SOME AREA THEOREMS AND A SPECIAL COEFFICIENT THEOREM

BY

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1. The first significant method used in the theory of univalent functions was the area principle. Its use led to the initial results on sharp bounds in the simple standard results for univalent functions. In the more modern developments of the theory the chief tools have been Löwner's parametric method, the method of the extremal metric, the method of contour integration, and the variational method. These methods have been employed to deal with a wide range of results, and frequently a given result has been obtained separately by the use of several or even all of them. Among them the method of the extremal metric and the method of contour integration share with the area principle the feature that an essential step in the procedure is the assertion that the integral of a positive function is nonnegative. The method of contour integration was first used by Grunsky [3]. He also used it to obtain some quite general relationships for the coefficients of univalent functions [4]. A closely related method has been used by Nehari [10]. The method of contour integration has also been used by Golusin, Schiffer and Spencer [13], and others.

In this paper we will observe that many of the results obtained by the method of contour integration, including Grunsky's coefficient inequalities, can be obtained by a direct application of the area principle. Indeed in all these cases the area principle provides a sharper inequality.

On the other hand many of these applications of the area principle are consequences of a general result which we obtain by the method of the extremal metric. Moreover there are general circumstances in which the latter applies while the former does not.

2. We will begin by recalling some notations and known results.

Let D be a domain of finite connectivity in the z-sphere containing the point at infinity, and let $\Sigma(D)$ be the class of functions $f(z)$ univalent in D, regular apart from a simple pole at the point at infinity where the Laurent expansion is given by

$$f(z) = z + c_0 + c_1/z + \cdots + c_n/z^n + \cdots .$$

Let $\Sigma'(D)$ denote the subclass of $\Sigma(D)$ of functions for which $c_0 = 0$. If D is the domain $|z| > 1$, we denote these classes simply by Σ and Σ'. Without loss of generality we can always assume D to be bounded by analytic curves.

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The Faber polynomials for a function in $\Sigma'(D)$ are defined as follows. To set up a consistent notation we write $f_1(z)$ for $f(z)$ and write its above Laurent expansion as

$$(1) \qquad f_1(z) = z + a_{11}/z + \cdots + a_{1n}/z^n + \cdots .$$

Then the m^{th} Faber polynomial F_m is a polynomial of degree m such that we have the Laurent expansion

$$f_m(z) = F_m(f_1(z)) = z^m + a_{m1}/z + \cdots + a_{mn}/z^n + \cdots .$$

These polynomials are uniquely determined. Let

$$U(z, w) = \log \frac{f(z) - f(w)}{z - w}$$

denote the function of (z, w) defined in $D \times D$ which takes the value zero at infinity, and let it have in the neighborhood of the point at infinity the development

$$U(z, w) = \sum_{m,n=1}^{\infty} d_{mn}\, z^{-m} w^{-n}.$$

It was proved by Schiffer [12] that

$$a_{mn} = -m\, d_{mn} .$$

The following result is due to Grunsky [4, p. 39].

I. *Given a polynomial*

$$Q_m(z) = x_m z^m + x_{m-1} z^{m-1} + \cdots + x_0 ,$$

there is a unique function $f^{(\alpha)}(z)$, *regular in* $\bar{D}$ *apart from a pole at the point at infinity where it has Laurent development with principal part (including constant term) given by* $Q_m(z)$, *such that* $\mathscr{I}(e^{-i\alpha} f^{(\alpha)}(z))$ *is constant on each boundary component of* D.

Making in this result the particular choice $Q_m(z) = z^m$ we obtain the particular function $X_m^{(\alpha)}(z)$ with Laurent development

$$X_m^{(\alpha)}(z) = z^m + a_{m1}^{(\alpha)}/z + \cdots + a_{mn}^{(\alpha)}/z^n + \cdots .$$

In terms of these we define functions with the Laurent developments indicated:

$$Y_m(z) = \tfrac{1}{2}(X_m^{(0)}(z) + X_m^{(\pi/2)}(z)) = z^m + b_{m1}/z + \cdots + b_{mn}/z^n + \cdots ,$$

$$Z_m(z) = \tfrac{1}{2}(X_m^{(0)}(z) - X_m^{(\pi/2)}(z)) = c_{m1}/z + \cdots + c_{mn}/z^n + \cdots .$$

Now we can state Grunsky's principal result.

II. *Let* $f_1(z)$ *be regular in the domain* D *apart from a simple pole at the point at infinity where it has the Laurent expansion* (1). *For* $f_1(z)$ *to be univalent in* D *it is necessary and sufficient that*

$$\left| \sum_{m,n=1}^{N} n(a_{mn} - b_{mn}) x_m x_n \right| \leqq \sum_{m,n=1}^{N} n c_{mn}\, \bar{x}_m x_n ,$$

where N is an arbitrary integer and x_m, $m = 1, \cdots, N$, are arbitrary complex numbers. In the particular case that D is the domain $|z| > 1$ these conditions reduce to

$$\left| \sum_{m,n=1}^{N} n a_{mn} x_m x_n \right| \leqq \sum_{n=1}^{N} n |x_n|^2.$$

Not long after Grunsky's work, Golusin [1] extended the Area Theorem to p-valent functions defined in $|z| > 1$. For our purposes we can interpret this as a result on univalent functions.

III. *Let $f \epsilon \Sigma$, let Q_m be a polynomial of degree m, and let $Q_m(f(z))$ have the Laurent development about the point at infinity*

$$Q_m(f(z)) = \sum_{n=-m}^{\infty} C_n z^{-n}.$$

Then

$$\sum_{n=-m}^{\infty} n |C_n|^2 \leqq 0. \tag{2}$$

Wolibner [15] proved that if conditions (2) obtain for all Q_m (m arbitrary) for a function f regular in $|z| > 1$ apart from a simple pole at the point at infinity, then f is univalent in $|z| > 1$.

It does not seem to have been mentioned in print that, in the case of the domain $|z| > 1$, Grunsky's inequalities are a direct consequence of Golusin's. Indeed if $f_1 \epsilon \Sigma'$ and Q_m is a polynomial of degree m, then

$$Q_m(f_1(z)) = \sum_{\mu=1}^{m} x_\mu z^\mu + \sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$$

for suitable complex x_μ. Hence

$$Q_m(f_1(z)) = \sum_{\mu=1}^{m} x_\mu f_\mu(z),$$

so that

$$a_\nu = \sum_{\mu=1}^{m} x_\mu a_{\mu\nu}, \qquad \nu = 1, 2, \cdots.$$

Here Golusin's inequality has the form

$$\sum_{\nu=1}^{\infty} \nu |a_\nu|^2 \leqq \sum_{\nu=1}^{m} \nu |x_\nu|^2.$$

On the other hand

$$\begin{aligned} \left| \sum_{\mu,\nu=1}^{m} \nu a_{\mu\nu} x_\mu x_\nu \right|^2 &= \left| \sum_{\nu=1}^{m} \nu x_\nu \sum_{\mu=1}^{m} x_\mu a_{\mu\nu} \right|^2 \\ &\leqq \left(\sum_{\nu=1}^{m} \nu |x_\nu|^2\right)\left(\sum_{\nu=1}^{m} \nu \left| \sum_{\mu=1}^{m} x_\mu a_{\mu\nu} \right|^2\right) \\ &= \left(\sum_{\nu=1}^{m} \nu |x_\nu|^2\right)\left(\sum_{\nu=1}^{m} \nu |a_\nu|^2\right) \leqq \left(\sum_{\nu=1}^{m} \nu |x_\nu|^2\right)^2, \end{aligned}$$

the last step using Golusin's inequality. Thus

$$\left| \sum_{\mu,\nu=1}^{m} \nu a_{\mu\nu} x_\mu x_\nu \right| \leqq \sum_{\nu=1}^{m} \nu |x_\nu|^2,$$

which is Grunsky's inequality. In the same way we could have derived a generalized inequality given by Golusin [2] who, however, did not obtain the result in this way. For arguments of this type see §4 and §5 below. On the other hand, as a sufficient criterion for univalence, Grunsky's result is sharper than that of Wolibner, indeed includes it as a special case.

Closely related to the preceding results are certain inequalities bearing on the values of univalent functions at a number of points. The first of these is due to Golusin [2].

IV. *Let $f \in \Sigma$, let $z_1, \cdots, z_n$ be points in $|z| > 1$, and let $\gamma_1, \cdots, \gamma_n$ be arbitrary complex numbers; then*

$$\left| \sum_{\mu,\nu=1}^{m} \gamma_\mu \gamma_\nu \log \frac{f(z_\mu) - f(z_\nu)}{z_\mu - z_\nu} \right| \leqq -\sum_{\mu,\nu=1}^{n} \gamma_\mu \bar{\gamma}_\nu \log (1 - z_\mu^{-1} \bar{z}_\nu^{-1}).$$

The second is due to Nehari [10].

V. *Let f be regular and univalent in $|z| < 1$, let $z_1, \cdots, z_n$ be points in $|z| < 1$, and let $\alpha_1, \cdots, \alpha_n$ be complex numbers satisfying $\sum_{\nu=1}^{n} \alpha_\nu = 0$; then*

$$\left| \sum_{\mu,\nu=1}^{n} \alpha_\mu \alpha_\nu \log \frac{f(z_\mu) - f(z_\nu)}{z_\mu - z_\nu} \right| \leqq -\sum_{\mu,\nu=1}^{n} \alpha_\mu \bar{\alpha}_\nu \log (1 - z_\mu \bar{z}_\nu).$$

It does not seem to have been recorded that Nehari's inequality is a simple consequence of Golusin's. In this connection see §5. It was observed by Shah [14] that the result IV is an easy consequence of Golusin's inequality III, and in this way he obtained a simple proof of Wolibner's sufficient condition. Nehari [10] utilized a similar concept in a connection related to Grunsky's sufficient condition.

3. Golusin's inequality III can be interpreted as an area theorem for univalent functions in a metric not the ordinary plane metric. This form does not include the application to p-valent functions, but on the other hand can be taken in certain respects in a more general context. Indeed we have

THEOREM 1. *Let $f \in \Sigma$, let g be an integral function, and let $g(f(z))$ have the Laurent expansion about the point at infinity*

$$g(f(z)) = \sum_{n=-\infty}^{\infty} C_n z^{-n}.$$

Then

$$\sum_{n=-\infty}^{\infty} n \, |C_n|^2 \leqq 0. \tag{3}$$

We consider in the w-plane the metric $|g'(w)| \, |dw|$ and the image curve Γ of the circle $|z| = r$, $r > 1$, under the mapping $w = f(z)$. The curve Γ bounds a domain D whose area in this metric is finite and nonnegative; thus

$$\iint_D |g'(w)|^2 \, dA_w \geqq 0$$

where dA_w denotes the element of Euclidean area in the w-plane. By a familiar variant of Green's formula this becomes

$$\frac{1}{2i} \int_\Gamma \overline{g(w)} \, dg(w) \geqq 0.$$

By transferring this integral to the z-plane it becomes

$$\frac{1}{2i}\int_{|z|=r}\overline{g(f(z))}\,dg(f(z)) \geqq 0,$$

that is,

$$\pi\sum_{n=-\infty}^{\infty} n\,|\,C_n\,|^2 r^{-2n} \leqq 0.$$

A standard argument yields the inequality (3). Essentially the same result with a slightly different formulation and proof appears in a paper of Lebedev and Milin [9, Lemma 1].

This result extends at once to mappings into a closed Riemann surface.

THEOREM 2. *Let $\mathfrak{R}$ be a closed Riemann surface, and let f be a univalent mapping of $|\,z\,| > 1$ into $\mathfrak{R}$ such that the point at infinity corresponds to the point P on $\mathfrak{R}$. Let the function g be single-valued and regular on $\mathfrak{R}$ apart from an isolated singularity at the point P. Let the function gf have the Laurent expansion about the point at infinity*

$$g(f(z)) = \sum_{n=-\infty}^{\infty} C_n\,z^{-n}.$$

Then

$$\sum_{n=-\infty}^{\infty} n\,|\,C_n\,|^2 \leqq 0.$$

The proof is unchanged except that each inequality referring to entities on $\mathfrak{R}$ must be understood as being expressed in terms of local uniformizing parameters.

4. We have seen how in the case of the domain $|\,z\,| > 1$ Golusin's inequality is superior to Grunsky's. However Grunsky's inequality extends to multiply-connected domains. We will now derive for such domains an inequality which plays the same role as that of Golusin.

Let D be a domain in the z-sphere containing the point at infinity bounded by analytic curves, and let $f(z)$ be a function in $\Sigma'(D)$ regular also on the boundary Γ of D where Γ is sensed so that each component has the counterclockwise sense in the z-plane. Let P be a polynomial of degree m. Then as in the proof of Theorem 1 we have

$$\frac{1}{2i}\int_{\Gamma}\overline{P(f(z))}\,dP(f(z)) \geqq 0. \tag{4}$$

Now we can express $P(f(z))$ in terms of the Faber polynomials associated with f as

$$P(f(z)) = \sum_{\mu=1}^{\infty} x_\mu\,f_\mu(z)$$

where this function has about the point at infinity the Laurent expansion

$$\sum_{\mu=1}^{m} x_\mu\,z^\mu + \sum_{\nu=0}^{\infty} a_\nu\,z^{-\nu}$$

where

$$a_\nu = \sum_{\mu=1}^{m} x_\mu\,a_{\mu\nu}\,, \qquad \nu = 1, 2, \cdots .$$

Further we can express it in terms of Grunsky's functions $Y_\mu(z)$ as

$$P(f(z)) = \sum_{\mu=1}^{m} x_\mu Y_\mu(z) + c_0 + r(z)$$

where c_0 is constant, $r(z)$ is regular in $\bar{D}$, vanishes at the point at infinity, and has about that point the Laurent expansion

$$\sum_{\nu=1}^{\infty} c_\nu z^{-\nu}$$

with

$$c_\nu = \sum_{\mu=1}^{m} x_\mu(a_{\mu\nu} - b_{\mu\nu}), \qquad \nu = 1, 2, \cdots .$$

Now we have

$$(5)\qquad \begin{aligned} \frac{1}{2\pi i}\int_\Gamma \overline{P(f(z))}\, dP(f(z)) = {} & \frac{1}{2\pi i}\int_\Gamma \left(\sum_{\mu=1}^{m} \bar{x}_\mu \bar{Y}_\mu(z)\right) d\left(\sum_{\mu=1}^{m} x_\mu Y_\mu(z)\right) \\ & + \frac{1}{2\pi i}\int_\Gamma \bar{r}(z)\, d\left(\sum_{\mu=1}^{m} x_\mu Y_\mu(z)\right) \\ & + \frac{1}{2\pi i}\int_\Gamma \left(\sum_{\mu=1}^{m} \bar{x}_\mu \bar{Y}_\mu(z)\right) dr(z) \\ & + \frac{1}{2\pi i}\int_\Gamma \bar{r}(z)\, dr(z). \end{aligned}$$

In this we observe that

$$\int_\Gamma \bar{Y}_\mu(z)\, dr(z) = \int_\Gamma Z_\mu(z)\, dr(z) = 0$$

and

$$\int_\Gamma \bar{r}(z)\, dY_\mu(z) = \int_\Gamma \bar{r}(z)\, d\bar{Z}_\mu(z) = \int_\Gamma \overline{r(z)\, dZ_\mu(z)} = 0.$$

Thus the second and third integrals on the right-hand side of equation (5) are zero. Moreover

$$\frac{1}{2\pi i}\int_\Gamma \bar{Y}_\mu(z)\, dY_\nu(z) = \frac{1}{2\pi i}\int_\Gamma Z_\mu(z)\, dY_\nu(z) = \nu c_{\mu\nu};$$

thus the first integral on the right-hand side of equation (5) is equal to

$$\sum_{\mu,\nu=1}^{m} \nu c_{\mu\nu} \bar{x}_\mu x_\nu .$$

It is clear that $r'(z)$ is in $L^2(D)$ (in the complex sense), and in fact

$$(6)\qquad \frac{1}{2\pi i}\int_\Gamma \bar{r}(z)\, dr(z) = -\frac{1}{\pi}\iint_D |\, r'(z)\, |^2\, dA_z .$$

Let $G_j(z)$, $j = 1, 2, \cdots$, be a complete orthonormal system of functions in the subspace Λ of $L^2(D)$ consisting of regular functions where we use the inner product for G, H

$$\frac{1}{\pi}\iint_D \bar{G}(z) H(z)\, dA_z .$$

Then

$$r'(z) = \sum_{j=1}^{\infty} \beta_j G_j(z)$$

where

$$\beta_j = \frac{1}{\pi} \iint_D \bar{G}_j(z) r'(z)\, dA_z .$$

We now have the following result.

THEOREM 3. *Let D be a domain in the z-sphere containing the point at infinity, f a function in $\Sigma'(D)$, and P a polynomial of degree m. Let*

$$P(f(z)) = \sum_{\mu=1}^{m} x_\mu f_\mu(z).$$

Then

$$\sum_{j=1}^{\infty} | \beta_j |^2 \leqq \sum_{\mu,\nu=1}^{m} \nu c_{\mu\nu} \bar{x}_\mu x_\nu . \tag{7}$$

For f and D satisfying the additional assumptions made initially, this result follows from inequality (4) and equations (5) and (6) on observing that

$$\frac{1}{\pi} \iint_D | r'(z) |^2 dA_z = \sum_{j=1}^{\infty} | \beta_j |^2.$$

The condition that D have analytic boundary curves can always be attained by auxiliary conformal mapping. If f is not then regular on Γ, the result is extended to it by standard approximation considerations; see in this connection [4], [11].

It is of interest to consider a certain special choice for the complete orthonormal system. We recall

$$\begin{aligned} \frac{1}{\pi} \iint_D \bar{Z}_\nu(z) Z_\mu(z)\, dA_z &= -\frac{1}{2\pi i} \int_\Gamma \bar{Z}_\nu(z)\, dZ_\mu(z) \\ &= -\frac{1}{2\pi i} \int_\Gamma Y_\nu(z)\, dZ_\mu(z) = \nu c_{\mu\nu} , \end{aligned}$$

and that $(\nu c_{\mu\nu})$, $\mu, \nu = 1, \cdots, M$, is a positive definite Hermitian matrix [4], which we denote by $\mathfrak{C}$. Thus the functions $Z_j'(z)$ are linearly independent elements of Λ, and we may take an orthonormal set formed from $Z_1'(z), \cdots, Z_M'(z)$ as our first M elements $G_1(z), \cdots, G_M(z)$. In other words, we know that there exists a nonsingular matrix $\mathfrak{T} = (t_{ik})$ such that

$$\mathfrak{T}'\mathfrak{C}\bar{\mathfrak{T}} = I$$

where I is an $M \times M$ unit matrix. Then we set

$$G_k(z) = \sum_{\mu=1}^{M} t_{\mu k} Z_\mu'(z), \qquad k = 1, \cdots, M,$$

the functions $G_k(z)$, $k > M$, being chosen to complete this orthonormal set. Now for this choice, if $1 \leqq k \leqq M$,

$$\beta_j = \frac{1}{\pi}\int\int_D \bar{G}_j(z)r'(z)\,dA_z = -\frac{1}{2\pi i}\int_\Gamma \sum_{\nu=1}^{M} \bar{t}_{\nu j}\,\bar{Z}_\nu(z)\,dr(z)$$
$$= \sum_{\nu=1}^{M} \bar{t}_{\nu j}\left(-\frac{1}{2\pi i}\int_\Gamma Y_\nu(z)\,dr(z)\right) = \sum_{\mu=1}^{m}\sum_{\nu=1}^{M} \nu(a_{\mu\nu} - b_{\mu\nu})x_\mu\,\bar{t}_{\nu j}\,.$$

Thus we obtain from Theorem 3

COROLLARY 1. *For arbitrary complex* x_μ, $\mu = 1, \cdots, m$,

$$(8) \qquad \sum_{j=1}^{M} \left| \sum_{\mu=1}^{m} \sum_{\nu=1}^{M} \nu(a_{\mu\nu} - b_{\mu\nu})x_\mu \bar{t}_{\nu j} \right|^2 \leqq \sum_{\mu,\nu=1}^{m} \nu c_{\mu\nu}\,\bar{x}_\mu x_\nu\,.$$

Now it is clear that the inequality (7) is sharper than Grunsky's inequality since he used also a second inequality in deriving his result. However it is of some interest that the inequality (8) also yields by simple formal manipulation Grunsky's inequality and in fact a generalized version of the latter. Indeed let (τ_{jk}) be the matrix τ inverse to $\mathfrak{T}$. Then if Y denotes the vector $(y_1, \cdots, y_M)$, we have

$$\bar{Y}'\mathfrak{C}Y = \bar{Y}'\bar{\tau}'\tau Y = \|\tau Y\|^2$$

where the last symbol denotes

$$\sum_{j=1}^{M} \left| \sum_{k=1}^{M} \tau_{jk}\,y_k \right|^2.$$

Finally we have

$$\left| \sum_{\mu=1}^{m} \sum_{\nu=1}^{M} \nu(a_{\mu\nu} - b_{\mu\nu})x_\mu y_\nu \right|^2$$
$$= \left| \sum_{j=1}^{M}\Big(\sum_{\mu=1}^{m} \sum_{\nu=1}^{M} \nu(a_{\mu\nu} - b_{\mu\nu})x_\mu \sum_{k=1}^{M} \bar{t}_{\nu j}\,\tau_{jk}\,y_k\Big) \right|^2$$
$$\leqq \Big(\sum_{j=1}^{M} \left| \sum_{k=1}^{M} \tau_{jk}\,y_k \right|^2\Big)\Big(\sum_{j=1}^{M} \left| \sum_{\mu=1}^{m} \sum_{\nu=1}^{M} \nu(a_{\mu\nu} - b_{\mu\nu})x_\mu \bar{t}_{\nu j} \right|^2\Big)$$
$$\leqq \Big(\sum_{\mu,\nu=1}^{m} \nu c_{\mu\nu}\,\bar{x}_\mu x_\nu\Big)\Big(\sum_{\mu,\nu=1}^{M} \nu c_{\mu\nu}\,\bar{y}_\mu y_\nu\Big).$$

We summarize these considerations in the following result.

COROLLARY 2. *For arbitrary complex* x_μ, $\mu = 1, \cdots, m$, y_μ, $\mu = 1, \cdots, M$,

$$(9) \quad \left| \sum_{\mu=1}^{m} \sum_{\nu=1}^{M} \nu(a_{\mu\nu} - b_{\mu\nu})x_\mu y_\nu \right| \leqq \Big(\sum_{\mu,\nu=1}^{m} \nu c_{\mu\nu}\,\bar{x}_\mu x_\nu\Big)^{1/2}\Big(\sum_{\mu,\nu=1}^{M} \nu c_{\mu\nu}\,\bar{y}_\mu y_\nu\Big)^{1/2}.$$

In case $M = m$, $x_\mu = y_\mu$, $\mu = 1, \cdots, m$, this is just Grunsky's inequality. In the special case where D is the domain $|z| > 1$ inequality (9) was obtained by Golusin [2]. We remark that the inequality (8) provides both a necessary and sufficient condition for univalence. Of course as a sufficient condition it is not as sharp as Grunsky's.

5. As we remarked at the end of §2, Shah observed that the result IV could be derived from Golusin's area result III. However it does not seem to have been recognized that both the results IV and V are even more directly consequences of suitable area theorems.

Let $f \in \Sigma$, and let z_j, $j = 1, \cdots, n$, be points in $|z| > 1$. Let $w_j = f(z_j)$, $j = 1, \cdots, n$. Let r be greater than one and such that all the points z_j lie in $|z| > r$. The complement E_r of the image of $|z| > r$ under f has positive area in the metric

$$\left| \sum_{j=1}^{n} \gamma_j (w - w_j)^{-1} \right| |dw|$$

where γ_j, $j = 1, \cdots, n$, are arbitrary complex constants. Thus

$$\iint_{E_r} \left| \sum_{j=1}^{n} \gamma_j (w - w_j)^{-1} \right|^2 dA_w \geqq 0,$$

and we can transform this into

$$\frac{1}{2i} \int_{\Gamma_r} \overline{\sum_{j=1}^{n} \gamma_j \log (w - w_j)}\, d\left(\sum_{j=1}^{n} \gamma_j \log (w - w_j) \right) \geqq 0$$

where Γ_r is the image of $|z| = r$ under f (taken in the counterclockwise sense), and where the branches of the logarithms may be chosen at will, each being single-valued in E_r. This may again be written as

$$(10) \quad \frac{1}{2i} \int_{|z|=r} \overline{\sum_{j=1}^{n} \gamma_j \log (f(z) - f(z_j))}\, d\left(\sum_{j=1}^{n} \gamma_j \log (f(z) - f(z_j)) \right) \geqq 0.$$

Now for each j, each branch of

$$\log \frac{f(z) - f(z_j)}{z - z_j}$$

is a single-valued function in $|z| > 1$ when defined suitably at z_j, and we may take this symbol to denote the branch vanishing at the point at infinity. Then we have an expansion as a function of two variables, valid for $|z| > 1$, $|\zeta| > 1$

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,l=1}^{\infty} d_{ml} z^{-m} \zeta^{-l}.$$

Further we can write

$$\log (f(z) - f(z_j)) = \log \frac{f(z) - f(z_j)}{z - z_j} + \log (z - z_j),$$

and this function has a Laurent expansion valid on $|z| = r$ derived as follows. We set

$$\log (z - z_j) = \text{const.} + \log (1 - z/z_j) = \text{const.} - \sum_{k=1}^{\infty} k^{-1} (z/z_j)^k.$$

Thus we have on $|z| = r$

$$(11) \quad \begin{aligned} &\log (f(z) - f(z_j)) \\ &\qquad = \sum_{m=1}^{\infty} \left(\sum_{l=1}^{\infty} d_{ml} z_j^{-l} \right) z^{-m} + \text{const.} - \sum_{k=1}^{\infty} k^{-1} (z/z_j)^k. \end{aligned}$$

Inserting the development (11) in inequality (10) for $j = 1, \cdots, n$ and dividing by π we obtain

$$(12)\quad \sum_{k=1}^{\infty} k^{-1}\Big| \sum_{j=1}^{n} \gamma_j z_j^{-k} \Big|^2 r^{2k} \geqq \sum_{m=1}^{\infty} m \Big| \sum_{j=1}^{n}\sum_{l=1}^{\infty} d_{ml}\, z_j^{-l}\gamma_j \Big|^2 r^{-2m}.$$

Letting r tend to one and performing an explicit reduction of the left-hand side of (12) for $r = 1$ we have the following result.

THEOREM 4. *Let $f \epsilon \Sigma$, let z_j, $j = 1, \cdots, n$, be points in $|z| > 1$, let the expansion*

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,l=1}^{\infty} d_{ml}\, z^{-m}\zeta^{-l}$$

be valid for $|z| > 1$, $|\zeta| > 1$, and let γ_j, $j = 1, \cdots, n$, be arbitrary complex constants. Then

$$\sum_{m=1}^{\infty} m \Big| \sum_{j=1}^{n} \sum_{l=1}^{\infty} d_{ml}\, z_j^{-l}\gamma_j \Big|^2 \leqq -\sum_{j,k=1}^{n} \gamma_j \bar{\gamma}_k \log\left(1 - z_j^{-1}\bar{z}_k^{-1}\right).$$

Now let $\zeta_1, \cdots, \zeta_N$ be further points of $|z| > 1$ which may coincide with certain of the z_j or not, and let $\lambda_1, \cdots, \lambda_N$ be arbitrary complex constants. Consider

$$\left| \sum_{k=1}^{N} \sum_{j=1}^{n} \lambda_k \gamma_j \log \frac{f(\zeta_k) - f(z_j)}{\zeta_k - z_j} \right|^2$$

where the term is understood to have the appropriate limiting value in case $\zeta_k = z_j$. If we rewrite this expression as follows, it is seen at once to satisfy the inequality

$$(13)\quad \begin{aligned} &\Big| \sum_{k=1}^{N} \sum_{j=1}^{n} \lambda_k \gamma_j \sum_{m,l=1}^{\infty} d_{ml}\, \zeta_k^{-m} z_j^{-l} \Big|^2 \\ &\quad \leqq \Big(\sum_{m=1}^{\infty} m^{-1} \Big| \sum_{k=1}^{N} \lambda_k \zeta_k^{-m} \Big|^2\Big)\Big(\sum_{m=1}^{\infty} m \Big| \sum_{j=1}^{n} \gamma_j \sum_{l=1}^{\infty} d_{ml}\, z_j^{-l} \Big|^2\Big). \end{aligned}$$

If on the right-hand side of inequality (13) we reduce the first term explicitly and apply Theorem 4 to the second term, we have proved the following result.

COROLLARY 3. *Let $f \epsilon \Sigma$, and let γ_j, $j = 1, \cdots, n$, λ_k, $k = 1, \cdots, N$, be arbitrary complex constants, z_j, $j = 1, \cdots, n$, ζ_k, $k = 1, \cdots, N$, points in $|z| > 1$. Then*

$$(14)\quad \begin{aligned} &\left| \sum_{k=1}^{N} \sum_{j=1}^{n} \lambda_k \gamma_j \log \frac{f(\zeta_k) - f(z_j)}{\zeta_k - z_j} \right| \\ &\quad \leqq \left[\left(\sum_{j,k=1}^{n} \gamma_j \bar{\gamma}_k \log(1 - z_j^{-1}\bar{z}_k^{-1}) \right)\left(\sum_{j,k=1}^{N} \lambda_j \bar{\lambda}_k \log(1 - \zeta_j^{-1}\bar{\zeta}_k^{-1}) \right) \right]^{1/2}. \end{aligned}$$

This result was obtained by Golusin [2] using Löwner's method and contains the result IV as a special case.

Next let the function $f(z)$ be regular and univalent for $|z| < 1$, and let z_j, $j = 1, \cdots, n$, be points in $|z| < 1$. Let $w_j = f(z_j)$, $j = 1, \cdots, n$. Then we can apply the same argument as before using now the metric

$$\Big| \sum_{j=1}^{n} \alpha_j (w - w_j)^{-1} \Big|\, |dw|$$

where α_j, $j = 1, \cdots, n$, are complex constants subject to the condition $\sum_{j=1}^{n} \alpha_j = 0$, so that the area of a neighborhood of the point at infinity is finite. Proceeding as before we obtain the inequality for $0 < r < 1$ and r sufficiently close to 1

$$(15)\quad \frac{1}{2i} \int_{|z|=r} \sum_{j=1}^{n} \overline{\alpha_j \log(f(z) - f(z_j))}\, d\left(\sum_{j=1}^{n} \alpha_j \log(f(z) - f(z_j))\right) \leqq 0.$$

We take the expansion, valid for $|z| < 1$, $|\zeta| < 1$,

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,l=0}^{\infty} A_{ml}\, z^m \zeta^l.$$

Then we have

$$\begin{aligned}(16)\qquad \sum_{j=1}^{n} \alpha_j \log(f(z) - f(z_j)) &= \sum_{j=1}^{n} \alpha_j \log \frac{f(z) - f(z_j)}{z - z_j} + \sum_{j=1}^{n} \alpha_j \log(z - z_j) \\ &= \sum_{j=1}^{n} \alpha_j \log \frac{f(z) - f(z_j)}{z - z_j} + \sum_{j=1}^{n} \alpha_j \log\left(1 - \frac{z_j}{z}\right) \\ &\quad + \sum_{j=1}^{n} \alpha_j \log z\end{aligned}$$

for suitable choices of the determinations. Thus the function (16) has on the circle $|z| = r$ the Laurent expansion

$$\sum_{m=1}^{\infty} \left(\sum_{j=1}^{n} \alpha_j \sum_{l=1}^{\infty} A_{ml}\, z_j^l\right) z^m + \text{const.} - \sum_{k=1}^{\infty} k^{-1} \left(\sum_{j=1}^{n} \alpha_j\, z_j^k\right) z^{-k}.$$

Inserting this in inequality (15), letting r tend to one and reducing we obtain

THEOREM 5. *Let $f(z)$ be regular and univalent in $|z| < 1$, let z_j, $j = 1, \cdots, n$, be points in $|z| < 1$, let the expansion*

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,l=0}^{\infty} A_{ml}\, z^m \zeta^l$$

be valid for $|z| < 1$, $|\zeta| < 1$, and let α_j, $j = 1, \cdots, n$, be complex constants such that $\sum_{j=1}^{n} \alpha_j = 0$. Then

$$\sum_{m=1}^{\infty} m \left| \sum_{j=1}^{n} \alpha_j \sum_{l=1}^{\infty} A_{ml}\, z_j^l \right|^2 \leqq -\sum_{j,k=1}^{n} \alpha_j \bar{\alpha}_k \log(1 - z_j \bar{z}_k).$$

With essentially the same proof as for Corollary 3 we obtain

COROLLARY 4. *Let $f(z)$ be regular and univalent in $|z| < 1$, and let α_j, $j = 1, \cdots, n$, β_k, $k = 1, \cdots, N$, be complex constants satisfying $\sum_{j=1}^{n} \alpha_j = 0$, $\sum_{k=1}^{N} \beta_k = 0$, z_j, $j = 1, \cdots, n$, ζ_k, $k = 1, \cdots, N$, points in $|z| < 1$. Then*

$$\begin{aligned}(17)\qquad &\left| \sum_{k=1}^{N} \sum_{j=1}^{n} \beta_k \alpha_j \log \frac{f(\zeta_k) - f(z_j)}{\zeta_k - z_j} \right| \\ &\qquad \leqq \left[\left(\sum_{j,k=1}^{n} \alpha_j \bar{\alpha}_k \log(1 - z_j \bar{z}_k)\right)\left(\sum_{j,k=1}^{N} \beta_j \bar{\beta}_k \log(1 - \zeta_j \bar{\zeta}_k)\right)\right]^{1/2}\end{aligned}$$

While this appears to be the first explicit record of this result, it should be observed that inequality (17) is an immediate consequence of inequality (14). Indeed we need merely apply that result to the function

$$((f(z^{-1}) - f(0))/f'(0))^{-1}$$

with $\lambda_k = \beta_k$, $k = 1, \cdots, N$, $\gamma_j = \alpha_j$, $j = 1, \cdots, n$, and reduce, using the conditions $\sum_{j=1}^{n} \alpha_j = 0$, $\sum_{k=1}^{N} \beta_k = 0$. In the same way the result V follows from the result IV.

6. We now wish to investigate the relationship between the area method and the method of the extremal metric, particularly in the form of the General Coefficient Theorem. As is well known the latter contains as special cases most of the standard results in the theory of univalent functions. Moreover it would provide a complete explicit solution to a much wider range of problems were it not for the fact that its application requires certain normalizations on initial coefficients of the functions considered. While the area method admits on the whole much less extensive application, no such restriction appears in the conditions for its use. We will now show that by somewhat strengthening other requirements appearing in the enunciation of the General Coefficient Theorem it is possible to drop the coefficient normalization restrictions found there. The result obtained does not include many of the most interesting consequences of the General Coefficient Theorem but does make possible the explicit treatment of a wide variety of new problems. It may be remarked that the method used here has already been applied in one special case [8]. There it provides new insights into the nature of span theorems and a considerable increase in their range of applicability.

We begin in the usual framework of a finite oriented Riemann surface $\mathfrak{R}$, a positive quadratic differential $Q(z)\,dz^2$ on $\mathfrak{R}$, and a family $\{\Delta\}$ of admissible domains Δ_j, $j = 1, \cdots, K$, on $\mathfrak{R}$ with respect to $Q(z)\,dz^2$. Then we enunciate the following definition.

Definition 1. *Let $\{f\}$ be a family of functions f_j, $j = 1, \cdots, K$, with the following properties*:

(i) *f_j maps Δ_j conformally into $\mathfrak{R}$,*

(ii) *if a pole or zero of odd order A of $Q(z)\,dz^2$ lies in Δ_j, $f_j(A) = A$,*

(iii) *$f_j(\Delta_j) \cap f_l(\Delta_l) = 0$, $j \neq l$, $j, l = 1, \cdots, K$.*

Then the family $\{f\}$ is said to admit a special admissible homotopy F into the identity if there exists a function $F(P, t)$ defined for $P \in \bigcup_{j=1}^{K} \Delta_j$, $0 \leqq t \leqq 1$, with values in $\mathfrak{R}$, continuous in both variables together satisfying the following conditions:

(a) *$F(P, 0) = f_j(P)$, $P \in \Delta_j$, $j = 1, \cdots, K$,*

(b) *$F(P, 1) = P$, $P \in \bigcup_{j=1}^{K} \Delta_j$,*

(c) *$F(P, t) = P$, P a pole or zero of odd order of $Q(z)\,dz^2$ in $\bigcup_{j=1}^{K} \Delta_j$, $0 \leqq t \leqq 1$,*

(d) $F(P, t) \neq Q$, Q *a pole or zero of odd order of* $Q(z)\,dz^2$ *in* $\mathfrak{R}$, $P \neq Q$, $0 \leqq t \leqq 1$.

The notion of deformation degree remains the same as before [6, p. 50]. However the function families considered are governed by the following definition.

DEFINITION 2. *Let* $\{\Delta\}$ *be an admissible family of domains* Δ_j, $j = 1, \cdots, K$, *on the finite oriented Riemann surface* $\mathfrak{R}$ *with respect to the positive quadratic differential* $Q(z)\,dz^2$. *Then by a special admissible family* $\{f\}$ *of functions* f_j, $j = 1, \cdots, K$, *associated with* $\{\Delta\}$ *we mean a family with the following properties:*

(i) f_j *maps* Δ_j *conformally into* $\mathfrak{R}$, $j = 1, \cdots, K$,

(ii) *if a pole or zero of odd order* A *of* $Q(z)\,dz^2$ *lies in* Δ_j, $f_j(A) = A$,

(iii) $f_j(\Delta_j) \cap f_l(\Delta_l) = 0$, $j \neq l$, $j, l = 1, \cdots, K$,

(iv) *the family* $\{f\}$ *admits a special admissible homotopy* F *into the identity.*

We are now ready to state our principal result.

THEOREM 6 (Special Coefficient Theorem). *Let* $\mathfrak{R}$ *be a finite oriented Riemann surface. Let* $Q(z)\,dz^2$ *be a positive quadratic differential on* $\mathfrak{R}$ *such that each branch of* $\int (Q(z))^{1/2}\,dz$ *is single-valued in a sufficiently small neighborhood of each pole of* $Q(z)\,dz^2$ *(with that pole deleted) of order greater than two. Let* $\{\Delta\}$ *be an admissible family of domains* Δ_j, $j = 1, \cdots, K$, *on* $\mathfrak{R}$ *relative to* $Q(z)\,dz^2$, *and* $\{f\}$ *a special admissible family of functions* f_j, $j = 1, \cdots, K$, *associated with* $\{\Delta\}$. *Let* $Q(z)\,dz^2$ *have double poles* $P_1, \cdots, P_r$ *and poles* $P_{r+1}, \cdots, P_n$ *of order greater than two. Let* P_j, $j \leqq r$, *lie in the domain* Δ_l, *and in terms of a local parameter* z *representing* P_j *as the point at infinity let* f_l *have the expansion*

$$f_l(z) = a^{(j)}z + a_0^{(j)} + \text{negative powers of } z, \tag{18}$$

and Q *the expansion*

$$Q(z) = \alpha^{(j)}z^{-2} + \text{higher powers of } z^{-1}. \tag{19}$$

For $j > r$, *let* ζ *denote a specifically chosen branch of* $\int (Q(z))^{1/2}\,dz$ *in a neighborhood of* P_j. *Let* $\gamma(P_j, L)$ *denote the antecedent on* $\mathfrak{R}$ *of the trace on the Riemann image of a neighborhood of* P_j *under* ζ *of a square in the* ζ*-plane of side* $2L$, *center at the origin, and with sides parallel to the real and imaginary axes, where* L *is to be sufficiently large that* $\gamma(P_j, L)$ *bounds a simply-connected neighborhood* $U(P_j, L)$ *of* P_j. *The curve* $\gamma(P_j, L)$ *is to be sensed so that* $U(P_j, L)$ *lies to its right. Let* P_j, $j > r$, *lie in the domain* Δ_l, *and let* ω *denote the single-valued function obtained in a neighborhood of* P_j *by substituting* $f_l(z)$ *for* z *in the function* ζ. *Let* Φ *denote the intersection of* $\bigcup_{j=1}^{K} \Delta_j$ *with the union of density domains in the trajectory structure of* $Q(z)\,dz^2$, *and let*

$$\Theta_j(L) = \Delta_j - \Phi - \bigcup_{k=r+1}^{n} \bar{U}(P_k, L), \qquad j = 1, \cdots, K.$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{K} \iint_{\Theta_i(L)} |(Q(f_i(z)))^{1/2} f_i'(z) - (Q(z))^{1/2}|^2 \, dA_z \\
(20) \qquad &\leqq -\Re\left(2\pi \sum_{j=1}^{r} \alpha^{(j)} \log a^{(j)}\right) \\
&\quad + \sum_{j=r+1}^{n} \left(\frac{1}{2i}\int_{\gamma(P_j,L)} (\bar{\omega} - \bar{\zeta})\, d(\omega - \zeta) - \mathscr{I}\int_{\gamma(P_j,L)} \omega \, d\zeta\right) + o(1).
\end{aligned}
$$

Here $\log a^{(j)} = \log |a^{(j)}| - i\, d(F, P_j)$, $j \leqq r$, *where* d *denotes the deformation degree. The roots* $(Q(f_i(z)))^{1/2}$, $(Q(z))^{1/2}$ *correspond under passage from* $f_i(z)$ *to* z *along the path curve for the deformation* F. *Finally* dA_z *denotes the element of area for the local parameter* z.

For poles of order two, P_j, $j \leqq r$, we define the curves $\gamma(P_j, L)$ and neighborhoods $U(P_j, L)$ as in [6, p. 60], $\gamma(P_j, L)$ being sensed so that $U(P_j, L)$ lies to its right. We define

$$\Delta_i(L) = \Delta_i - \bigcup_{j=1}^{n} \bar{U}(P_j, L), \qquad i = 1, \cdots, K,$$

and denote $f_i(\Delta_i(L))$ by $\Delta_i'(L)$, $i = 1, \cdots, K$. We will estimate the area of $\bigcup_{i=1}^{K} \Delta_i'(L)$ in the Q-metric $|Q(z)|^{1/2} |dz|$ from above in terms of the area of $\bigcup_{i=1}^{K} \Delta_i(L)$, also in the Q-metric. As usual this is done by determining the change in area arising from the displacement of each boundary curve $\gamma(P_j, L)$, $j = 1, \cdots, n$, under its mapping by the appropriate function in $\{f\}$. In the case of a pole P_j, $r + 1 \leqq j \leqq n$, of order greater than two, the desired quantity is given by

$$\frac{1}{2i}\int_{\gamma(P_j,L)} (\bar{\omega}\, d\omega - \bar{\zeta}\, d\zeta).$$

At the corresponding point in the proof of the General Coefficient Theorem we immediately made further estimates, but now we leave the expression in this form. The corresponding quantity for a pole P_j, $j \leqq r$, of order two is given by

$$2\pi\Re(|\alpha^{(j)}| \log a^{(j)}) + o(1)$$

as in the proof in [6, §4.4]. Thus we have the evaluation

$$
\begin{aligned}
&\sum_{i=1}^{K} \iint_{\Delta_i'(L)} dA \leqq \sum_{i=1}^{K} \iint_{\Delta_i(L)} dA + \sum_{j=1}^{r} 2\pi\Re(|\alpha^{(j)}| \log a^{(j)}) \\
(21) \qquad &\qquad + \sum_{j=r+1}^{n} \frac{1}{2i}\int_{\gamma(P_j,L)} (\bar{\omega}\, d\omega - \bar{\zeta}\, d\zeta) + o(1)
\end{aligned}
$$

where dA denotes the element of area in the Q-metric.

We now obtain an estimate in the opposite direction by use of a modified form of the method of the extremal metric. In this connection we study separately each type of basic domain associated with the trajectory structure of $Q(z)\,dz^2$; see [7, Theorem 1]. Let Φ' denote the image of Φ under the respective functions in $\{f\}$, i.e., $\bigcup_{i=1}^{K} f_i(\Delta_i \cap \Phi)$. At present the best we can do is to assert as in [6, p. 57] that

$$\iint_{\Phi'} dA \geqq \iint_{\Phi} dA. \tag{22}$$

Next let $\mathfrak{D}$ be a ring domain in the trajectory structure. Its intersection with $\bigcup_{i=1}^{K} \Delta_i$ consists of one or several doubly-connected domains possibly slit along trajectory arcs. We slit the totality of these along an arc of an orthogonal trajectory. Then any branch of $\int (Q(z))^{1/2}\,dz$ will map this configuration onto a rectangle, which by choice of a suitable branch ζ we may take to be ($\zeta = \xi + i\eta$)

$$0 < \xi < \lambda, \qquad 0 < \eta < h,$$

this being slit along certain horizontal segments which may extend its full length. We now define as follows a mapping on this slit rectangle R. From a point of R we pass back to a point P on $\mathfrak{R}$ by the inverse of the chosen branch of $\int (Q(z))^{1/2}\,dz$, perform the mapping f_i corresponding to the appropriate domain Δ_i such that $P \in \Delta_i$, and map again by the branch of $\int (Q(z))^{1/2}\,dz$ obtained by continuation of the chosen branch from P to $f_i(P)$ along the path $F(P, t)$, $1 \geqq t \geqq 0$. We denote the mapping so obtained by $\omega = \phi(\zeta)$.

We know that for any trajectory τ lying in $\mathfrak{D} \cap \bigcup_{i=1}^{K} \Delta_i$ and its image τ' under the appropriate mapping f_i we have

$$\int_{\tau} (Q(z))^{1/2}\,dz = \int_{\tau'} (Q(z))^{1/2}\,dz$$

where the branches are chosen according to the prescription just given. Thus we have

$$\int_{\sigma(\eta)} \mathcal{R}\phi'(\zeta)\,d\xi = \lambda,$$

where $\sigma(\eta)$ is the intersection of R with the line $\mathcal{I}\zeta = \eta$, for all but a finite number of values of η in $0 < \eta < h$. Integrating with respect to η over $(0, h)$ we have

$$\iint_{R} \mathcal{R}\phi'(\zeta)\,dA_\zeta = \lambda h = \iint_{R} dA_\zeta. \tag{23}$$

Similarly for a circle domain $\mathfrak{C}$ if we denote the image of $\mathfrak{C} \cap \bigcup_{i=1}^{K} \Delta_i(L)$ slit along an orthogonal trajectory under an appropriate branch of $\int (Q(z))^{1/2}\,dz$ by $\mathfrak{C}(L)$ and define the mapping $\phi(\zeta)$ by the same prescription as above, we have

$$\iint_{\mathfrak{C}(L)} \mathcal{R}\phi'(\zeta)\,dA_\zeta = \iint_{\mathfrak{C}(L)} dA_\zeta. \tag{24}$$

Now let P_j be a pole of order m_j greater than two. Our hypotheses actually require that m_j be even, so at least four. With P_j are associated $m_j - 2$ end domains which we denote by $\mathfrak{E}_q$, $q = 1, \cdots, m_j - 2$ and which are taken to be numbered in cyclic order about P_j, so that under the single-valued branch ζ of $\int (Q(z))^{1/2} dz$ chosen in the enunciation of the theorem $\mathfrak{E}_1$ is mapped into an upper half-plane. Let P_j lie in the domain Δ_i. We denote $\mathfrak{E}_q \cap \Delta_i(L)$ by $\mathfrak{E}_q(L)$, $q = 1, \cdots, m_j - 2$. Its image under the given branch ζ will be denoted by $E_q(L)$ and consists of a rectangle, possibly provided with a finite number of horizontal slits

$$-L < \xi < L, \qquad \lambda_q < \eta < L \tag{25}$$

or

$$-L < \xi < L, \qquad -L < \eta < \lambda_q, \tag{26}$$

according as j is odd or even, where λ_q may be positive or negative. The mapping from ζ to ω associated with P_j in the enunciation of the theorem extends to a single-valued function defined throughout $E_q(L)$ which we denote by $\phi(\zeta)$ once again. Then for all but a finite number of values of η in the range (25) or (26) we have

$$\int_{\sigma(\eta)} \mathfrak{R}\phi'(\zeta)\, d\xi = \mathfrak{R}(\phi(A(\eta)) - \phi(B(\eta)))$$

where $\sigma(\eta)$ is the intersection of $E_q(L)$ with the line $\mathfrak{I}\zeta = \eta$ and $A(\eta)$, $B(\eta)$ are the points of this line on $\xi = L$, $\xi = -L$ respectively. Integrating with respect to η over the appropriate interval (25) or (26) we have

$$\iint_{E_q(L)} \mathfrak{R}\phi'(\zeta)\, dA_\zeta = \int \mathfrak{R}\omega\, d\mathfrak{I}\zeta = \iint_{E_q(L)} dA_\zeta + \int (\mathfrak{R}\omega - \mathfrak{R}\zeta)\, d\mathfrak{I}\zeta \tag{27}$$

where the line integrals on the right-hand side may be taken over the three sides of $E_q(L)$ arising from arcs on $\gamma(P_j, L)$.

Finally let $\mathfrak{S}$ be a strip domain which has boundary elements arising from poles P_q and P_l which may be distinct or coincident and of any (even) order greater than or equal to two. These poles must lie in the same domain, say Δ_i, of the family $\{\Delta\}$, and $\mathfrak{S}$ can meet no domain in $\{\Delta\}$ other than Δ_i. We denote $\mathfrak{S} \cap \Delta_i(L)$ by $\mathfrak{S}(L)$. A suitable determination of $\int (Q(z))^{1/2} dz$ maps $\mathfrak{S}$ onto a strip S given by

$$0 < \mathfrak{I}\zeta < \lambda$$

(λ positive) where the boundary element of $\mathfrak{S}$ arising from P_q corresponds to the boundary point of S at infinity in whose neighborhood $\mathfrak{R}\zeta$ becomes positively infinite. Evidently the present choice of determination may differ from those used to define the neighborhoods $U(P_j, L)$, $j = q, l$. Thus under this mapping, to $\mathfrak{S}(L)$ corresponds a domain $S(L)$ consisting of a rectangle ($\zeta = \xi + i\eta$)

$$-L + b < \xi < L + a, \qquad 0 < \eta < \lambda$$

provided perhaps with a finite number of horizontal slits where a, b are real numbers possibly either positive or negative. Let $\sigma(\eta)$ denote the horizontal segment lying in $S(L)$ on the line $\mathfrak{I}\zeta = \eta$, defined for all but a finite number of values of η in $0 < \eta < \lambda$. Let $A(\eta)$, $B(\eta)$ be the end points of $\sigma(\eta)$ on the lines $\xi = L + a$, $\xi = -L + b$ respectively. Let us denote by $\omega = \phi(\zeta)$ the mapping induced by f_i on S, apart from any slits occurring in $S(L)$, in the following manner. From a point ζ we pass back to a point P on $\mathfrak{R}$ by the inverse of the present branch of $\int (Q(z))^{1/2}\, dz$, perform the mapping f_i and map again by the branch of $\int (Q(z))^{1/2}\, dz$ obtained by continuation of the previous branch from P to $f_i(P)$ along the path $F(P, t)$, $1 \geqq t \geqq 0$. The current determination ζ_2 of $\int (Q(z))^{1/2}\, dz$ is related to the determination ζ_1 used to define the neighborhood $U(P_j, L)$, $j = q, l$, by a relation $\zeta_2 = c \pm \zeta_1$ where c is a constant. Let $\delta_j = \pm 1$ according as we have $\pm\zeta_1$. If P_j, $j = q, l$, is a pole of order two, the mapping ϕ is represented asymptotically by the expansion

$$\omega = \zeta + \delta_j(\alpha^{(j)})^{1/2} \log a^{(j)} + o(1)$$

(we note in this case $\delta_q = 1$, $\delta_l = -1$) where $(\alpha^{(j)})^{1/2}$ is the root with positive real part and $\log a^{(j)}$ has the determination given in the statement of Theorem 6.

Now

$$\int_{\sigma(\eta)} \mathfrak{R}\phi'(\zeta)\, d\xi = \mathfrak{R}(\phi(A(\eta)) - \phi(B(\eta))),$$

and integrating with respect to η over the interval $(0, \lambda)$ we have

$$\iint_{S(L)} \mathfrak{R}\phi'(\zeta)\, dA_\zeta = \iint_{S(L)} dA_\zeta + \rho_q + \rho_l + o(1) \tag{28}$$

where for $j = q, l$

$$\rho_j = \lambda(\alpha^{(j)})^{1/2} \log a^{(j)}$$

if P_j has order 2, and

$$\rho_j = \int (\mathfrak{R}\omega - \mathfrak{R}\zeta)\, d\mathfrak{I}\zeta$$

if P_j has order greater than two with the integral taken over the arc of $\gamma(P_j, L)$ in $\mathfrak{S}$. It is clear that in this last term we may take ζ and ω to have the determinations canonically associated with the pole P_j.

Now let

$$\Lambda_i(L) = \Delta_i(L) - \Phi, \qquad \Lambda_i'(L) = \Delta_i'(L) - \Phi'.$$

Then from the inequalities (21) and (22) we find

$$\begin{aligned}\sum_{i=1}^{K} \iint_{\Lambda_i'(L)} dA \leqq \sum_{i=1}^{K} \iint_{\Lambda_i(L)} dA + \sum_{j=1}^{r} 2\pi\mathfrak{R}(|\alpha^{(j)}| \log a^{(j)}) \\ + \sum_{j=r+1}^{n} \frac{1}{2i} \int_{\gamma(P_j, L)} (\bar{\omega}\, d\omega - \bar{\zeta}\, d\zeta) + o(1).\end{aligned} \tag{29}$$

We subtract from (29) twice the sum of equations (23), (24), (27), and (28) and recall that

$$\iint_{\Lambda_i'(L)} dA = \sum \iint |\phi'(\zeta)|^2 \, dA_\zeta$$

with the integrals in the sum on the right-hand side being taken over the portions of basic domains making up $\Lambda_i(L)$. In this way we obtain

$$\begin{aligned}(30)\qquad \sum \iint |\phi'(\zeta) - 1|^2 \, dA_\zeta &\leqq -\mathcal{R}\left(2\pi \sum_{j=1}^{r} \alpha^{(j)} \log a^{(j)}\right) \\ &+ \sum_{j=r+1}^{n} \left(\frac{1}{2i}\int_{\gamma(P_j,L)} (\bar{\omega}\, d\omega - \bar{\zeta}\, d\zeta) - 2\int_{\gamma(P_j,L)} (\mathcal{R}\omega - \mathcal{R}\zeta)\, d\mathcal{I}\zeta\right) + o(1)\end{aligned}$$

with the integrals in the sum on the left-hand side being taken over all portions of basic domains making up $\bigcup_{i=1}^{K} \Lambda_i(L)$.

The second term on the right-hand side of (30) reduces by elementary operations to

$$\sum_{j=r+1}^{n} \left(\frac{1}{2i}\int_{\gamma(P_j,L)} (\bar{\omega} - \bar{\zeta})\, d(\omega - \zeta) - \mathcal{I}\int_{\gamma(P_j,L)} \omega\, d\zeta\right).$$

Making this substitution in (30) and taking account of the meaning of $\phi'(\zeta)$ we see that this reduces to the form

$$\begin{aligned}&\sum_{i=1}^{K} \iint_{\Delta_i(L)} |(Q(f_i(z)))^{1/2} f_i'(z) - (Q(z))^{1/2}|^2 \, dA_z \\ (31)\qquad &\leqq -\mathcal{R}\left(2\pi \sum_{j=1}^{r} \alpha^{(j)} \log a^{(j)}\right) \\ &\quad + \sum_{j=r+1}^{n} \left(\frac{1}{2i}\int_{\gamma(P_j,L)} (\bar{\omega} - \bar{\zeta})\, d(\omega - \zeta) - \mathcal{I}\int_{\gamma(P_j,L)} \omega\, d\zeta\right) + o(1).\end{aligned}$$

Letting the curve $\gamma(P_j, L), j = 1, \cdots, r$, shrink to the point P_j, we obtain the enunciation of Theorem 6.

Remark. If $Q(z)\, dz^2$ has only poles of order greater than two, and the family $\{\Delta\}$ consists of a single simply-connected domain Δ_1 which contains no zeros of $Q(z)\, dz^2$ of odd order, or if $\mathfrak{R}$ is the z-sphere, $Q(z)\, dz^2$ has a single pole and no zeros of odd order, and $\{\Delta\}$ consists of a single domain Δ_1 either simply- or multiply-connected, the result of Theorem 6 reduces to a form of the area principle. Indeed the left-hand side of inequality (20) becomes

$$\sum_{j=1}^{n} \frac{1}{2i}\int_{\gamma(P_j,L)} (\bar{\omega} - \bar{\zeta})\, d(\omega - \zeta) - \frac{1}{2i}\int_{\Gamma} (\bar{\omega} - \bar{\zeta})\, d(\omega - \zeta)$$

where the second term is taken over the boundary γ of Δ_1 sensed to have Δ_1 to its right and is understood in a limiting sense if necessary. Thus our result takes the form

$$(32) \qquad -\frac{1}{2i}\int_{\Gamma}(\bar{\omega}-\bar{\zeta})\,d(\omega-\zeta) \leqq -\sum_{j=1}^{n}\mathscr{I}\int_{\gamma(P_j,L)}\omega\,d\zeta + o(1).$$

The left-hand side evidently expands out to

$$-\frac{1}{2i}\int_{\Gamma}\bar{\omega}\,d\omega - \frac{1}{2i}\int_{\Gamma}\bar{\zeta}\,d\zeta + \frac{1}{2i}\int_{\Gamma}(\bar{\omega}\,d\zeta + \bar{\zeta}\,d\omega).$$

Now

$$\frac{1}{2i}\int_{\Gamma}(\bar{\omega}\,d\zeta + \bar{\zeta}\,d\omega) = \frac{1}{2i}\int_{\Gamma}(\bar{\omega}\,d\zeta - \omega\,d\bar{\zeta}) = -\mathscr{I}\int_{\Gamma}\omega\,d\bar{\zeta}$$

$$= -\mathscr{I}\int_{\Gamma}\omega\,d\zeta = -\sum_{j=1}^{n}\mathscr{I}\int_{\gamma(P_j,L)}\omega\,d\zeta.$$

Moreover

$$\frac{1}{2i}\int_{\Gamma}\bar{\zeta}\,d\zeta = \frac{1}{2i}\int_{\Gamma}\bar{\zeta}\,d\bar{\zeta} = 0.$$

Performing the proper cancellations in inequality (32) and letting L tend to infinity we obtain

$$\frac{1}{2i}\int_{\Gamma}\bar{\omega}\,d\omega \geqq 0.$$

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