# ON A PROBLEM OF STÖRMER

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## 1. Introduction

Let $q_1 < q_2 < \cdots < q_t$ be a given set of $t$ primes, and let $Q$ be the set of all numbers

$$q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_t^{\alpha_t} \qquad (\alpha_i \geqq 0, \quad i = 1(1)t)$$

generated by these primes. We consider the question of finding pairs $(S, S + 1)$ of consecutive integers such that both $S$ and $S + 1$ belong to $Q$. Since it is obvious that no such pair exists unless $q_1 = 2$, we are at the same time asking about those members of $Q$ which are triangular numbers. Interest in such pairs dates back to the 18$^{\text{th}}$ century and seems to have been awakened by their usefulness in calculating logarithms of integers to great accuracy. Gauss notes for example that

$$9800 = 2^3 \cdot 5^2 \cdot 7^2, \qquad 9801 = 3^4 \cdot 11^2.$$

Such pairs lead to sets of "nearly" dependent logarithms of primes. For instance the number

$$\begin{aligned} K &= \log 63927525376 - \log 63927525375 \\ &= 13 \log 2 - 3 \log 3 - 3 \log 5 - 7 \log 7 \\ &\qquad + 4 \log 11 + \log 13 - \log 23 + \log 41, \end{aligned}$$

which cannot be zero because of the unique factorization theorem, is, however, less than $1.56427 \cdot 10^{-11}$.

Another use for such pairs is in finding particular solutions of diophantine equations of the form

$$Ax^n - By^m = 1.$$

For example the equation

$$x^2 - 14y^3 = 1$$

has the solution (55, 6) because of the pair (3024, 3025). In a recent proof of some results on the distribution of consecutive pairs of higher residues, many hundreds of such pairs were used with $t$ ranging up to 32 [1].

The problem proposed and solved by Størmer [2] is that of finding *all* pairs $(S, S + 1)$ both belonging to the given set $Q$. He showed that there are indeed only a finite number of such pairs, and that they can be found in a nontentative way by solving $3^t - 2^t$ Pell equations. He gave all 23 pairs that go with the set

$$Q : 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4}.$$

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It follows from Størmer's procedure that the number of pairs cannot exceed $3^t - 2^t$.

The mere finiteness of the number of such pairs follows from the celebrated theorem of Thue as Thue [7] himself noted in 1908. However this nonconstructive argument fails to furnish the actual pairs. An upper bound of $3^{2t+1}$ for the number of pairs follows from "certain results on diophantine cubics" according to a recent remark of Nagell [3].

The large number of Pell equations required by Størmer's method makes it impractical except for very limited values of $t$. The purpose of this paper is to present an alternative to Størmer's algorithm requiring the solution of only $2^t - 1$ Pell equations. It follows from the new procedure that the number of pairs cannot exceed $(q_t + 1)(2^t - 1)/2$ when $q_t > 3$. It is also possible to give an upper limit for the largest possible pair in terms of the given $q$'s.

Størmer's procedure depends on his interesting lemma to the effect that if $x^2 - Dy^2 = 1$, and if all the prime factors of $y$ divide $D$, then $(x, y)$ is the fundamental solution of this Pell equation. The present method makes use of the multiple solutions of the Pell equation and their characteristic prime factors. The theory [4] is that of Lucas's function $U_n$, but in this particular case rather more can be proved in a simpler self-contained elementary treatment.

Although in the present method the number of Pell equations to be solved is drastically reduced, a complete set of pairs corresponding to a given set $Q$ still may represent a great deal of calculation, with quite large numbers appearing frequently. We have made these calculations for the most useful case in which $q_i$ is the $i^{\text{th}}$ prime and $t = 13$, that is, for the set

$$q_1 = 2, \quad q_2 = 3, \quad q_3 = 5, \quad \cdots, \quad q_{13} = 41.$$

The results are tabulated with the expectation that they will be of future use.

The computer used was the IBM 704 at the University of California Computer Center at Berkeley.

## 2. The Lucas function $U_n$

The exact procedure for solving Størmer's problem is contained in Theorem 1. The proof of the theorem justifying the procedure is approached by way of five lemmas dealing with the multiple solutions of the Pell equation

$$x^2 - Dy^2 = 1. \tag{1}$$

It is assumed that the reader is familiar with the classical method of finding the fundamental or least positive solution $(x_1, y_1)$ of (1) by means of the continued fraction expansion of the square root of $D$ (see [5]). The $n^{\text{th}}$ multiple solution $(x_n, y_n)$ is then given by

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n \quad (n = 0, 1, 2, 3, \cdots).$$

For brevity we write

$$\alpha = x_1 + y_1\sqrt{D}, \qquad \beta = x_1 - y_1\sqrt{D},$$

so that

$$\alpha + \beta = 2x_1, \qquad \alpha\beta = 1, \qquad \alpha - \beta = 2y_1\sqrt{D},$$

and

$$2x_n = \alpha^n + \beta^n, \qquad 2y_n\sqrt{D} = \alpha^n - \beta^n.$$

We also introduce

$$U_n = y_n/y_1 = (\alpha^n - \beta^n)/(\alpha - \beta).$$

It will be convenient later to introduce the number $M$ defined by

$$M = \max(3, (q_t + 1)/2).$$

The following identities are easily verified

(2) $$x_{2n} = 2x_n^2 - 1,$$

(3) $$U_{2n} = 2x_n U_n,$$

(4) $$x_{m\pm n} = x_m x_n \pm D y_m y_n,$$

(5) $$U_{m\pm n} = x_n U_m \pm x_m U_n,$$

(6) $$U_n = \sum_{i\geqq 0} \binom{n}{2i+1} D^i y_1^{2i} x_1^{n-1-2i},$$

(7) $$x_n = \sum_{i\geqq 0} \binom{n}{2i} D^i y_1^{2i} x_1^{n-2i},$$

(8) $$U_{mn} = \sum_{i\geqq 0} \binom{n}{2i+1} D^i U_m^{2i+1} y_1^{2i} x_m^{n-1-2i}.$$

Let $p \geqq 2$ be a prime, and let $w(p) = w$ be the "rank of apparition" of $p$ in the sequence $\{U_n\}$, that is, the least positive $j$ for which $U_j$ is divisible by $p$. Lemma 1 shows that $w$ exists. By (5) we see that the set of all subscripts $j$ for which $p$ divides $U_j$ is a module. Hence $p$ divides $U_n$ if and only if $w$ divides $n$.

LEMMA 1 (Law of Apparition). *$w(2) = 2$; $w(p) = p$ if $p$ divides $Dy_1$. For any other prime $p$, $w(p)$ divides $(p - \varepsilon)/2$, where*

$$\varepsilon = \left(\frac{D}{p}\right) \equiv D^{(p-1)/2} \pmod{p}.$$

*Proof.* $U_1 = 1$, $U_2 = 2x_1$. Hence $w(2) = 2$. If $p$ divides $Dy_1$, then (6) gives

(9) $$U_n \equiv n x_1^{n-1} \pmod{p}.$$

Since

$$x_1^2 = 1 + Dy_1^2 \equiv 1 \pmod{p},$$

it follows from (9) that $U_p$ is the first $U$ to be divisible by $p$. Finally suppose $p > 2$, and $p$ does not divide $Dy_1$. Then (6) gives for $n = p$

(10) $$U_p \equiv D^{(p-1)/2} y_1^{p-1} \equiv \varepsilon \pmod{p},$$

because of the divisibility of the binomial coefficients by $p$. Similarly (7) gives

$$x_p \equiv x_1^p \equiv x_1 \pmod{p}. \tag{11}$$

Using (5), (10), and (11) we have

$$U_{p-\varepsilon} = U_p x_1 - \varepsilon x_p \equiv x_1\{U_p - \varepsilon\} \equiv 0 \pmod{p},$$

$$x_{p-\varepsilon} = x_p x_1 - \varepsilon D y_p y_1 \equiv x_1^2 - \varepsilon^2 D y_1^2 \equiv 1 \pmod{p}.$$

Now by (2)

$$2x_{(p-\varepsilon)/2}^2 - 1 = x_{p-\varepsilon} \equiv 1 \pmod{p}.$$

Hence $p$ does not divide $x_{(p-\varepsilon)/2}$. But by (3)

$$2x_{(p-\varepsilon)/2} U_{(p-\varepsilon)/2} \equiv U_{p-\varepsilon} \equiv 0 \pmod{p}.$$

Thus $p$ divides $U_{(p-\varepsilon)/2}$. By the remark preceding the lemma, $w(p)$ divides $(p - \varepsilon)/2$.

LEMMA 2. *Let $p > 3$ be a prime dividing $Dy_1$. Then $U_p \equiv p \pmod{p^2}$.*

*Proof.* By (6), with $n = p$,

$$U_p \equiv p x_1^{p-1} + \binom{p}{3} D y_1^2 x_1^{p-3} \pmod{D y_1^2}.$$

Since $p > 3$, and since $p$ divides $Dy_1$ but not $x_1$, we have

$$U_p \equiv p x_1^{p-1} \equiv p \pmod{p^2}.$$

The condition $p > 3$ is necessary since $U_3 = 15$ if $D = 3$ and $U_3 = 99$ if $D = 6$.

LEMMA 3 (Law of Repetition). *Let $\lambda \geqq 0$, and let $k$ be an integer not divisible by the prime $p$. Let $p^a$, $a > 0$, be the highest power of $p$ dividing $U_m$. Then the highest power of $p$ dividing $U_{kmp^\lambda}$ is $p^{a+\lambda}$.*

*Proof.* It is clearly sufficient to establish the lemma for $\lambda = 0$ and $\lambda = 1$ as the rest follows by repeated application of the case $\lambda = 1$.

For $\lambda = 0$ we set $n = k$ in (8) and obtain

$$U_{km} \equiv k U_m x_m^{k-1} \pmod{U_m^3}.$$

Since $U_m$ and $x_m$ are relatively prime, it follows that $U_{km}$ and $U_m$ contain the same highest power, $p^a$, of $p$. For $\lambda = 1$ we set $n = kp$ in (8) and obtain

$$U_{kmp} \equiv kp U_m x_m^{kp-1} \pmod{U_m^3}.$$

This shows that $U_{kmp}$ is divisible by $p^{a+1}$ but not by $p^{a+2}$.

## 3. The function $G_n$

We now introduce a factor $G_n$ of $U_n$ defined as follows

$$G_1 = 1,$$
$$G_2 = \alpha + \beta = 2x_1 = U_2,$$
$$G_3 = \alpha^2 + \alpha\beta + \beta^2 = U_3,$$

and in general for $n > 1$

$$G_n = \prod_h \{\alpha - \beta \exp(2\pi i h/n)\}$$

where $h$ ranges over all $\phi(n)$ numbers $< n$ and prime to $n$. It is clear that $G_n$ is an integer, being a symmetric function of $\alpha$ and $\beta$ and of the primitive $n^{\text{th}}$ roots of unity. In fact

$$U_n = \prod_{\delta | n} G_\delta$$

where the product ranges over the divisors of $n$. We distinguish two kinds of prime factors of $G_n$. A prime factor of $G_n$ which divides $n$ is called *intrinsic*. The other prime factors of $G_n$ are called *extrinsic*.

LEMMA 4. *If $G_n$ has an intrinsic prime factor $p$, then $p$ is the largest prime factor of $n$. If $n > 3$, $G_n$ is not divisible by $p^2$.*

*Proof.* Let $d$ be the greatest common divisor of $G_n$ and $n$. If $d = 1$, there is nothing to prove. If $d > 1$, let $p$ be any prime factor of $d$, and let $w = w(p)$ be the rank of apparition of $p$ in the sequence $U$. Since $p$ divides $G_n$ and hence $U_n$, it follows that $w$ divides $n$. Let

$$n = kwp^\lambda \qquad (\lambda \geqq 0, \quad p \nmid k).$$

We first show that $k = 1$. In fact if $k > 1$, the integer

$$U_n/U_{n/k} = \prod_{\delta | n, \delta \nmid n/k} G_\delta$$

is divisible by $G_n$ and hence by $p$. But by the Law of Repetition (Lemma 3), $U_n/U_{n/k}$ is not divisible by $p$. Hence $k = 1$, and

$$n = wp^\lambda \qquad (\lambda \geqq 0).$$

By Lemma 1, $p \geqq w$. Thus $p$ is the largest prime factor of $n$. It remains to show that if $n > 3$, $G_n$ is not divisible by $p^2$. Suppose the contrary, and suppose that $\lambda > 0$. Then the ratio

$$U_{wp^\lambda}/U_{wp^{\lambda-1}}$$

would be divisible by $G_n$ and hence by $p^2$. But Lemma 3 denies this. Hence $\lambda = 0$ and $n = w$. Since $p \mid n$, $p \leqq w$. But $p \geqq w$. Hence $p = w = n > 3$. By Lemma 2, $G_n = G_p = U_p$ is not divisible by $p^2$. This establishes the lemma.

LEMMA 5. *If $n > 3$, $y_n$ is divisible by a prime $\geqq 2n - 1$.*

*Proof.* Let

$$n = \prod_{i=1}^t p_i^{\alpha_i}$$

be the factorization of $n$ into its prime factors of which the prime $p_t$ is the largest. Then

$$\phi(n) = \prod_{i=1}^{t} p_i^{\alpha_i - 1}(p_i - 1) \geqq p_t - 1.$$

Hence

$$| G_n | = \prod_h | \alpha - \beta \exp (2\pi i h/n) | > (\alpha - \beta)^{\phi(n)}$$
$$= (2y_1\sqrt{D})^{\phi(n)} > 2^{p_t - 1} \geqq p_t .$$

Therefore, by Lemma 4, $G_n$ has an extrinsic prime factor $p^*$. Let $w = w(p^*)$ be rank of apparition of $p^*$. Since $p^*$ divides $G_n$ and hence $U_n$, $w$ divides $n$. Suppose, if possible, that $w < n$, so that $G_n$ divides the integer

$$U_n/U_w = \prod_{\delta | n, \delta \nmid w} G_\delta .$$

Then $p^*$ divides this ratio. But $p^*$, being extrinsic, does not divide $n$ or $w$ and so, by Lemma 3, $U_n/U_w$ is not divisible by $p^*$. This contradiction proves that $w = n$. But then $p^* \neq w$ since $p^*$ does not divide $n$. Therefore by Lemma 1, $w$, and hence $n$, divides $\frac{1}{2}(p^* \pm 1)$. Thus $p^* \geqq 2n - 1$. But $p^*$ divides $G_n$, which divides $U_n$, which in turn divides $y_n = U_n y_1$. This proves the lemma.

## 4. The procedure

We are now in a position to prove the following theorem.

THEOREM 1. *Let*

$$2 = q_1 < q_2 < \cdots < q_t$$

*be a given set of $t$ primes. Let $Q$ be the set of numbers of the form*

$$q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_t^{\alpha_t} \qquad (\alpha_i \geqq 0, \quad i = 1(1)t),$$

*and let $Q'$ be the subset of all $2^t - 1$ square-free members of $Q$ with the exception of 2. Let $S$ be an integer such that both $S$ and $S + 1$ belongs to $Q$. Then $S = (x_n - 1)/2$ where $(x_n, y_n)$ is a solution of the Pell equation*

$$x^2 - 2\Delta y^2 = 1 \tag{12}$$

*in which*

$$\Delta \in Q', \qquad 1 \leqq n \leqq M, \qquad y_n \in Q. \tag{13}$$

*Conversely, if $(x_n, y_n)$ is a solution of (12) subject to conditions (13), then $S = (x_n - 1)/2$ and $S + 1$ both belong to $Q$.*

*Proof.* Suppose first that $(x_n, y_n)$ satisfies (12) and (13). Then, since $x_n$ is odd and $y_n$ is even,

$$S(S + 1) = (x_n^2 - 1)/4 = 2\Delta(y_n/2)^2 \in Q.$$

On the other hand, suppose that $S(S + 1) \in Q$, so that

$$S(S + 1) = 2q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_t^{\alpha_t} \tag{14}$$

where

$$\alpha_i = \varepsilon_i + 2\beta_i, \qquad \varepsilon_i = 0, 1 \qquad (i = 1(1)t).$$

Furthermore let

$$x = 2S + 1, \qquad y = 2q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t} \in Q, \qquad \Delta = q_1^{\varepsilon_1} q_2^{\varepsilon_2} \cdots q_t^{\varepsilon_t} \in Q'.$$

Multiplying (14) by 4 we see that

$$4S^2 + 4S = x^2 - 1 = 2\Delta y^2.$$

Hence each such $S$ leads to some solution $(x, y)$ of (12) in which $y$ and $\Delta$ belong to $Q$ and $Q'$ respectively. As is well known, $(x, y)$ must be $(x_n, y_n)$ for some $n \geqq 1$. It remains to show that $n \leqq M$.

Suppose, instead, that $n > M$. Applying Lemma 5 we conclude that $y_n$ is divisible by a prime $p$ such that

$$p \geqq 2n - 1 > 2M - 1 \geqq q_t .$$

Hence $y_n$ is not a member of $Q$, contrary to fact. Thus $n \leqq M$.

Størmer considered also the question of finding two members of $Q$ differing by 2, and Nagell [3] that of two members of $Q$ differing by 4. The present method extends to both these cases. In fact we have the following counterparts of Theorem 1.

THEOREM 2. *Let*

$$q_1 < q_2 < \cdots < q_t$$

*be a given set of $t$ primes, and let $Q$ be the set of numbers generated by them. Let $Q'$ be the subset of all square-free members of $Q$. Let $S$ be a number such that both $S$ and $S + 2$ belong to $Q$. Then $S = x_n - 1$ where $(x_n, y_n)$ is a solution of the Pell equation*

$$x^2 - Dy^2 = 1 \tag{15}$$

*in which*

$$1 < D \in Q', \qquad 1 \leqq n \leqq M, \qquad y_n \in Q. \tag{16}$$

*Conversely, if $(x_n, y_n)$ is a solution of* (15) *subject to* (16), *then both $S = x_n - 1$ and $S + 2$ belong to $Q$.*

THEOREM 3. *Let*

$$q_1 < \cdots < q_t$$

*be a set of odd primes, and let $Q$ be the set of numbers generated by them. Let $Q'$ denote the set of all square-free members of $Q$ of the form $8m + 5$. If both $S$ and $S + 4$ belong to $Q$, then $S = \xi_n - 2$ where $(\xi_n, \eta_n)$ is the $n^{th}$ solution, in order of magnitude, of the equation*

$$\xi^2 - D\eta^2 = 4 \tag{17}$$

*where*

$$D \in Q' \text{ and is such that (17) has a solution in odd integers } (\xi, \eta), \tag{18}$$
$$1 \leqq n \leqq M, \qquad n \not\equiv 0 \pmod 3, \qquad \eta_n \in Q.$$

*Conversely, if* $(\xi_n, \eta_n)$ *is a solution of* (17) *in odd integers subject to* (18), *then* $S = \xi_n - 2$ *and* $S + 4$ *both belong to* $Q$.

The proofs of Theorems 2 and 3 are similar to that of Theorem 1. In each case use is made of Lemma 5.

## 5. Bounds

These theorems give immediately upper bounds for the number of numbers $S$ such that $S$ and $S + d$ have their prime factors taken from a set of $t$ primes for $d = 1, 2, 4$. In fact this number cannot exceed $M$ times the number of Pell equations involved. Thus we have

THEOREM 4. *For* $d = 1, 2$, *let* $N_d(t)$ *denote the number of pairs of numbers differing by d whose product has its prime factors restricted to a given set of t primes of which the largest is* $q_t$. *Then*

$$N_d(t) \leqq M(2^t - 1).$$

THEOREM 5. *Let* $N_4(t)$ *denote the number of pairs of odd numbers differing by* 4 *whose product has its prime factors taken from a set of odd primes*

$$q_1 < q_2 < \cdots < q_t. \tag{19}$$

*Then*

$$N_4(t) \leqq h2^t(M + \tfrac{1}{2})/3$$

*where* $h = \frac{1}{2}$ *if the set* (19) *contains a prime of the form* $8n + 5$ *and at least one prime of the form* $8n + 3$ *or* $8n + 7$; $h = 1$ *if* (19) *contains at least one prime of the form* $8n + 5$ *but no prime of the form* $8n + 3$ *or* $8n + 7$; $h = \frac{1}{2}$ *if* (19) *contains primes of both forms* $8m + 3$ *and* $8m + 7$ *but no prime of the form* $8m + 5$; *and finally* $h = 0$ *otherwise.*

It is possible to use Theorems 1, 2, 3 to obtain upper bounds for the largest pairs. For this we use a theorem of Hua [6]:

THEOREM 6. *Let D be a positive nonsquare integer congruent to* 0 *or* 1 *modulo* 4. *Let* $(\xi_1, \eta_1)$ *be the least positive solution of the equation*

$$\xi^2 - D\eta^2 = 4. \tag{20}$$

*Let*

$$\theta = \tfrac{1}{2}(\xi_1 + \eta_1\sqrt{D}).$$

*Then*

$$\log \theta < \tfrac{1}{2}(2 + \log D)\sqrt{D}.$$

LEMMA 6. *Let D be a positive nonsquare integer, and let* $(x_n, y_n)$ *be the*

$n^{th}$ *multiple solution of* (1). *If* $D \equiv 0, 1 \pmod 4$, *let* $(\xi_n, \eta_n)$ *be the* $n^{th}$ *solution of* (20). *Then*

$$\log (x_n + y_n\sqrt{D}) < n(2 + \log (4D))\sqrt{D}, \tag{21}$$

$$\log \{\tfrac{1}{2}(\xi_n + \eta_n\sqrt{D})\} < \tfrac{n}{2}(2 + \log D)\sqrt{D}. \tag{22}$$

*Proof.* The inequality (22) is an immediate consequence of Theorem 6 and the fact that

$$\tfrac{1}{2}(\xi_n + \eta_n\sqrt{D}) = \theta^n.$$

To prove (21) we note that $(2x, y)$ is a solution of $\xi^2 - 4D\eta^2 = 4$ if and only if $(x, y)$ is a solution of (1). Therefore

$$\log (x_n + y_n\sqrt{D}) = n \log (x_1 + y_1\sqrt{D}) = n \log \{\tfrac{1}{2}(2x_1 + y_1\sqrt{(4D)})\}.$$

Applying Theorem 6 with $D$ replaced by $4D$ gives

$$\log (x_n + y_n\sqrt{D}) < n(2 + \log (4D))\sqrt{D}.$$

We can now easily prove the following inequalities.

THEOREM 7. *Let* $S_1$ *be the largest* $S$ *such that* $S(S + 1)$ *has all its prime factors taken from the set*

$$q_1 < q_2 < \cdots < q_t.$$

*Then*

$$\log S_1 < M\{2 + \log (8P)\}\sqrt{(2P)} - \log 4$$

*where*

$$P = q_1 q_2 \cdots q_t.$$

*Proof.* By Theorem 1, $S_1$ will correspond to some value of $2\Delta$ with $\Delta \in Q'$ (so that $\Delta \leqq P$), and to some value of $n \leqq M$. Hence

$$2 S_1 = x_n - 1 < \tfrac{1}{2}(x_n + y_n\sqrt{(2\Delta)}) \leqq \tfrac{1}{2}(x_M + y_M\sqrt{(2\Delta)}).$$

By (21)

$$\log 4 + \log S_1 < M(2 + \log 8\Delta)\sqrt{(2\Delta)}.$$

The theorem now follows from the inequality $\Delta \leqq P$.

THEOREM 8. *Let* $S_2$ *be the largest* $S$ *such that* $S(S + 2)$ *has all its prime factors taken from the set*

$$3 \leqq q_1 < q_2 < \cdots < q_t.$$

*Then*

$$\log S_2 < M\{2 + \log (4P)\}\sqrt{P} - \log 2$$

*where*

$$P = q_1 q_2 \cdots q_t.$$

This is proved in the same way from Theorem 2 and (21).

THEOREM 9. *Let* $S_4$ *be the largest* $S$ *such that* $S(S + 4)$ *has all its prime* *factors taken from the set*

$$3 \leqq q_1 < q_2 < \cdots < q_t .$$

*Then, if* $S_4$ *exists,*

$$\log S_4 < M'[\log 2 + \tfrac{1}{2}(2 + \log P')\sqrt{P'}] - \log 2$$

*where* $P'$ *is the largest product of* $q$*'s that is congruent to* 5 *modulo* 8 *and* $M'$ *is the largest integer* $\leqq (q_t + 1)/2$ *not divisible by* 3.

This follows from Theorem 3 and (22).

Of course, these inequalities and even those of Theorems 4 and 5 are very weak. The actual values of $N_1(t)$ and $S_1 = S_1(t)$ for the case in which $q_k$ is the $k^{\text{th}}$ prime are given for $t \leqq 13$ in Table A. In contrast, for $t = 13$, Theorems 4 and 7 give

$$N_1(13) \leqq 172011, \qquad S_1(13) < 10^{10^{9.925}}.$$

TABLE A

| $t$ | $q_t$ | $N_1(t)$ | $S_1(t)$ | $t$ | $q_t$ | $N_1(t)$ | $S_1(t)$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 8 | 19 | 167 | 11859210 |
| 2 | 3 | 4 | 8 | 9 | 23 | 241 | 11859210 |
| 3 | 5 | 10 | 80 | 10 | 29 | 345 | 177182720 |
| 4 | 7 | 23 | 4374 | 11 | 31 | 482 | 1611308699 |
| 5 | 11 | 40 | 9800 | 12 | 37 | 653 | 3463199999 |
| 6 | 13 | 68 | 123200 | 13 | 41 | 869 | 63927525375 |
| 7 | 17 | 108 | 336140 | | | | |

## 6. Remarks on procedure

The following remarks may be of use to the reader who may wish to apply Theorems 1, 2, or 3 to a given set of $q$'s. Tables of the solutions of the Pell equation are so limited that it becomes necessary to use a digital computer except for very small $t$ and $q_t$. As is well known, solutions of the Pell equation may be exceedingly large even for small $D$, so one must be prepared for multiprecise arithmetic operations, that is, one must use subroutines which perform addition, multiplication, and square-root of numbers which occupy many hundreds of machine words.

The successive solutions $(x_n, y_n)$ are quickly found recursively by means of the familiar relations

$$x_{m+1} = 2x_1 x_m - x_{m-1}, \qquad y_{m+1} = 2x_1 y_m - y_{m-1},$$

once the continued fraction procedure has produced the fundamental solution $(x_1, y_1)$.

To decide whether or not $y_n$ belongs to $Q$, it is only necessary to test $y_n$ for divisibility by each of the $q_i$, removing at each step whatever powers of

$q_i$ it may chance to contain. If at any step the quotient becomes unity, then $y_n \in Q$, if not, $y_n \notin Q$.

Since every $y_n$ is divisible by $y_1$, it is useless to examine multiple solutions if $y_1$ does not belong to $Q$. More generally, if $y_m$ does not belong to $Q$, then neither does $y_{km}$. These facts, incorporated in the routine, eliminate a great deal of multiprecise testing of large $y$'s for membership in $Q$.

In dealing with the very large values of $D$ that the method requires, one is running the risk of having an intolerably long period in the continued fraction for $\sqrt{D}$. Indeed it is not uncommon for the period to be more than $\sqrt{D}$. In such a case the value of $y_1$ is apt to exceed

$$\exp\{\pi^2\sqrt{D}/\log 4096\}.$$

Had this occurred for any one of the large values of $D$ encountered in our examination of the case

$$q_1 = 2, \quad q_2 = 3, \quad \cdots, \quad q_{13} = 41,$$

we would have had to abandon the project. As it was, the longest period experienced was 7922, the period corresponding to

$$\begin{aligned} D &= 43464323361030 \\ &= 2\cdot3\cdot5\cdot11\cdot13\cdot17\cdot19\cdot23\cdot29\cdot31\cdot37\cdot41. \end{aligned}$$

Apparently, for $D$ a product of small primes, one may expect unusually short periods, a fortunate phenomenon for our method.

If for some $D$ the continued fraction turns out to have a long period, the value of $y_1$ would be very large, and so it is almost certain that $y_1$ does not belong to $Q$. We can find the highest power of each $q_i$ dividing $y_1$, without calculating $y_1$ itself, by simply carrying out the calculation of the convergents of the continued fraction modulo $m_1, m_2, \cdots$ where each $m$ is a suitably chosen product of powers of $q$'s and each $m$ is a single machine word. In this way a great deal of multiprecise arithmetic is avoided. If we know the highest power of $q_i$ contained in $y_1$ and the length $K$ of the period, it is easy to prove that $y_1$ must be divisible by some prime greater than $q_t$. In fact, $y_1$ exceeds the $K^{\text{th}}$ Fibonacci number, which is almost sure to be greater than the product of powers of $q_i$ actually dividing $y_1$.

## 7. Description of tables

We append three tables described as follows.

Table I gives all 869 numbers $N$ greater than 1 such that $N(N - 1)$ has no prime factor greater than 41. Table I is divided into two parts. In Table IA the 869 numbers in question are classified according to the largest prime factor of $N(N - 1)$. Table IB gives the 251 numbers $N$ greater than $10^5$ such that $N(N - 1)$ has no prime factor greater than 41 and, for each such $N$, gives the exponents of the primes in the factorization of $N/(N - 1)$.

Thus the entry

| N | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 116964 | 2 | 4 | | −3 | −1 | | | 2 | | | −1 | | |

in Table IB means that

$$116963 = 7^3 \cdot 11 \cdot 31, \qquad 116964 = 2^2 \cdot 3^4 \cdot 19^2.$$

Table II gives all 101 odd numbers $N$ greater than 1 such that $N(N-2)$ has no prime factor greater than 31. In Table IIA these numbers are classified according to the largest prime factor of $N(N-2)$, while Table IIB gives the factorization of $N/(N-2)$ for those $N$ greater than $10^5$.

Table III gives all 99 odd numbers $N$ greater than 3 such that $N(N-4)$ has no prime factor greater than 31. In Table IIIA these numbers are classified according to the largest prime factor of $N(N-4)$, while Table IIIB gives the factorization of $N/(N-4)$ for those $N$ greater than $10^5$.

The corresponding factorizations for values of $N$ less than $10^5$ can be readily supplied from [8].

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TABLE IA

Integers $N$ greater than 1 such that the largest prime factor of $N(N-1)$ is the $t^{\text{th}}$ prime number, $t \leqq 13$

| $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ | | $t = 5$ | | |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 50 | 11 | 99 | 540 |
| | 4 | 6 | 8 | 64 | 12 | 100 | 3025 |
| | 9 | 10 | 15 | 126 | 22 | 121 | 9801 |
| | | 16 | 21 | 225 | 33 | 176 | |
| | | 25 | 28 | 2401 | 45 | 243 | |
| | | 81 | 36 | 4375 | 55 | 385 | |
| | | | 49 | | 56 | 441 | |

| $t = 6$ | | | | $t = 7$ | | | | |
|---|---|---|---|---|---|---|---|---|
| 13 | 91 | 364 | 4225 | 17 | 136 | 442 | 1225 | 5832 |
| 14 | 105 | 625 | 6656 | 18 | 154 | 561 | 1275 | 12376 |
| 26 | 144 | 676 | 10648 | 34 | 170 | 595 | 1701 | 14400 |
| 27 | 169 | 729 | 123201 | 35 | 221 | 715 | 2058 | 28561 |
| 40 | 196 | 1001 | | 51 | 256 | 833 | 2431 | 31213 |
| 65 | 325 | 1716 | | 52 | 273 | 936 | 2500 | 37180 |
| 66 | 351 | 2080 | | 85 | 289 | 1089 | 2601 | 194481 |
| 78 | 352 | 4096 | | 120 | 375 | 1156 | 4914 | 336141 |

| $t = 8$ | | | | $t = 9$ | | | | |
|---|---|---|---|---|---|---|---|---|
| 19 | 343 | 2432 | 14365 | 23 | 391 | 1863 | 8281 | 71875 |
| 20 | 361 | 2926 | 23409 | 24 | 392 | 2024 | 8625 | 75141 |
| 39 | 400 | 3136 | 27456 | 46 | 460 | 2025 | 10626 | 76545 |
| 57 | 456 | 3250 | 28900 | 69 | 484 | 2185 | 11271 | 104329 |
| 76 | 476 | 4200 | 43681 | 70 | 507 | 2300 | 11662 | 122452 |
| 77 | 495 | 5776 | 89376 | 92 | 529 | 2646 | 12168 | 126225 |
| 96 | 513 | 5929 | 104976 | 115 | 576 | 2737 | 16929 | 152881 |
| 133 | 969 | 5985 | 165376 | 161 | 736 | 3060 | 19551 | 202125 |
| 153 | 1216 | 6175 | 228096 | 162 | 760 | 3381 | 21505 | 264385 |
| 171 | 1331 | 6860 | 601426 | 208 | 875 | 3520 | 21736 | 282625 |
| 190 | 1445 | 10241 | 633556 | 231 | 897 | 3888 | 23276 | 328510 |
| 209 | 1521 | 10830 | 709632 | 253 | 1105 | 4693 | 25025 | 2023425 |
| 210 | 1540 | 12636 | 5909761 | 276 | 1197 | 4761 | 25921 | 4096576 |
| 286 | 1729 | 13377 | 11859211 | 300 | 1288 | 5083 | 43264 | 5142501 |
| 324 | 2376 | 14080 | | 323 | 1496 | 7866 | 52326 | |

| $t = 10$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 29 | 145 | 320 | 609 | 1015 | 1683 | 2465 | 3510 | 5916 | 9802 |
| 30 | 175 | 378 | 638 | 1045 | 2001 | 2640 | 4641 | 6670 | 10557 |
| 58 | 204 | 406 | 726 | 1276 | 2002 | 2755 | 4785 | 7106 | 11340 |
| 88 | 232 | 494 | 783 | 1450 | 2176 | 2784 | 4901 | 7425 | 12006 |
| 116 | 261 | 551 | 784 | 1596 | 2205 | 3249 | 5104 | 7889 | 12673 |
| 117 | 290 | 552 | 841 | 1625 | 2262 | 3451 | 5888 | 8671 | 13225 |

TABLE IA (*Continued*)

$t = 10$ (continued)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 13311 | 24795 | 47125 | 158950 | 240787 | 949026 | 2697696 | 96059601 |
| 13312 | 25840 | 53361 | 166635 | 244036 | 1163800 | 4004001 | 177182721 |
| 13456 | 27000 | 72501 | 168751 | 303601 | 1235169 | 4090625 | |
| 19228 | 30625 | 83521 | 176001 | 410670 | 1243840 | 8268800 | |
| 20736 | 30856 | 87465 | 176176 | 418761 | 1625625 | 10556001 | |
| 23751 | 35322 | 136851 | 184093 | 613089 | 1852201 | 18085705 | |

$t = 11$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 31 | 528 | 1519 | 5797 | 11781 | 29792 | 116964 | 453376 | 3897166 |
| 32 | 589 | 1520 | 6076 | 11935 | 31465 | 122265 | 459173 | 14753025 |
| 63 | 621 | 1768 | 6138 | 12122 | 31900 | 174097 | 509796 | 16093000 |
| 93 | 651 | 1860 | 6293 | 13300 | 32799 | 175770 | 773605 | 76271625 |
| 125 | 714 | 2016 | 6325 | 13455 | 41262 | 178126 | 863940 | 80061345 |
| 155 | 806 | 2233 | 6480 | 15625 | 42688 | 190464 | 912951 | 133920000 |
| 156 | 837 | 2945 | 6728 | 17577 | 49011 | 207576 | 1147125 | 181037025 |
| 187 | 868 | 2976 | 7657 | 19251 | 58311 | 212382 | 1154440 | 370256250 |
| 217 | 900 | 3565 | 7905 | 19344 | 78337 | 227448 | 1255501 | 1611308700 |
| 248 | 931 | 3751 | 7936 | 19965 | 96876 | 240065 | 1594176 | |
| 280 | 961 | 3876 | 8092 | 21142 | 98736 | 245025 | 2307361 | |
| 341 | 1024 | 3969 | 8464 | 22816 | 102487 | 260338 | 2310400 | |
| 342 | 1054 | 4186 | 8526 | 23375 | 108376 | 268801 | 2345057 | |
| 435 | 1210 | 4960 | 8960 | 23716 | 111321 | 278784 | 3206269 | |
| 465 | 1365 | 4992 | 9425 | 24025 | 111476 | 288145 | 3301376 | |
| 496 | 1426 | 5643 | 10881 | 27405 | 116281 | 314433 | 3346110 | |

$t = 12$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 37 | 741 | 2553 | 7696 | 20350 | 49248 | 120176 | 466830 | 2598400 |
| 38 | 851 | 2738 | 8991 | 23200 | 50025 | 143375 | 469568 | 2772225 |
| 75 | 925 | 2850 | 9177 | 26011 | 55056 | 155585 | 494209 | 2893401 |
| 111 | 962 | 3146 | 9251 | 28750 | 56203 | 156066 | 675584 | 3930400 |
| 112 | 1000 | 3220 | 9361 | 28861 | 60606 | 161875 | 777925 | 4765600 |
| 148 | 1036 | 3256 | 10693 | 29601 | 67600 | 164836 | 787176 | 5538975 |
| 185 | 1184 | 3367 | 11914 | 33264 | 68783 | 165649 | 812890 | 6615675 |
| 186 | 1296 | 3553 | 12321 | 34225 | 71485 | 198912 | 837200 | 6770556 |
| 222 | 1332 | 3626 | 13690 | 34596 | 77441 | 208495 | 923521 | 7105000 |
| 260 | 1369 | 3627 | 13950 | 35816 | 78625 | 227070 | 986272 | 7475000 |
| 297 | 1444 | 3774 | 14652 | 37962 | 80920 | 254449 | 1000000 | 7491169 |
| 407 | 1480 | 4256 | 15873 | 38962 | 82621 | 285418 | 1055241 | 13147876 |
| 408 | 1665 | 4625 | 16170 | 41515 | 85064 | 319125 | 1341250 | 14080573 |
| 481 | 1666 | 5291 | 16576 | 42625 | 88320 | 348726 | 1510785 | 21386001 |
| 630 | 1702 | 5292 | 17205 | 43401 | 93093 | 360640 | 1763125 | 27994681 |
| 666 | 1925 | 5440 | 17576 | 44955 | 93500 | 378880 | 1771561 | 50481025 |
| 667 | 2109 | 5625 | 18241 | 45696 | 108780 | 390166 | 2085136 | 71843751 |
| 703 | 2146 | 6993 | 19500 | 47916 | 108928 | 443556 | 2417876 | 308915776 |
| 704 | 2295 | 7105 | 19684 | 48841 | 117624 | 446369 | 2560845 | 3463200000 |

TABLE IA (*Continued*)

$t = 13$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 41 | 1395 | 6273 | 22100 | 64125 | 228781 | 1050625 | 9174816 |
| 42 | 1518 | 6561 | 22386 | 70357 | 243049 | 1082565 | 9222500 |
| 82 | 1600 | 6601 | 23001 | 76384 | 275808 | 1104376 | 9458086 |
| 124 | 1681 | 6930 | 24150 | 76875 | 284376 | 1152921 | 10491040 |
| 165 | 1682 | 7176 | 24273 | 81345 | 330625 | 1205646 | 13745537 |
| 205 | 1805 | 7216 | 27676 | 81549 | 386631 | 1294371 | 14235529 |
| 246 | 1886 | 7750 | 28126 | 81796 | 389500 | 1319626 | 19826576 |
| 247 | 1887 | 8569 | 29233 | 82369 | 395200 | 1362636 | 24601600 |
| 287 | 2091 | 8856 | 29602 | 82944 | 412091 | 1413721 | 25836889 |
| 288 | 2255 | 9472 | 30381 | 91840 | 432345 | 1437501 | 25872148 |
| 369 | 2296 | 10045 | 31488 | 101270 | 453871 | 1536640 | 27005265 |
| 370 | 2542 | 10374 | 32800 | 103156 | 461825 | 1600313 | 30138076 |
| 451 | 2584 | 10660 | 40426 | 103936 | 466089 | 1729750 | 30944914 |
| 493 | 2625 | 11440 | 40960 | 106191 | 476749 | 1740000 | 32517265 |
| 533 | 2665 | 13776 | 41041 | 121771 | 482448 | 1946721 | 36315136 |
| 575 | 2871 | 14145 | 41328 | 130340 | 524800 | 2185300 | 40750802 |
| 616 | 3690 | 14801 | 41616 | 134850 | 536239 | 2267916 | 41808151 |
| 697 | 3773 | 15376 | 46208 | 136161 | 589744 | 2304324 | 43075585 |
| 780 | 4060 | 15457 | 47151 | 142885 | 610204 | 2351350 | 85459375 |
| 820 | 4264 | 16400 | 48750 | 151250 | 638001 | 2825761 | 119094300 |
| 1025 | 4551 | 16524 | 52480 | 152685 | 643126 | 2829124 | 132663168 |
| 1026 | 4675 | 16606 | 53505 | 153791 | 679042 | 3063808 | 293635441 |
| 1148 | 4921 | 17425 | 56376 | 156333 | 728365 | 3331251 | 415704576 |
| 1189 | 4961 | 17836 | 60516 | 174825 | 769120 | 3453840 | 876219201 |
| 1190 | 5084 | 17918 | 61009 | 186592 | 798721 | 3556996 | 1075774401 |
| 1312 | 5577 | 19721 | 63427 | 203320 | 1011840 | 4588311 | 45105689161 |
| 1353 | 6069 | 19845 | 63714 | 212381 | 1048576 | 5267025 | 63927525376 |

TABLE IB

Integers $N$ greater than 100,000 such that $N(N-1)$ has no prime factor greater than 41, with factorizations of $N/(N-1)$

| $N$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 101270 | 1 | | 1 | −1 | | 1 | −1 | 1 | −1 | | | −1 | 1 |
| 102487 | −1 | −1 | | 1 | 4 | | | −1 | | −1 | −1 | | |
| 103156 | 2 | −1 | −1 | | | −1 | 1 | | −2 | | | 1 | 1 |
| 103936 | 9 | −1 | −1 | 1 | | −2 | | | | 1 | | | −1 |
| 104329 | −3 | −4 | | −1 | | | 2 | 2 | −1 | | | | |
| 104976 | 4 | 8 | −2 | | | −1 | −1 | −1 | | | | | |
| 106191 | −1 | 5 | −1 | −1 | | | | 1 | 1 | | | −1 | −1 |
| 108376 | 3 | −1 | −3 | | | | −2 | 1 | 1 | | 1 | | |
| 108780 | 2 | 1 | 1 | 2 | −2 | | | | | −1 | −1 | 1 | |
| 108928 | 7 | −2 | | −2 | | −1 | | −1 | 1 | | | 1 | |
| 111321 | −3 | 3 | −1 | 1 | −2 | | | 1 | −1 | | 1 | | |
| 111476 | 2 | | −2 | −3 | | −1 | | | | 1 | 2 | | |

TABLE IB (*Continued*)

| $N$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 116281 | −3 | −2 | −1 | | 2 | | −1 | −1 | | | 2 | | |
| 116964 | 2 | 4 | | −3 | −1 | | | 2 | | | −1 | | |
| 117624 | 3 | 1 | | | −1 | 2 | −2 | | | 1 | | −1 | |
| 120176 | 4 | | −2 | 1 | −1 | | | −1 | −1 | 1 | | 1 | |
| 121771 | −1 | −3 | −1 | | −1 | 1 | 1 | 1 | | 1 | | | −1 |
| 122265 | −3 | 2 | 1 | | 1 | 1 | −1 | 1 | | −1 | −1 | | |
| 122452 | 2 | −1 | | −4 | 3 | | −1 | | 1 | | | | |
| 123201 | −6 | 6 | −2 | −1 | −1 | 2 | | | | | | | |
| 126225 | −4 | 3 | 2 | −3 | 1 | | 1 | | −1 | | | | |
| 130340 | 2 | | 1 | 3 | −1 | | −2 | 1 | | | | | −1 |
| 134850 | 1 | 1 | 2 | | −1 | −1 | | | −1 | 1 | 1 | | −1 |
| 136161 | −5 | 4 | −1 | | | | | | −1 | | | −1 | 2 |
| 136851 | −1 | 1 | −2 | −1 | 2 | 1 | −1 | | −1 | 1 | | | |
| 142885 | −2 | −6 | 1 | −2 | | | 1 | | | | | | 2 |
| 143375 | −1 | | 3 | −3 | −1 | | | −1 | | | 1 | 1 | |
| 151250 | 1 | | 4 | −1 | 2 | | −1 | | | | −1 | | −1 |
| 152685 | −2 | 4 | 1 | −2 | | 1 | | −1 | | 1 | | | −1 |
| 152881 | −4 | −1 | −1 | −2 | | −1 | 2 | | 2 | | | | |
| 153791 | −1 | | −1 | −1 | 2 | −3 | | | | | 1 | | 1 |
| 155585 | −6 | | 1 | | −1 | −1 | −1 | | | 2 | | 1 | |
| 156066 | 1 | 1 | −1 | −4 | | −1 | | 1 | | | | 2 | |
| 156333 | −2 | 1 | | | −2 | | −1 | −1 | | | 1 | | 2 |
| 158950 | 1 | −3 | 2 | −1 | 1 | | 2 | | | −2 | | | |
| 161875 | −1 | −2 | 4 | 1 | | | −1 | | −2 | | | 1 | |
| 164836 | 2 | −4 | −1 | 2 | −1 | | | | | 2 | | −1 | |
| 165376 | 9 | −3 | −3 | −2 | | | 1 | 1 | | | | | |
| 165649 | −4 | −1 | | −1 | 2 | | −1 | | | −1 | | 2 | |
| 166635 | −1 | 2 | 1 | 1 | | −2 | −1 | | 2 | −1 | | | |
| 168751 | −1 | −3 | −5 | | 1 | | | | 2 | 1 | | | |
| 174097 | −4 | −3 | | 2 | 1 | −1 | 1 | 1 | | | −1 | | |
| 174825 | −3 | 3 | 2 | 1 | | −1 | | | | | | 1 | −2 |
| 175770 | 1 | 4 | 1 | 1 | −1 | | | −1 | | −2 | 1 | | |
| 176001 | −7 | 1 | −3 | 1 | −1 | | 2 | | | 1 | | | |
| 176176 | 4 | −5 | −2 | 1 | 2 | 1 | | | | −1 | | | |
| 178126 | 1 | −1 | −5 | | | 2 | 1 | −1 | | | 1 | | |
| 184093 | −2 | −1 | | 2 | | 1 | 2 | | −2 | −1 | | | |
| 186592 | 5 | −1 | | 3 | | | 1 | | | | | −1 | −2 |
| 190464 | 11 | 1 | | −2 | | −2 | | | −1 | | 1 | | |
| 194481 | −4 | 4 | −1 | 4 | −1 | −1 | −1 | | | | | | |
| 198912 | 8 | 1 | | 1 | | | | −3 | | −1 | | 1 | |
| 202125 | −2 | 1 | 3 | 2 | 1 | −3 | | | −1 | | | | |
| 203320 | 3 | −2 | 1 | | | 1 | 1 | −1 | 1 | −1 | | | −1 |

TABLE IB (*Continued*)

| $N$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 207576 | 3 | 3 | −2 | | | | | −2 | −1 | | 2 | | |
| 208495 | −1 | −6 | 1 | 2 | −1 | −1 | | | 1 | | | 1 | |
| 212381 | −2 | | −1 | −1 | | 1 | 1 | | | | 2 | −1 | −1 |
| 212382 | 1 | 5 | | | | −1 | −1 | 1 | 1 | | −2 | | |
| 227070 | 1 | 3 | 1 | | | | −1 | −2 | | 2 | | −1 | |
| 227448 | 3 | 7 | | | −1 | 1 | | | −1 | −1 | −1 | | |
| 228096 | 8 | 4 | −1 | −4 | 1 | | | −1 | | | | | |
| 228781 | −2 | −2 | −1 | 3 | | | | | 1 | 1 | −1 | | −1 |
| 240065 | −6 | | 1 | 1 | −2 | | | 3 | | | −1 | | |
| 240787 | −1 | −3 | | −3 | | −1 | | 2 | 1 | 1 | | | |
| 243049 | −3 | −1 | | | | −1 | 2 | −1 | | 2 | | | −1 |
| 244036 | 2 | −2 | −1 | | −1 | 2 | −1 | 2 | | −1 | | | |
| 245025 | −5 | 4 | 2 | | 2 | −1 | | −1 | | | −1 | | |
| 254449 | −4 | −3 | | | | 1 | | −1 | 2 | | −1 | 1 | |
| 260338 | 1 | −1 | | −3 | −1 | 1 | 1 | 1 | −1 | | 1 | | |
| 264385 | −6 | −5 | 1 | | 2 | | −1 | 1 | 1 | | | | |
| 268801 | −9 | −1 | −2 | −1 | | 1 | | | 1 | 1 | 1 | | |
| 275808 | 5 | 1 | | −1 | | 2 | 1 | | | | −2 | | −1 |
| 278784 | 8 | 2 | | | 2 | | −1 | | −2 | | −1 | | |
| 282625 | −12 | −1 | 3 | 1 | | | 1 | 1 | −1 | | | | |
| 284376 | 3 | 1 | −5 | −1 | | −1 | 2 | | | | | | 1 |
| 285418 | 1 | −3 | | 1 | −1 | | | 1 | | 1 | −2 | 1 | |
| 288145 | −4 | −3 | 1 | | 1 | 2 | | | −1 | −1 | 1 | | |
| 303601 | −4 | −1 | −2 | | −1 | | | 2 | −1 | 2 | | | |
| 314433 | −6 | 2 | | 2 | | | −3 | | 1 | | 1 | | |
| 319125 | −2 | 1 | 3 | | | −1 | −1 | −2 | 1 | | | 1 | |
| 328510 | 1 | −3 | 1 | 1 | | 1 | | 2 | −3 | | | | |
| 330625 | −7 | −2 | 4 | −1 | | | | | 2 | | | | −1 |
| 336141 | −2 | 2 | −1 | −5 | | 3 | 1 | | | | | | |
| 348726 | 1 | 1 | −2 | 1 | | −1 | | 2 | 1 | −1 | | −1 | |
| 360640 | 6 | −3 | 1 | 2 | | | | −2 | 1 | | | −1 | |
| 378880 | 11 | −1 | 1 | | | | −2 | −1 | −1 | | | 1 | |
| 386631 | −1 | 2 | −1 | 1 | | | 1 | 2 | −1 | | | | −2 |
| 389500 | 2 | −1 | 3 | | −2 | | | 1 | | −1 | | −1 | 1 |
| 390166 | 1 | −1 | −1 | 1 | | | | −1 | | 1 | 2 | −2 | |
| 395200 | 6 | −4 | 2 | −1 | | 1 | −1 | 1 | | | | | −1 |
| 410670 | 1 | 5 | 1 | −2 | | 2 | −2 | | | −1 | | | |
| 412091 | −1 | | −1 | −2 | | | | 1 | 2 | −2 | | | 1 |
| 418761 | −3 | 2 | −1 | 1 | | | 2 | −2 | 1 | −1 | | | |
| 432345 | −3 | 1 | 1 | | −1 | | −3 | 1 | | | | 1 | 1 |
| 443556 | 2 | 4 | −1 | −1 | | | | −1 | −1 | −1 | | 2 | |
| 446369 | −5 | | | 1 | 2 | −1 | 1 | | | −1 | 1 | −1 | |

TABLE IB (*Continued*)

| $N$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 453376 | 8 | −2 | −3 | 1 | 1 | −1 | | | 1 | | −1 | | |
| 453871 | −1 | −3 | −1 | | 4 | | | | | | 1 | | −2 |
| 459173 | −2 | | | −1 | 1 | 3 | | 1 | −2 | | −1 | | |
| 461825 | −10 | | 2 | 2 | −1 | 1 | | | | 1 | | | −1 |
| 466089 | −3 | 1 | | −2 | | 1 | 1 | 1 | | −1 | | 1 | −1 |
| 466830 | 1 | 3 | 1 | 1 | −1 | 1 | | 1 | | | −1 | −2 | |
| 469568 | 6 | | | −3 | 1 | | | | 1 | 1 | | −2 | |
| 476749 | −2 | −2 | | 1 | | 3 | −1 | −1 | | | 1 | | −1 |
| 482448 | 4 | 1 | | −1 | | | | 1 | 2 | | | | −3 |
| 494209 | −7 | −3 | | | −1 | −1 | | 2 | | | | 2 | |
| 509796 | 2 | 2 | −1 | 2 | −1 | −1 | 2 | | −1 | | −1 | | |
| 524800 | 9 | −4 | 2 | | −1 | | | −1 | | | −1 | | 1 |
| 536239 | −1 | −2 | | | 1 | | | | | 1 | −3 | | 2 |
| 589744 | 4 | −2 | | −1 | −1 | | | | −1 | 1 | 1 | −1 | 1 |
| 601426 | 1 | −7 | −2 | 2 | −1 | | 1 | 2 | | | | | |
| 610204 | 2 | −1 | | 1 | −2 | | | 1 | | | 1 | 1 | −2 |
| 613089 | −5 | 6 | | −2 | | | −1 | | −1 | 2 | | | |
| 633556 | 2 | −3 | −1 | 1 | 3 | −1 | 1 | −2 | | | | | |
| 638001 | −4 | 2 | −3 | 1 | −1 | 1 | | 1 | | −1 | | | 1 |
| 643126 | 1 | −1 | −4 | −3 | 1 | | | | 1 | | 1 | | 1 |
| 675584 | 8 | | | 1 | | 1 | | −1 | | 1 | −2 | −1 | |
| 679042 | 1 | −2 | | 2 | −1 | 2 | | −3 | | | | | 1 |
| 709632 | 10 | 2 | | 1 | 1 | −3 | −1 | −1 | | | | | |
| 728365 | −2 | −1 | 1 | −1 | 1 | −1 | 1 | 1 | −1 | −1 | | | 1 |
| 769120 | 5 | −1 | 1 | | 1 | −2 | | 1 | 1 | | | −1 | −1 |
| 773605 | −2 | −3 | 1 | 1 | | −1 | | −1 | 1 | −1 | 2 | | |
| 777925 | −2 | −4 | 2 | −4 | | | | | | 2 | | 1 | |
| 787176 | 3 | 2 | −2 | | | 1 | | | −1 | 2 | | −2 | |
| 798721 | −12 | −1 | −1 | 1 | 2 | −1 | | | 1 | | | | 1 |
| 812890 | 1 | −3 | 1 | −1 | −1 | 3 | −1 | | −1 | | | 1 | |
| 837200 | 4 | | 2 | 1 | −3 | 1 | −1 | | 1 | | | −1 | |
| 863940 | 2 | 1 | 1 | 1 | 2 | | 1 | | | −1 | −3 | | |
| 912951 | −1 | 5 | −2 | | | 1 | 2 | −1 | | | −2 | | |
| 923521 | −7 | −1 | −1 | | | −1 | | | | | 4 | −1 | |
| 949026 | 1 | 1 | −2 | −1 | −1 | 1 | −1 | | 3 | −1 | | | |
| 986272 | 5 | −1 | | 2 | −3 | −1 | 1 | −1 | | | | 1 | |
| 1000000 | 6 | −3 | 6 | −1 | −1 | −1 | | | | | | −1 | |
| 1011840 | 7 | 1 | 1 | | | | 1 | | −1 | −1 | 1 | −1 | −1 |
| 1048576 | 20 | −1 | −2 | | −1 | | | | | | −1 | | −1 |
| 1050625 | −11 | −3 | 4 | | | | | −1 | | | | | 2 |
| 1055241 | −3 | 3 | −1 | | 2 | | 1 | 1 | −1 | | −1 | −1 | |
| 1082565 | −2 | 9 | 1 | −1 | 1 | | | | −1 | | | | −2 |

TABLE IB (*Continued*)

| $N$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1104376 | 3 | −1 | −4 | 1 | | 1 | | −1 | | | −1 | 1 | 1 |
| 1147125 | −2 | 1 | 3 | 1 | −1 | | | 1 | 1 | −2 | −1 | | |
| 1152921 | −3 | 1 | −1 | 2 | 1 | | | −1 | 1 | | 1 | −1 | −1 |
| 1154440 | 3 | −3 | 1 | 2 | −1 | −2 | | 1 | −1 | | 1 | | |
| 1163800 | 3 | −2 | 2 | −3 | 1 | −1 | | | 2 | −1 | | | |
| 1205646 | 1 | 1 | −1 | −3 | | 2 | | −1 | | 1 | | −1 | 1 |
| 1235169 | −5 | 5 | | | −3 | 1 | 1 | | 1 | −1 | | | |
| 1243840 | 6 | −1 | 1 | | | 2 | −1 | | 1 | −3 | | | |
| 1255501 | −2 | −4 | −3 | | | 2 | 1 | 1 | 1 | | −1 | | |
| 1294371 | −1 | 2 | −1 | −1 | −1 | 2 | | | 1 | | | 1 | −2 |
| 1319626 | 1 | −3 | −3 | 1 | 2 | | −1 | 1 | −1 | | | | 1 |
| 1341250 | 1 | −1 | 4 | −1 | | −1 | −3 | | | 1 | | 1 | |
| 1362636 | 2 | 3 | −1 | | 1 | | −2 | | −1 | | 1 | 1 | −1 |
| 1413721 | −3 | −3 | −1 | −1 | −1 | | −1 | | | 2 | | | 2 |
| 1437501 | −2 | 1 | −6 | | | 1 | | | −1 | 1 | 1 | | 1 |
| 1510785 | −7 | 3 | 1 | | −1 | | | 2 | | −1 | 1 | −1 | |
| 1536640 | 7 | −1 | 1 | 4 | | −1 | | | | | −2 | | −1 |
| 1594176 | 6 | 1 | −2 | | −2 | | −1 | 2 | 1 | | −1 | | |
| 1600313 | −3 | | | −1 | 1 | 1 | −1 | 2 | | | 1 | | −2 |
| 1625625 | −3 | 2 | 4 | −2 | −1 | −1 | 2 | | | −1 | | | |
| 1729750 | 1 | −1 | 3 | −3 | 1 | | 1 | | | | | 1 | −2 |
| 1740000 | 5 | 1 | 4 | | | | | | | 1 | −1 | −2 | −1 |
| 1763125 | −2 | −1 | 4 | 1 | −1 | 1 | | −2 | | | 1 | −1 | |
| 1771561 | −3 | −2 | −1 | −1 | 6 | | | −1 | | | | −1 | |
| 1852201 | −3 | −3 | −2 | −3 | | 1 | 3 | | | 1 | | | |
| 1946721 | −5 | 1 | −1 | 2 | | | 1 | 1 | −3 | | | | 1 |
| 2023425 | −13 | 2 | 2 | | | −1 | 1 | −1 | 2 | | | | |
| 2085136 | 4 | −1 | −1 | | | −1 | −2 | 4 | | | | −1 | |
| 2185300 | 2 | −5 | 2 | | | 1 | −1 | | −2 | | | | 2 |
| 2267916 | 2 | 1 | −1 | 3 | | −1 | | 1 | −1 | 1 | | −1 | −1 |
| 2304324 | 2 | 2 | | −2 | 2 | | | | 2 | | −1 | −1 | −1 |
| 2307361 | −5 | −1 | −1 | 4 | −1 | | | −1 | −1 | | 2 | | |
| 2310400 | 8 | −2 | 2 | −2 | | −2 | | 2 | | | −1 | | |
| 2345057 | −5 | | | −1 | 1 | 1 | | −2 | 2 | −1 | 1 | | |
| 2351350 | 1 | −4 | 2 | −1 | −1 | −1 | | | | −1 | 1 | 1 | 1 |
| 2417876 | 2 | | −3 | | | | 1 | | −1 | −2 | 2 | 1 | |
| 2560845 | −2 | 1 | 1 | 1 | −3 | −1 | | | | 3 | | −1 | |
| 2598400 | 9 | −5 | 2 | 1 | | | −2 | | | 1 | | −1 | |
| 2697696 | 5 | 2 | −1 | −3 | −2 | −1 | 1 | 1 | | 1 | | | |
| 2772225 | −8 | 4 | 2 | −2 | | −1 | −1 | | | | | 2 | |
| 2825761 | −5 | −1 | −1 | −1 | | | | | | −2 | | | 4 |
| 2829124 | 2 | −2 | | | −1 | | −1 | | | 4 | | | −2 |

TABLE IB (*Continued*)

| $N$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2893401 | −3 | 10 | −2 | 2 | | | −1 | | −1 | | | −1 | |
| 3063808 | 14 | −2 | | | 1 | | 1 | −2 | −1 | | | | −1 |
| 3206269 | −2 | −2 | | | 1 | −2 | −1 | 1 | 2 | 1 | −1 | | |
| 3301376 | 13 | | −3 | −4 | −1 | 1 | | | | | 1 | | |
| 3331251 | −1 | 2 | −5 | 1 | 2 | −1 | | 1 | 1 | | | | −1 |
| 3346110 | 1 | 9 | 1 | | | −1 | 1 | −2 | −1 | | −1 | | |
| 3453840 | 4 | 4 | 1 | | | 1 | −3 | −1 | | | | −1 | 1 |
| 3556996 | 2 | −1 | −1 | | | −1 | −1 | | 2 | −1 | | −1 | 2 |
| 3897166 | 1 | −1 | −1 | 3 | | 1 | −2 | 1 | 1 | −1 | −1 | | |
| 3930400 | 5 | −2 | 2 | | −1 | | 3 | | | −1 | | −2 | |
| 4004001 | −5 | 2 | −3 | −1 | −1 | −1 | | | 2 | 2 | | | |
| 4090625 | −8 | | 5 | 1 | 1 | | 1 | −1 | | −2 | | | |
| 4096576 | 6 | −4 | −2 | −1 | 2 | | −2 | | 2 | | | | |
| 4588311 | −1 | 1 | −1 | 6 | | 1 | | −2 | | | −1 | | −1 |
| 4765600 | 5 | −2 | 2 | 1 | | | | −1 | 1 | −1 | −2 | 1 | |
| 5142501 | −2 | 3 | −4 | 2 | −2 | 2 | −1 | | 1 | | | | |
| 5267025 | −4 | 6 | 2 | −1 | | | 2 | | | | −1 | −1 | −1 |
| 5538975 | −1 | 1 | 2 | −1 | | 2 | −2 | 1 | 1 | | | −2 | |
| 5909761 | −8 | −5 | −1 | | 2 | 2 | 2 | −1 | | | | | |
| 6615675 | −1 | 7 | 2 | | 2 | −2 | | | −2 | | | −1 | |
| 6770556 | 2 | 2 | −1 | | −2 | 1 | 1 | −2 | 1 | | −1 | 1 | |
| 7105000 | 3 | −1 | 4 | 2 | −2 | | | | −2 | 1 | | −1 | |
| 7475000 | 3 | | 5 | −3 | | 1 | | −1 | 1 | | −1 | −1 | |
| 7491169 | −5 | −2 | | 2 | | | 2 | −1 | 2 | | | −2 | |
| 8268800 | 10 | | 2 | −2 | −1 | | 1 | 1 | −2 | −1 | | | |
| 9174816 | 5 | 3 | −1 | 1 | | −1 | −1 | −2 | −1 | | | 1 | 1 |
| 9222500 | 2 | | 4 | 1 | −3 | −2 | 1 | | | | 1 | | −1 |
| 9458086 | 1 | −1 | −1 | −1 | 4 | −3 | 1 | 1 | | | | | −1 |
| 10491040 | 5 | −9 | 1 | 1 | | −1 | 1 | 1 | | 1 | | | −1 |
| 10556001 | −5 | 4 | −3 | −1 | | −1 | | 4 | | −1 | | | |
| 11859211 | −1 | −4 | −1 | 1 | −4 | 1 | | 4 | | | | | |
| 13147876 | 2 | −2 | −3 | 4 | | −1 | | | | −1 | −1 | 2 | |
| 13745537 | −7 | | | −1 | | 1 | 1 | | −2 | −1 | | 1 | 2 |
| 14080573 | −2 | −2 | | | −1 | 4 | 1 | | | 1 | −2 | −1 | |
| 14235529 | −3 | −1 | | 6 | 2 | | −1 | | −1 | | | −1 | −1 |
| 14753025 | −8 | 2 | 2 | 1 | −1 | −2 | 1 | 1 | | 1 | −1 | | |
| 16093000 | 3 | −4 | 3 | 1 | 2 | −1 | −1 | 1 | | −1 | −1 | | |
| 18085705 | −3 | −1 | 1 | −3 | 1 | −3 | 1 | | 1 | 2 | | | |
| 19826576 | 4 | | −2 | 2 | 3 | | | 1 | −1 | −2 | | | −1 |
| 21386001 | −4 | 1 | −3 | 2 | | 1 | −2 | 2 | | | 1 | −1 | |
| 24601600 | 10 | −2 | 2 | | −2 | | | −1 | | −1 | 2 | | −1 |
| 25836889 | −3 | −1 | | −1 | −2 | 2 | 2 | | 2 | | −1 | | −1 |

TABLE IB (*Continued*)

| $N$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 25872148 | 2 | -2 | | -3 | | | -2 | 3 | 1 | -1 | | | 1 |
| 27005265 | -4 | 3 | 1 | 1 | -2 | -1 | 1 | | | -1 | | -1 | 2 |
| 27994681 | -3 | -3 | -1 | -2 | 2 | 2 | | | -2 | | | 2 | |
| 30138076 | 2 | -5 | -2 | | -2 | | 2 | | | 2 | 1 | | -1 |
| 30944914 | 1 | -1 | | 1 | 1 | 2 | -1 | | -2 | 1 | -1 | -1 | 1 |
| 32517265 | -4 | -1 | 1 | | 1 | -1 | | 1 | | 2 | -1 | 1 | -2 |
| 36315136 | 13 | -11 | -1 | | 1 | 1 | | | | | 1 | | -1 |
| 40750802 | 1 | | | -3 | | -2 | 1 | -1 | 1 | | 1 | -1 | 2 |
| 41808151 | -1 | -6 | -2 | 1 | 1 | | 1 | 1 | | | -1 | -1 | 2 |
| 43075585 | -11 | -3 | 1 | 1 | | | | -1 | | 1 | 1 | 2 | -1 |
| 50481025 | -7 | -1 | 2 | 4 | -1 | | -1 | -1 | | 2 | | -1 | |
| 71843751 | -1 | 2 | -6 | 3 | -2 | | 1 | -1 | | | | 2 | |
| 76271625 | -3 | 9 | 3 | | -3 | -1 | | -1 | | -1 | 1 | | |
| 80061345 | -5 | 3 | 1 | 4 | -2 | 1 | | 1 | -1 | -1 | -1 | | |
| 85459375 | -1 | -4 | 5 | -1 | -1 | -1 | -1 | | 1 | 1 | -1 | | 1 |
| 96059601 | -4 | 8 | -2 | -2 | 4 | -2 | | | | -1 | | | |
| 119094300 | 2 | 5 | 2 | | | 2 | -2 | -1 | -2 | 1 | | | -1 |
| 132663168 | 7 | 2 | | -1 | 1 | -1 | | 2 | | 1 | -2 | -1 | -1 |
| 133920000 | 8 | 3 | 4 | | | | -2 | -1 | | -3 | 1 | | |
| 177182721 | -11 | 6 | -1 | | -3 | -1 | 2 | | | 2 | | | |
| 181037025 | -5 | 4 | 2 | -1 | | 2 | | | 2 | -2 | -2 | | |
| 293635441 | -4 | -2 | -1 | -3 | 1 | | 1 | | | -1 | 1 | 3 | -1 |
| 308915776 | 6 | -4 | -2 | -1 | | 6 | | -1 | | | -1 | -1 | |
| 370256250 | 1 | 1 | 5 | 2 | -7 | 1 | | -1 | | | 1 | | |
| 415704576 | 9 | 1 | -2 | 1 | -3 | -1 | | | 1 | | -2 | | 2 |
| 876219201 | -6 | 4 | -2 | | 2 | 2 | | -2 | 2 | | | -1 | -1 |
| 1075774401 | -6 | 2 | -2 | | | 2 | | | -2 | 4 | -1 | | -1 |
| 1611308700 | 2 | 6 | 2 | -4 | -1 | -2 | | -2 | 1 | | 2 | | |
| 3463200000 | 8 | 2 | 5 | -5 | | 1 | -2 | | -1 | | -1 | 1 | |
| 45105689161 | -3 | -5 | -1 | -1 | | 2 | 2 | -1 | -1 | | 4 | -1 | -1 |
| 63927525376 | 13 | -3 | -3 | -7 | 4 | 1 | | | -1 | | | | 1 |

## TABLE IIA

Odd integers $N$ greater than 1 such that the largest prime factor of $N(N-2)$ is the $t^{\text{th}}$ prime, $t \leqq 11$

| $t = 2$ | $t = 3$ | $t = 4$ | $t = 5$ | $t = 6$ | $t = 7$ |
|---|---|---|---|---|---|
| 3 | 5 | 7 | 11 | 13 | 17 |
| | 27 | 9 | 35 | 15 | 51 |
| | | 245 | 77 | 65 | 119 |
| | | | | 275 | 121 |
| | | | | 847 | 189 |
| | | | | 1575 | 1377 |

| $t = 8$ | $t = 9$ | $t = 10$ | $t = 11$ | |
|---|---|---|---|---|
| 19 | 23 | 29 | 31 | 86275 |
| 21 | 25 | 87 | 33 | 130977 |
| 57 | 117 | 145 | 93 | 203205 |
| 135 | 209 | 147 | 95 | 2509047 |
| 171 | 255 | 377 | 155 | 3322055 |
| 247 | 299 | 437 | 343 | 287080367 |
| 325 | 345 | 495 | 405 | |
| 363 | 1127 | 667 | 527 | |
| 627 | 1311 | 2873 | 529 | |
| 665 | 2187 | 8381 | 715 | |
| 1617 | 2277 | 9947 | 899 | |
| 3213 | 2875 | 12675 | 1085 | |
| 3971 | 3705 | 14877 | 1521 | |
| | 6877 | 16445 | 1955 | |
| | 8075 | 24565 | 2697 | |
| | 9317 | 41327 | 3627 | |
| | 18515 | 45619 | 4125 | |
| | 41745 | 87725 | 5425 | |
| | 57477 | 184877 | 7163 | |
| | 1128127 | | 19437 | |
| | 1447875 | | 22477 | |

## TABLE IIB

Odd integers $N$ greater than 100,000 such that $N(N-2)$ has no prime factor greater than 31, with the factorization of $N/(N-2)$

| $N$ | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
|---|---|---|---|---|---|---|---|---|---|---|
| 130977 | 5 | −2 | 2 | 1 | −2 | | | | | −1 |
| 184877 | −1 | −3 | 5 | 1 | | −1 | | | −1 | |
| 203205 | 1 | 1 | −2 | −1 | −1 | | 1 | 1 | −1 | 1 |
| 1128127 | | −5 | 3 | 1 | 1 | | −2 | 1 | | |
| 1447875 | 4 | 3 | −1 | 1 | 1 | −1 | | −3 | | |
| 2509047 | 2 | −1 | −4 | −1 | | 1 | −1 | 2 | | 1 |
| 3322055 | −7 | 1 | −2 | 2 | | 2 | 1 | | | −1 |
| 287080367 | −1 | −1 | 5 | −2 | −1 | | 1 | −3 | 1 | 1 |

TABLE IIIA

Odd integers $N$ greater than 3 such that the largest prime factor of $N(N-4)$ is the $t^{\text{th}}$ prime number, $t \leqq 11$

| $t = 3$ | $t = 4$ | $t = 5$ | $t = 6$ | $t = 7$ |
|---|---|---|---|---|
| 5 | 7 | 11 | 13 | 17 |
| 9 | 25 | 15 | 39 | 21 |
| | 49 | 81 | 121 | 55 |
| | | 125 | 147 | 85 |
| | | | 169 | 225 |
| | | | 4459 | 429 |
| | | | | 459 |
| | | | | 14161 |
| | | | | 21879 |

| $t = 8$ | $t = 9$ | $t = 10$ | | $t = 11$ | |
|---|---|---|---|---|---|
| 19 | 23 | 29 | 10469 | 31 | 98553 |
| 95 | 27 | 33 | 21025 | 35 | 112999 |
| 99 | 69 | 91 | 294151 | 221 | 117649 |
| 175 | 119 | 207 | 442225 | 279 | 212629 |
| 247 | 165 | 319 | 8254129 | 345 | 344379 |
| 289 | 441 | 323 | | 403 | 10439037 |
| 361 | 529 | 609 | | 589 | |
| 935 | 625 | 667 | | 837 | |
| 2299 | 1449 | 729 | | 841 | |
| 3553 | 1729 | 845 | | 1089 | |
| 6175 | 1863 | 1131 | | 1705 | |
| 60025 | 2695 | 1309 | | 1771 | |
| 121125 | 7429 | 1425 | | 2639 | |
| | 12397 | 1885 | | 4437 | |
| | 13689 | 2527 | | 15345 | |
| | 54625 | 2875 | | 27625 | |
| | 110565 | 3861 | | 58125 | |

TABLE IIIB

Odd integers $N$ greater than 100,000 such that $N(N-4)$ has no prime factor greater than 31, with the factorization of $N/(N-4)$

| $N$ | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
|---|---|---|---|---|---|---|---|---|---|---|
| 110565 | 5 | 1 | 1 | −1 | 1 | | −1 | −2 | | |
| 112999 | −6 | −1 | | | | 3 | | 1 | | −1 |
| 117649 | −1 | −1 | 6 | −1 | | | | −1 | | −1 |
| 121125 | 1 | 3 | −1 | −3 | −1 | 1 | 1 | | | |
| 212629 | −5 | −3 | −1 | | | | 3 | | | 1 |
| 294151 | −2 | | −2 | 3 | 1 | 1 | | −1 | −1 | |
| 344379 | 1 | −4 | 1 | | | | −1 | 2 | −1 | 1 |
| 442225 | −1 | 2 | 2 | | −1 | −1 | 2 | −1 | −1 | |
| 8254129 | −2 | −3 | | −1 | 4 | 2 | | −1 | −1 | |
| 10439037 | 5 | | 1 | −4 | | 1 | 2 | −1 | | −1 |