nonnegative f_n of norm one in $l_p(\Sigma)$ such that $\| f_n \circ \varphi \|_p \to 1$.*

(iii) *There exist (the same) nonnegative f_n of norm one in $l_p(\Sigma)$ such that*

$$\| f_n - f_n \circ \delta\sigma \|_p \to 0 \qquad \textit{for each } \sigma \textit{ in } P_\varphi .$$

(iv) $\lambda_p(\varphi) = 1$.

(v) 1 *is in the spectrum of* $\circ \varphi$.

Proof. Lemma 3(iii) shows that (v) implies (iv) implies (i). (i) implies (ii) because the sum defining $\| f \circ \varphi \|_p$ is not decreased if f is replaced by $| f |$.

(ii) implies (iii). Given a sequence or net $(f_n, n \in \Delta)$ which fails to satisfy (iii), then for any right unit u in P_φ and some s_0 in P_φ we have the conclusion that $\| f_n \circ \delta u - f_n \circ \delta s_0 \|_p$ does not tend to zero. By uniform rotundity of $l_p(\Sigma)$ (see [2, pp. 112–113]) it follows that

$$\| \varphi(u) f_n \circ \delta u + \varphi(s_0) f_n \circ \delta s_0 \|_p / (\varphi(u) + \varphi(s_0))$$

can not tend to 1 so must have some upper limit $1 - \eta < 1$. Then

$$\| f_n \circ \varphi \|_p = \| \varphi(u) f_n \circ \delta u + \varphi(s_0) f_n \circ \delta s_0 + \text{other terms} \|_p$$

has an upper limit $\leqq 1 - \eta(\varphi(u) + \varphi(s_0)) < 1$, so (ii) fails for $(f_n, n \in \Delta)$ if (iii) fails.

(iii) implies (v). Choose $(f_n, n \in \Delta)$ to satisfy (iii). Then

$$\| f_n - f_n \circ \varphi \|_p = \| \sum_{s \in \Sigma} \varphi(s)(f_n - f_n \circ \delta s) \|_p \leqq \sum_{s \in \Sigma} \varphi(s) \| f_n - f_n \circ \delta s \|_p ,$$

which tends to zero. Hence $\circ (\delta u - \varphi)$ has no bounded inverse; that is, 1 is in the spectrum of $\circ \varphi$.

Note that the nature of this proof, especially the crux of it, (ii) implies (iii), can be applied to a more geometrical result.

LEMMA 5. *If B is a uniformly rotund space, if T is a linear operator from B into B of norm $\leqq 1$, if $0 < \alpha < 1$, and if $U = \alpha I + (1 - \alpha)T$, then the following conditions are equivalent:* (i) $\| U \| = 1$. (ii) 1 *is in the spectrum of* U. (iii) 1 *is in the spectrum of* T.

Because $\| U \| \leqq 1$, (ii) implies (i) as before. If $\| U \| = 1$, then there exist f_n of norm one in B such that $\| Uf_n \| \to 1$, so $\| \alpha f_n + (1 - \alpha)Tf_n \| \to 1$; then $\| (f_n + Tf_n)/2 \| \to 1$. By uniform rotundity, $\| f_n - Tf_n \| \to 0$, so $I - T$ can not have a bounded inverse; that is, 1 is in the spectrum of T. This proves (i) implies (iii). To see that (iii) implies (ii) note that because $I - U = (1 - \alpha)(I - T)$, the operator $I - U$ has no inverse if and only if $I - T$ has no inverse, so 1 is in the spectrum of U if and only if it is in the spectrum of T.

Let us restate Lemma 5 in a more geometric form.

LEMMA 6. *Let $\mathcal{S}$ be the unit sphere in the algebra of continuous linear operators from a uniformly rotund Banach space B into B. Then an element T of $\mathcal{S}$ has* 1 *in its spectrum if and only if the segment connecting T to I (the identity operator) is a subset of $\mathcal{S}$.*

If Γ is a subset of Σ, define $\mathcal{G}(\Gamma)$ to be the smallest subset of Σ which contains Γ and contains with any two of s, t, and st the third.

(The family of sets $\mathcal{G} \supseteq \Gamma$ closed under the given process has Σ as a member, and the intersection of all such $\mathcal{G}$ has the desired property; hence $\mathcal{G}(\Gamma)$ exists.)

Note that if Γ is nonempty, then $U \subseteq \mathcal{G}(\Gamma)$, for γ in Γ and u in U imply $\gamma u = \gamma \in \Gamma$, so $u \in \mathcal{G}(\Gamma)$.

For each $p \geqq 1$ and each net $(f_n, n \in \Delta)$ of nonnegative elements of norm one in $l_p(\Sigma)$, define

$$Z_p(f_n) = \{\sigma : \lim_n \| f_n - f_n \circ \delta\sigma \|_p = 0\}.$$

LEMMA 7. (i) *If $p' \geqq 1$ and $F_n = f_n^{p/p'}$, then $Z_p(f_n) = Z_{p'}(F_n)$.*

(ii) *U, the set of right units of Σ, is contained in $Z_p(f_n)$.*

(iii) *$Z_p(f_n) = \mathcal{G}(Z_p(f_n))$, so for any nonempty subset Γ of $Z_p(f_n)$ it follows that $\mathcal{G}(\Gamma) \subseteq Z_p(f_n)$.*

Proof. (i) This follows from Lemma 2 by working from f_n to $\varphi_n = f_n^p$, and then on to F_n using $F_n^{p'} = \varphi_n$.

(ii) If $u \in U$, then $\circ\, \delta u$ is the identity in $l_p(\Sigma)$.

(iii) If $s, t \in Z = Z_p(f_n)$, then

$$\| f_n - f_n \circ \delta st \| \leqq \| f_n - f_n \circ \delta t \| + \| (f_n - f_n \circ \delta s) \circ \delta t \| \to 0.$$

If s and $st \in Z$, then

$$\| f_n - f_n \circ \delta t \| \leqq \| f_n - f_n \circ \delta st \| + \| (f_n \circ \delta s - f_n) \circ \delta t \| \to 0.$$

If t and $st \in Z$, then

$$\| f_n - f_n \circ \delta s \| = \| f_n \circ \delta t - f_n \circ \delta st \| \to \| f_n - f_n \| = 0.$$

LEMMA 8. *If $p > 1$ and $(f_n, n \in \Delta)$ is a net of nonnegative elements of norm one in $l_p(\Sigma)$, then $Z_p(f_n) = \bigcup_\varphi P_\varphi$, where the union is taken over the set of all probability densities φ such that $\| f_n \circ \varphi \|_p \to 1$ and P_φ meets U.*

The proof of equivalence of (ii) and (iii) of Lemma 4 proves this also.

4. Proofs of principal theorems

Proof of Theorem 1. The proof of equivalence of (a) with (g) is the "right" part of the equivalence of amenability and strong amenability [1, §5, Theorem 1]. That (f) implies (g) is easily shown by the methods which prove a group is strongly right amenable if every finitely generated subgroup is [1, §4, (K)]. By the definition of right strong amenability (g) is equivalent to the existence of a net $(\varphi_n, n \in \Delta)$, of (finite) probability densities on Σ such that for each σ in Σ, $\lim_n \| \varphi_n - \varphi_n \circ \delta\sigma \|_1 = 0$; this is the assertion that $Z_1(\varphi_n) = \Sigma$. But Lemma 7 (i) allows this to be transferred to $f_n = \varphi_n^{1/p}$ in $l_p(\Sigma)$. This is the same as the condition that every $\circ\, \delta\sigma$ has 1 in its spectrum. By Lemma 4 this implies (b), (b) implies (c), and (c) implies (d). (e) is a formal weakening of (d), so we need only prove (e) implies (f). This proof is based on an idea of Granirer [5, Theorem E_1, pp. 50–51].

For each countable set ξ there is by (e) a φ such that $P_\varphi \cap U \neq \emptyset$, $P_\varphi \supseteq \xi$, and $\| \circ \varphi \|_{p \to p} = 1$. Therefore there is a sequence of elements f_n of norm one in $l_p(\Sigma)$ such that $\| f_n \circ \varphi \|_p \to 1$. By Lemma 4, (iii) implies (ii), $\| f_n - f_n \circ \delta\sigma \|_p \to 0$ if $\sigma \in P_\varphi \supseteq \xi$. Setting $\varphi_n = f_n^{\,p}$, Lemma 2 shows that $\| \varphi_n - \varphi_n \circ \delta\sigma \|_1 \to 0$ if $\sigma \in \xi$.

Now let $\xi_1 = \xi$; then the above method produces a sequence $\varphi_{1,n}$ of probability densities converging to right invariance for each σ in ξ_1. If $\xi_1 \subseteq \xi_2 \subseteq \cdots \subseteq \xi_m$ are countable subsets of Σ, the same construction will give a corresponding sequence $\varphi_{m,n}$ such that $\lim_n \| \varphi_{m,n} - \varphi_{m,n} \circ \delta\sigma \| = 0$ for each σ in ξ_m. Then let $\xi_{m+1} = \xi_m \cup \bigcup_n P_{\varphi_{m,n}}$, and construct the next sequence for this new countable set. Let Σ' be the smallest semigroup containing all the ξ_m; then Σ' is countable and contains all $P_{\varphi_{m,n}}$ and ξ. To show that Σ' is right amenable, enumerate $\bigcup_m \xi_m$ as a sequence $s_1, s_2, \cdots, s_k, \cdots$. For each k there is an $m = m(k)$ such that all s_i, $i \leqq k$, are in ξ_m. Then there is a ψ_k among the terms of the corresponding sequence $\varphi_{m,n}$ such that

$$\| \psi_k - \psi_k \circ \delta s_i \| < 1/k \qquad \text{for } i = 1, 2, \cdots, k.$$

Hence $\lim_k \| \psi_k - \psi_k \circ \delta s \| = 0$ if $s \in \mathcal{S}(\bigcup_m \xi_m) \supseteq \Sigma'$. Because P_{ψ_k} also $\subseteq \Sigma'$, this proves [1, Lemma 1, p. 522] that Σ' is right amenable; it was already known to be countable and to contain ξ.

Proof of Theorem 2. (*b*) of Theorem 1, restated, implies that the convex hull of the set of all $\circ\, \delta s$ consists exclusively of elements $\circ\, \varphi$ with 1 in the spectrum.

By Lemma 6 every $\circ\, \varphi$ is connected by a straight line segment lying in $\mathcal{S}$

to I. If $T \in K$, the closed convex hull of the $\circ\,\delta s$, then there exist φ_n such that $\| \circ\,\varphi_n - T \| \to 0$. Hence for each α between 0 and 1 the sequence $\alpha I + (1 - \alpha) \circ \varphi_n$ tends to $\alpha I + (1 - \alpha)T$, so every point on the closed segment from I to T is in $\mathcal{S}$. By Lemma 5, 1 is in the spectrum of T.

Proof of Theorem 3. This is part of the proof of Theorem 1. It begins with "(ii) implies (iii)" from Lemma 4 and is completed with Lemma 2 (iii).

To prove Theorem 4 recall that for groups amenability is equivalent to right amenability [1, §4, Theorem 1], and that a group is amenable if and only if every subgroup, or every finitely generated subgroup, is amenable [1, §4, (D) and (K)].

The equivalence of (f′) with (e′) is very much like the corresponding proof in Theorem 1. The mapping $\sigma \leftrightarrow \sigma^{-1}$ in G interchanges left and right both for convolutions and for invariance of means. Since (f′) is invariant under such a change, the "left" versions (a_l), $\cdots$, (h_l) are equivalent to the original conditions when Σ is a group.

5. Examples and remarks

A. *The adjoint of* $\circ\,\varphi$. It is easily calculated that if G is a group and if for φ in $l_1(G)$ the element φ^* of $l_1(G)$ is defined by: $\varphi^*(g)$ = complex conjugate of $\varphi(g^{-1})$, then $\circ\,\varphi^*$ interpreted in $l_q(G)$ is the adjoint of $\circ\,\varphi$ in $l_p(G)$. Hence for each probability distribution φ on G the operator $\circ\,\varphi$ is self-adjoint on $l_2(G)$ if and only if φ is symmetric, that is $\varphi(g^{-1}) = \varphi(g)$ for all g in G; this is the case studied by Kesten ([6] and [7]). In this case the spectrum of $\circ\,\varphi$ is real, and 1 is in the spectrum of $\circ\,\varphi$ if and only if 1 is the supremum of the spectrum of $\circ\,\varphi$.

B. *Random walk and convolution.* When Σ is a semigroup, a random walk W_φ on Σ may be defined by assigning to σ and τ the number $\varphi(\tau)$ as the probability of taking the step from σ to $\sigma\tau$. When Σ is a group, the walk W_φ is called symmetric if φ is symmetric.

If ψ and φ are probability densities on Σ, then $\psi \circ \varphi$ is quite naturally described as the transform of the density ψ by the walk W_φ. Indeed $\psi \circ \varphi$ assigns to a point τ the sum of products of the probability $\psi(s)$ of being at s by the probability $\varphi(t)$ that the step from s to $st = \tau$ will be taken by the walk. Kesten [6] represents this walk in a group G by a matrix $((m_{st}))$ where $m_{st} = \varphi(s^{-1}t)$. His φ is symmetric so his matrix is symmetric, $m_{st} = m_{ts}$ for all s, t in G. Then he defines the linear operator M from $l_{2(G)}$ into $l_{2(G)}$ by means of the matrix $((m_{st}))$

$$[Mf](s) = \sum_{t \in G} m_{st}\, f(t) = \sum_{t \in G} \varphi(s^{-1}t) f(t) = \sum_{t \in G} \varphi(t^{-1}s) f(t).$$

Calculation of $f \circ \varphi$ in a group shows that

$$[f \circ \varphi](s) = \sum_{tz=s} f(t)\varphi(z) = \sum_{t \in G} f(t)\varphi(t^{-1}s).$$

Hence *Kesten's* Mf *is* $f \circ \varphi$ *whenever* φ *is a symmetric probability density on a group.*

C. In general semigroups convolution by φ need not be everywhere defined nor closed, nor bounded, nor need the conditions of the theorems remain equivalent for general semigroups.

C1. *An abelian (therefore amenable) semigroup where* 1 *is in the point spectrum of* $\circ \varphi$. Let T be an infinite set containing a point 0, and let T be made into a trivial semigroup $\mathfrak{Z}$ by the rule: for all s, t in T the product $st = 0$. Then for any φ and f, $[f \circ \varphi](x) = 0$ unless $x = 0$; also

$$[f \circ \varphi](0) = \textstyle\sum_{st=0} f(s)\varphi(t) = \sum_s f(s) \sum_t \varphi(t) = \sum_s f(s)$$

whenever this sum is defined, that is, whenever $f \in l_1(\Sigma)$. Hence $f \circ \varphi$ is defined if and only if $f \in l_1(\mathfrak{Z})$, and then $f \circ \varphi = (\sum_s f(s))\delta 0$.

Then $\delta 0 = \delta 0 \circ \varphi$, so 1 is in the point spectrum of $\circ \varphi$.

For $p > 1$ all λ are in the spectrum of $\circ \varphi$ in $l_p(\mathfrak{Z})$. We already know that 1 is in the spectrum. Suppose that for a $\lambda \neq 1$ there were a T_λ defined and linear and continuous on $l_p(\mathfrak{Z})$ such that for each f in $l_1(\mathfrak{Z})$, the domain of definition of $\lambda I - \varphi$,

$$T^\lambda(\lambda I - \circ \varphi)f = f = (\lambda I - \circ \varphi)T^\lambda f.$$

Then

$$f = \lambda T^\lambda f - (\textstyle\sum_s f(s))T^\lambda \delta 0,$$

or

$$(\textstyle\sum_s f(s))T^\lambda \delta 0 = \lambda T^\lambda f - f.$$

The right side depends continuously on f in the l_p norm, but the left will not unless $T^\lambda \delta 0 = 0$. On the other hand, $\delta 0 = T^\lambda(\lambda \delta 0 - \delta 0)$, so $T^\lambda \delta 0 = \delta 0/(\lambda - 1) \neq 0$. Hence all λ are in the spectrum of $\circ \varphi$ in $l_p(\mathfrak{Z})$.

C2. *A left-but-not-right-amenable semigroup where* 1 *is in the point spectrum of* $\circ \varphi$. Let L be an infinite set, and let $\mathfrak{L}$ be the semigroup obtained from it by defining $st = t$ for every ordered pair of elements s, t of L. As $\mathfrak{L}$ has more than one element, there are no right invariant means or right units nor any ghost of right cancellation. Also for each f for which the sum converges unconditionally and for each σ in $\mathfrak{L}$

$$f \circ \delta\sigma = (\textstyle\sum_s f(s))\delta\sigma,$$

so, as in $\mathfrak{Z}$, convolution of an f by a φ is defined if and only if $f \in l_1(\mathfrak{L})$. If φ is a probability density, then

$$f \circ \varphi = \textstyle\sum_\sigma \varphi(\sigma) f \circ \delta\sigma = \sum_\sigma \varphi(\sigma)(\sum_s f(s))\delta\sigma = (\sum_s f(s))\varphi.$$

Then $\varphi \circ \varphi = \varphi$, so 1 is a characteristic number of $\circ \varphi$ corresponding to the characteristic vector φ. To show that every $\lambda \neq 1$ is also in the spectrum of $\circ \varphi$ in $l_p(\mathfrak{L})$, merely repeat the argument of C1 replacing $\delta 0$ by φ.

Note that in both these semigroups 0 is the characteristic number of $\circ \varphi$

in $l_1(\Sigma)$ corresponding to all elements of the hyperplane $\{f : \sum_s f(s) = 0\}$. There are no other points in the spectrum of $\circ\varphi$ in $l_1(\mathfrak{Z})$ or $l_1(\mathfrak{L})$.

D. *The special case* $p = 1$ *of* (b), (c), *and* (d) *of Theorem* 1 *is true for every semigroup.* $\| \circ\varphi \|_{1\to 1} = 1$ because $\| \psi \circ \varphi \|_1 = \| \psi_1 \| \| \varphi_1 \|$ whenever φ and ψ are nonnegative elements of $l_1(\Sigma)$. To show that $\circ\varphi$ has 1 in its spectrum, for each n let $f_n = (\varphi + \varphi\circ\varphi + \cdots + \varphi \circ \cdots \circ \varphi)/n$. Then f_n is also a probability density, and

$$\| f_n \circ \varphi - f_n \| = \| \varphi - \varphi \circ \cdots \circ \varphi \| /n \leqq 2/n \to 0.$$

Hence 1 is in the spectrum of $\circ\varphi$, and as in §3, these conditions make the spectral radius of $\circ\varphi$ equal to 1.

E. *Semigroups satisfying the hypotheses of Theorem* 1 *which are not groups.*

E1. Let $\mathfrak{R}_m$ be a set of elements (m can be any cardinal number), and define products so that all elements are right units, that is, $ab = a$ for each pair a, b of elements of $\mathfrak{R}_m$. Then $\mathfrak{R}_m$ also has right cancellation, and $\circ\varphi$ is the identity operation in $l_p(\mathfrak{R}_m)$ for every probability density φ and number $p \geqq 1$. Hence all the conditions of Theorem 1 are trivially verified for $\mathfrak{R}_m$.

E2. If Σ is any semigroup with right cancellation and a right unit, for example, if Σ is a group, then $\mathfrak{R}_m \times \Sigma$ has right cancellation and right unit. Multiplication is defined coordinatewise, of course, so $(u, s)(u', s') = (u, ss')$. Then, if v is any right unit in Σ, (u, v) is a right unit for $\mathfrak{R}_m \times \Sigma$. If Σ is a group, every $\circ\delta(u, s)$ is an isometry of every $l_p(\mathfrak{R}_m \times \Sigma)$ on itself.

E3. If Σ = set of nonnegative integers under the usual addition operation, let F be any function from $\mathfrak{R}_m$ into Σ such that there is at least one u with $F(u) = 0$. Let Σ' be the subsemigroup of $\mathfrak{R}_m \times \Sigma$ consisting of all these elements (u, i) with $i \geqq F(u)$. Then Σ' is a semigroup with cancellation and right unit which satisfies all the conditions (a)–(h).

E4. If in example E2, F is any function from $\mathfrak{R}_m$ into Σ, let Σ' be the set of ordered pairs $\{(u, s) : s \in F(u)\cdot\Sigma\}$. Then Σ' is a subsemigroup of $\mathfrak{R}_m \times \Sigma$ with right cancellation. Σ' has a right unit if and only if there is a u in $\mathfrak{R}_m$ and a right unit v in Σ such that $v \in F(u)\cdot\Sigma$ (so $(u, v) \in \Sigma'$).

This example includes those of E2 and E3 as special cases. Σ' will be right amenable if Σ is.

E5. It is to be noted that if Σ is right amenable in examples E2 and E4, then $\mathfrak{R}_m \times \Sigma$ and Σ' satisfy all the conditions (a)–(h) of the principal theorems. In particular, if Σ is an amenable group, then $\mathfrak{R}_m \times \Sigma$ is right amenable but not left amenable.

F. *Generalizations to be studied further.* The extension of this to Orlicz spaces is a natural problem for a thesis, and one of my students is currently at work on it. The relation of this study to that of groups studied by Dieudonné [3] is also to be carried forward. Much of the above discussion, especially the lemmas, can be carried out for measurable or continuous probability distributions on locally compact groups, but some of the proofs involving measurability give trouble when uncountable directed systems and nets are used. Also that difference between atomic and nonatomic measures which is responsible for the Riemann-Lebesgue theorem also interferes with direct generalizations of the theorems. Again this is under investigation.

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