ARITHMETIC PROBLEMS CONCERNING CAUCHY'S FUNCTIONAL EQUATION[1]

BY

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Introduction

The present study concerns some modifications of the functional equation

$$f(x + y) = f(x) + f(y)$$

which arose in connection with certain problems on additive arithmetic functions. An arithmetic function $F(n)$ $(n = 1, 2, \cdots)$ is said to be *additive* provided that $F(mn) = F(m) + F(n)$ whenever $(m, n) = 1$. In [2] Erdös found that if the additive function $F(n)$ is nondecreasing, i.e., $F(n) \leqq F(n + 1)$ for all n, then it must be of the form $F(n) = C \log n$. This result was rediscovered by Moser and Lambek [3], and recently further proofs were given by Schoenberg [4] and Besicovitch [1].

Erdös' remarkable characterization of the function $\log n$ raises the following question: Let $p_1, p_2, \cdots, p_k$ be a given set of k $(\geqq 2)$ distinct prime numbers. Let $F(n)$ be defined on the set A of integers n which allow no prime divisors except those among $p_1, \cdots, p_k$, and let $F(n)$ be additive, i.e.,

$$(1) \qquad F(p_1^{u_1} p_2^{u_2} \cdots p_k^{u_k}) = F(p_1^{u_1}) + F(p_2^{u_2}) + \cdots + F(p_k^{u_k}).$$

If we assume $F(n)$ to be nondecreasing over the set A, is it still true that $F(n) = C \log n$?

One of us having communicated this question to Erdös, received in reply a letter dated February 13, 1961, in which Erdös states, with brief indications of proofs, that the answer to the above question is affirmative if $k = 3$ and negative if $k = 2$. We shall deal with these results below under more general assumptions. The negative answer for $k = 2$ is already established by any counterexample, a particularly simple one being

$$(2) \qquad F(n) = [\log n / \log p_1] + [\log n / \log p_2].$$

Indeed, it is easy to verify that this particular monotone $F(n)$ satisfies (1), for $k = 2$, while it is not of the form $C \cdot \log n$, for $n = p_1^{u_1} p_2^{u_2}$ (see also Section 12).

At this point we change notations. If we write $F(e^x) = f(x)$, $\log p_i = \alpha_i$, the relation (1) becomes

$$(3) \quad f(u_1 \alpha_1 + \cdots + u_k \alpha_k) = f(u_1 \alpha_1) + \cdots + f(u_k \alpha_k) \quad (u_i \text{ integers} \geqq 0).$$

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We shall study the monotone solutions of this functional equation under various assumptions concerning the number k and the components α_i.

The paper is divided into three parts. In Part I we assume that the α_i are natural integers, the discussion belonging to the elements of number theory. In Part II we assume that $k = 3$ and that the ratios α_1/α_2, α_2/α_3, α_1/α_3 are irrational, the result being that the monotone solutions of (3) are linear (Theorem 2.1). In Part III we study the case when $k = 2$ assuming that α_1/α_2 is irrational. Nonlinear monotone solutions of (3) always exist in this case, and they will all be constructed (Theorem 3.1). It will be shown that every monotone solution $f(x)$ admits monotone extensions satisfying the unrestricted functional equation

$$(4) \quad f(n_1 \alpha_1 + n_2 \alpha_2) = f(n_1 \alpha_1) + f(n_2 \alpha_2) \qquad (n_1, n_2 \text{ arbitrary integers})$$

(Theorem 3.2). We actually determine all monotone extensions of $f(x)$ which are solutions of (4) (Theorem 3.3). In particular we obtain the conditions for the unicity of this extension (Corollary 3.1). In the course of this discussion we describe all monotone solutions of the equation (4) (Section 11).

Parts II and III contain the main results and may be read independently of Part I.

I. The Rational Case

1. The main result and some auxiliary propositions

In the present rational case all lower case Latin letters, except the functional symbol f, belong to the ring Z of rational integers. Let $a_1, a_2, \cdots, a_k$ ($k \geqq 2$) be given positive integers such that $(a_1, \cdots, a_k) = 1$. We wish to determine all solutions of the functional equation

$$(1.1) \quad f(u_1 a_1 + \cdots + u_k a_k) = f(u_1 a_1) + \cdots + f(u_k a_k) \quad (u_i \geqq 0),$$

where $f(x)$ is defined on the set

$$S = \{\textstyle\sum u_i a_i \mid u_1 \geqq 0, \cdots, u_k \geqq 0\}.$$

Clearly S contains all sufficiently large positive integers.

These solutions are described by the following theorem.

THEOREM 1.1. *Let* $d_i = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k)$ $(i = 1, \cdots, k)$. *To every solution* $f(x)$ *of* (1.1) *there correspond uniquely a real number* λ *and* k *periodic functions* $\varphi_i(x)$ $(x \in Z)$ *of period* d_i, *respectively, with* $\varphi_i(0) = 0$ $(i = 1, \cdots, k)$ *such that*

$$(1.2) \qquad f(x) = \lambda x + \varphi_1(x) + \cdots + \varphi_k(x)$$

if $x \in S$. *Moreover,* (1.2) *extends the solution* $f(x)$ *to all integers* x *such that the extended* $f(x)$ *satisfies* (1.1) *for all integral* u_i.

Conversely, every function $f(x)$ *defined by* (1.2), *where* λ *is a constant and the*

$\varphi_i(x)$ have the properties described above, is a solution of (1.1) *for all integral values of the u_i.*

Until further notice $f(x)$ will denote a solution of (1.1). We need a number of auxilliary propositions the first of which is

1°. *If*

$$f(P) = 0, \quad \text{where} \quad P = a_1 a_2 \cdots a_k, \tag{1.3}$$

then

$$f(ma_i a_j) = 0 \quad \text{if} \quad i \neq j, \quad m \geqq 0. \tag{1.4}$$

Indeed, let us show that $f(ma_1 a_2) = 0$. In particular (1.1) implies

$$f(u_1 a_1 + u_2 a_2) = f(u_1 a_1) + f(u_2 a_2) \qquad (u_1 \geqq 0, u_2 \geqq 0). \tag{1.5}$$

If we set $\mu = f(a_1 a_2)$, then by (1.5)

$$f(2a_1 a_2) = f(a_2 a_1 + a_1 a_2) = f(a_2 a_1) + f(a_1 a_2) = 2\mu,$$

$$f(3a_1 a_2) = f(2a_2 a_1 + a_1 a_2) = f(2a_2 a_1) + f(a_1 a_2) = 3\mu,$$

and by induction we obtain $f(ma_1 a_2) = m\mu$. If we choose $m = a_3 \cdots a_k$, then $m\mu = f(P) = 0$ by (1.3). Thus $\mu = 0$, and (1.4) is established.

Returning to the general situation we observe that

$$\omega(x) = f(x) - xf(P)/P$$

is also a solution of (1.1). Evidently $\omega(P) = 0$ so that by 1°

$$\omega(ma_i a_j) = 0 \quad \text{if} \quad i \neq j, \quad m \geqq 0. \tag{1.6}$$

Writing $\lambda = f(P)/P$ we find

$$f(x) = \lambda x + \omega(x). \tag{1.7}$$

The "reduced" solutions $\omega(x)$ enjoy the following property:

2°. *If for a given i*

$$\textstyle\sum_1^k u_j a_j \equiv \sum_1^k u_j' a_j \pmod{d_i} \qquad (u_j \geqq 0,\ u_j' \geqq 0), \tag{1.8}$$

then

$$\omega(u_i a_i) = \omega(u_i' a_i). \tag{1.9}$$

It suffices to prove (1.9) for $i = 1$. Now (1.8) implies $u_1 a_1 \equiv u_1' a_1 \pmod{d_1}$, and since $(a_1, d_1) = 1$, we conclude that $u_1 \equiv u_1' \pmod{d_1}$. Since $d_1 = (a_2, \cdots, a_k)$, we can write $u_1' - u_1 = \sum_2^k w_j a_j$ and also $w_j = v_j - v_j'$, where $v_j \geqq 0$, $v_j' \geqq 0$. We thus obtain the relation

$$u_1 + \textstyle\sum_2^k v_j a_j = u_1' + \sum_2^k v_j' a_j.$$

On multiplying both sides by a_1 we may write

$$\omega\left(u_1 a_1 + \textstyle\sum_2^k v_j a_1 a_j\right) = \omega\left(u_1' a_1 + \sum_2^k v_j' a_1 a_j\right),$$

whence from the functional equation and (1.6) we obtain

$$\omega(u_1 a_1) + \sum_2^k \omega(v_j a_1 a_j) = \omega(u_1' a_1) + \sum_2^k \omega(v_j' a_1 a_j),$$

$$\omega(u_1 a_1) = \omega(u_1' a_1),$$

which proves our statement.

2. Proof of Theorem 1.1

For every $x = \sum a_i u_i \in S$ we define a function $\varphi_1(x)$ by

$$\varphi_1(x) = \omega(a_1 u_1). \tag{1.10}$$

Lemma 2° shows that $\varphi_1(x)$ is well-defined in S. Indeed, if $x \in S$ can be written as $x = \sum a_i u_i$ and also as $x = \sum a_i u_i'$, then (1.8) holds, which implies (1.9) for $i = 1$, that is, $\omega(a_1 u_1) = \omega(a_1 u_1')$.

Moreover, if x and x', both in S, are such that $x \equiv x' \pmod{d_1}$, then by Lemma 2°, $\varphi_1(x) = \varphi_1(x')$. We finally observe that every residue class mod d_1, contains elements in S, in fact already elements of the subset $a_1 u_1$ $(u_1 \geqq 0)$, because $(a_1, d_1) = 1$. We may therefore extend the definition of $\varphi_1(x)$ to *all integers* obtaining a function having the period d_1.

Similarly we define a function $\varphi_i(x)$ $(i = 1, \cdots, k)$ over all integers, having the period d_i, with the property that

$$\varphi_i(x) = \omega(a_i u_i) \quad \text{if} \quad x = \sum a_j u_j \in S, \tag{1.11}$$

and in particular

$$\varphi_i(0) = 0. \tag{1.12}$$

If $x = \sum a_i u_i \in S$, then the functional equation gives $\omega(x) = \omega(\sum a_i u_i) = \sum \omega(a_i u_i) = \sum \varphi_i(x)$; hence

$$\omega(x) = \varphi_1(x) + \cdots + \varphi_k(x). \tag{1.13}$$

We now use this relation (1.13) to extend the definition of the reduced solution $\omega(x)$ to all integers x. Let us show that this implies

$$\omega(a_i u_i) = \varphi_i(a_i u_i) \qquad (u_i \in Z)\cdot \tag{1.14}$$

This is indeed clear from (1.11) if $u_i \geqq 0$, while if $u_i < 0$, then from the defining relation (1.13), $\omega(a_i u_i) = \sum_j \varphi_j(a_i u_i) = \varphi_i(a_i u_i)$. It follows that the extended $\omega(x)$ satisfies the unrestricted functional equation

$$\omega(\sum a_i u_i) = \sum \omega(a_i u_i) \qquad (u_i \in Z),$$

for by (1.13) and (1.14)

$$\omega(\sum a_j u_j) = \sum_i \varphi_i(\sum_j a_j u_j) = \sum \varphi_i(a_i u_i) = \sum \omega(a_i u_i).$$

We may now complete a proof of Theorem 1.1 in a few lines: The direct part is established by (1.7) and the representation (1.13). The converse follows from the fact that each $\varphi_i(x)$ is a solution of the unrestricted functional equation: $\varphi_i(\sum a_j u_j) = \varphi_i(a_i u_i) = \sum_j \varphi_i(a_j u_j)$.

In view of (1.12) we see that the periodic function $\varphi_i(x)$ depends on $d_i - 1$ arbitrary parameters, and therefore the most general solution (1.2) depends on $1 + \sum(d_i - 1)$ arbitrary parameters. In particular (1.2) will reduce to the linear solution λx if and only if all $d_i = 1$. We state this as

COROLLARY 1. *The functional equation* (1.1) *has only linear solutions* $f(x) = \lambda x$ *if and only if every* $k - 1$ *among the numbers* $a_1, \cdots, a_k$ *are relatively prime.*

3. Monotone solutions

Because the monotonicity of the solutions $f(x)$ will play an important role in Parts II and III, we wish to describe also here those solutions (1.2) of (1.1) which are nondecreasing functions of x. The least slope among the slopes of the sides of the polygonal graph of the function $\varphi_i(x)$ $(x \in Z)$ is given by

$$(1.15) \qquad -\mu_i = \min_{x \in Z} \Delta\varphi_i(x) \qquad (\Delta\varphi_i(x) = \varphi_i(x+1) - \varphi_i(x)).$$

Evidently $\mu_i \geqq 0$. If in (1.2) we choose λ so that $\lambda \geqq \sum \mu_i$, then $f(x)$ is certainly nondecreasing since for all x

$$\Delta f(x) = \lambda + \sum \Delta\varphi_i(x) \geqq \lambda - \sum \mu_i \geqq 0.$$

The converse is also true as stated in

THEOREM 1.2. *The solution of* (1.1) *defined by* (1.2) *is nondecreasing if and only if the quantities defined by* (1.15) *satisfy the inequality*

$$(1.16) \qquad \textstyle\sum_1^k \mu_i \leqq \lambda.$$

Indeed, let the minimum (1.15) be reached for $x \equiv c_i \pmod{d_i}$ so that $\Delta\varphi_i(c_i) = -\mu_i$ $(i = 1, \cdots, k)$. Since the moduli $d_1, \cdots, d_k$ are pairwise relatively prime, we may by the Chinese remainder theorem find an integer ξ such that $\xi \equiv c_i \pmod{d_i}$ for all i. But then $\Delta\varphi_i(\xi) = -\mu_i$ for all i. Now the monotonicity of $f(x)$ gives

$$0 \leqq \Delta f(\xi) = \lambda + \sum \Delta\varphi_i(\xi) = \lambda - \sum \mu_i,$$

which proves the theorem.

II. THE THREE-DIMENSIONAL MODULE

4. Statement of the problem and a few lemmas

We start with the following lemma.

LEMMA 2.1. *Let* θ *be given,* $0 < \theta < 1$, *and let us consider the set of reals*

$$S^* = \{x = u + v\theta \mid u \geqq 0, v \geqq 0, u, v \text{ integers}\}.$$

Let $F(x)$ *be real-valued and defined on a set* E *of reals,* $E \supset S^*$, *subject to the conditions*

$$(2.1) \qquad F(u + v\theta) = F(u) + F(v\theta) \quad \text{if} \quad u \geqq 0, \quad v \geqq 0,$$

(2.2) $F(x)$ *is nondecreasing in* E.

Then the limit

(2.3) $$\lambda = \lim_{x\to\infty, x\in E} F(x)/x$$

exists, $\lambda \geqq 0$.

Proof. Since $0 < \theta < 1$, we may associate with every integer $v \geqq 0$, an integer w such that $v < w\theta < v + 1$. From (2.1) and (2.2) we obtain

$$F(u + v) \leqq F(u + w\theta) = F(u) + F(w\theta) \leqq F(u) + F(v + 1)$$

and

$$F(u) + F(v) \leqq F(u) + F(w\theta) = F(u + w\theta) \leqq F(u + v + 1).$$

Combining these we obtain

$$F(u) + F(v - 1) \leqq F(u + v) \leqq F(u) + F(v + 1) \qquad (u \geqq 0, v \geqq 1).$$

In this relation we replace u by $u + (j - 1)v$ for $j = 1, \cdots, m$, obtaining $F(u + (j - 1)v) + F(v - 1) \leqq F(u + jv) \leqq F(u + (j - 1)v) + F(v + 1)$. Summing these for $j = 1, \cdots, m$ we obtain after cancellations

$$F(u) + mF(v - 1) \leqq F(u + mv) \leqq F(u) + mF(v + 1),$$

whence

(2.4) $$\frac{F(u) + mF(v - 1)}{u + mv} \leqq \frac{F(u + mv)}{u + mv} \leqq \frac{F(u) + mF(v + 1)}{u + mv}.$$

We keep v fixed and divide an arbitrary positive integer n by v obtaining $n = u + mv$. If we now let $n \to \infty$, also the quotient $m \to \infty$, while the remainder u remains bounded, being $< v$. Under these circumstances the relations (2.4) show that $F(n)/n$ remains bounded as $n \to \infty$, and

$$\lambda' = \liminf F(n)/n, \qquad \lambda'' = \limsup F(n)/n$$

are both finite. Given an arbitrary $\varepsilon > 0$ we now select v so that

$$F(v + 1)/v = (F(v + 1)/(v + 1))\cdot((v + 1)/v) < \lambda' + \varepsilon.$$

Letting in (2.4) $n \to \infty$ through a sequence of values n_k such that $\lim F(n_k)/n_k = \lambda''$, we obtain in the limit that $\lambda'' < \lambda' + \varepsilon$. Therefore $\lambda'' \leqq \lambda'$. Hence $\lambda' = \lambda''$, and the existence of $\lambda = \lim_{n\to\infty} F(n)/n$ is established. If now $x \in E$, $x \geqq 1$, let $n \leqq x < n + 1$, so that (2.2) implies $F(n)/(n + 1) \leqq F(x)/x \leqq F(n + 1)/n$. Letting $x \to \infty$ we obtain (2.3), and the proof of the lemma is complete.

We turn now to a study of our functional equation. Given three real numbers $\alpha > 0$, $\beta > 0$, $\gamma \geqq 0$, $(\alpha \neq \beta)$, we consider the set of reals

(2.5) $$S = \{x = u\alpha + v\beta + w\gamma \mid u, v, w \text{ integers} \geqq 0\}.$$

Let $f(x)$ be real-valued, with domain S, satisfying the following conditions:

$$f(u\alpha + v\beta + w\gamma) = f(u\alpha) + f(v\beta) + f(w\gamma) \quad (u, v, w \geqq 0), \tag{2.6}$$

$$f(x) \text{ is nondecreasing in } S. \tag{2.7}$$

Our objective is to show that $f(x)$ must be a linear function under appropriate additional assumptions on α, β, and γ.

If $\alpha > \beta$, say, it is clear that Lemma 2.1 becomes applicable. Introducing the function $F(y) = f(\alpha y)$ and setting $\theta = \beta/\alpha$ we conclude from (2.6) and (2.7) that $F(y)$ satisfies (2.1), and that it is nondecreasing on the set $E = \{y = u + v\theta + w(\gamma/\alpha) \mid u, v, w \geqq 0\}$. We therefore obtain the following corollary.

COROLLARY 2.1. *The assumptions* (2.6) *and* (2.7) *imply the existence of*

$$\lim_{x\to\infty, x\in S} f(x)/x = \lambda \qquad (\lambda \geqq 0). \tag{2.8}$$

We now define a new function $\omega(x)$ by

$$f(x) = \lambda x + \omega(x) \qquad (x \in S) \tag{2.9}$$

and observe that it enjoys the following properties:

$$\omega(u\alpha + v\beta + w\gamma) = \omega(u\alpha) + \omega(v\beta) + \omega(w\gamma) \quad (u, v, w \geqq 0), \tag{2.10}$$

$$(\omega(y) - \omega(x))/(y - x) \geqq -\lambda \quad \text{if} \quad x \in S,\ y \in S,\ x \neq y, \tag{2.11}$$

$$\lim_{x\to\infty, x\in S} \omega(x)/x = 0. \tag{2.12}$$

Indeed, (2.10), (2.11), and (2.12) are respectively equivalent to (2.6), (2.7), and (2.8) by virtue of the defining relation (2.9).

An essential step in the study of $\omega(x)$ is contained in

LEMMA 2.2. *Let u and u' be given integers, $u \geqq 0$, $u' \geqq 0$. For any pair of integers h and k (greater than, equal to, or less than zero), the inequality*

$$\frac{\omega(u\alpha) - \omega(u'\alpha)}{(u - u')\alpha + h\beta + k\gamma} \geqq -\lambda \tag{2.13}$$

holds, provided that the denominator of the fraction does not vanish.

Proof. For every integer $j \geqq 1$ let

$$\begin{aligned} v &= jh, & v' &= (j-1)h & &\text{if} \quad h \geqq 0, \\ v &= (j-1)|h|, & v' &= j|h| & &\text{if} \quad h < 0, \\ w &= jk, & w' &= (j-1)k & &\text{if} \quad k \geqq 0, \\ w &= (j-1)|k|, & w' &= j|k| & &\text{if} \quad k < 0. \end{aligned}$$

We also define

$$\begin{aligned} \eta &= +1 \ \text{ if } \ h \geqq 0, & \eta' &= +1 \ \text{ if } \ k \geqq 0, \\ &= -1 \ \text{ if } \ h < 0, & &= -1 \ \text{ if } \ k < 0. \end{aligned}$$

Notice that in any case v, v', w, w' are all nonnegative, and that $v - v' = h$, $w - w' = k$. Applying the inequality (2.11) with

$$y = u\alpha + v\beta + w\gamma \quad \text{and} \quad x = u'\alpha + v'\beta + w'\gamma$$

and using (2.10), we obtain

$$\frac{\omega(u\alpha) - \omega(u'\alpha) + \omega(v\beta) - \omega(v'\beta) + \omega(w\gamma) - \omega(w'\gamma)}{(u - u')\alpha + h\beta + k\gamma} \geqq -\lambda.$$

Adding together these inequalities for $j = 1, \cdots, n$ we obtain

$$\frac{n(\omega(u\alpha) - \omega(u'\alpha)) + \eta\omega(n \mid h \mid \beta) + \eta'\omega(n \mid k \mid \gamma)}{(u - u')\alpha + h\beta + k\gamma} \geqq -n\lambda.$$

Dividing both sides by n and letting $n \to \infty$ we see that this relation turns into (2.13), in view of the limit relation (2.12), and this terminates our proof.

5. The main result

The main result of this section is

THEOREM 2.1. *Let $f(x)$ have the properties* (2.5), (2.6), *and* (2.7). *If the numbers α, β, γ are positive and all three ratios α/β, α/γ, β/γ are irrational, then $f(x) = \lambda x$ on S.*

Proof. We apply Lemma 2.2 for a fixed $u \geqq 0$ and $u' = 0$ obtaining

$$\omega(u\alpha)/(u\alpha + h\beta + k\gamma) \geqq -\lambda \tag{2.14}$$

for any integers h and k, provided that the denominator does not vanish. The ratio β/γ being irrational, the set of values of $h\beta + k\gamma$ is everywhere dense. This means that if we choose $\varepsilon > 0$ we can find integers h and k such that $0 < u\alpha + h\beta + k\gamma < \varepsilon$, and now (2.14) shows that $\omega(u\alpha) \geqq -\lambda\varepsilon$. But we can likewise find integers h and k such that $-\varepsilon < u\alpha + h\beta + k\gamma < 0$, in which case (2.14) implies $\omega(u\alpha) \leqq \lambda\varepsilon$. Hence $\omega(u\alpha) = 0$ for every $u \geqq 0$. By the symmetry of our assumptions on α, β, and γ we likewise obtain $\omega(v\beta) = 0$ ($v \geqq 0$) and $\omega(w\gamma) = 0$ ($w \geqq 0$). Finally (2.10) shows that $x \epsilon S$ implies $\omega(x) = 0$. Now (2.9) shows that indeed $f(x) = \lambda x$ if $x \epsilon S$.

The assumptions of Theorem 2.1 are surely satisfied if α, β, γ are positive and rationally independent. Therefore also the result of our Introduction concerning monotone additive functions ($k = 3$) is settled.

III. THE TWO-DIMENSIONAL MODULE

6. Decomposition of a reduced solution into its periodic components $\varphi(x)$ and $\psi(x)$

Throughout this last part of the paper we assume that α and β are positive, that the ratio α/β is irrational, and finally that the third component $\gamma = 0$. The set (2.5) now reduces to

(3.1) $$S = \{x = u\alpha + v\beta \mid u, v \text{ nonnegative integers}\}.$$

We study functions $f(x)$ defined in S subject to the following conditions:

(3.2) $$f(u\alpha + v\beta) = f(u\alpha) + f(v\beta) \qquad (u \geqq 0, v \geqq 0),$$

(3.3) $$f(x) \quad \text{is nondecreasing in } S.$$

All results derived in Part II which allow the assumption $\gamma = 0$ become applicable, in particular Corollary 2.1, the relations (2.9) to (2.12), and Lemma 2.2. For convenience we repeat a few of these here:

(3.4) $$f(x) = \lambda x + \omega(x) \qquad (x \in S),$$

(3.5) $$\omega(u\alpha + v\beta) = \omega(u\alpha) + \omega(v\beta) \qquad (u \geqq 0, v \geqq 0),$$

(3.6) $$(\omega(y) - \omega(x))/(y - x) \geqq -\lambda \quad \text{if} \quad x \in S, \quad y \in S, \quad x \neq y.$$

We now define a new function $\varphi(x)$ by the following conditions:

(3.7) $\varphi(v\beta) = \omega(v\beta) \quad (v \geqq 0)$,

(3.8) $\varphi(x)$ has the period α, i.e., if $\varphi(x_0)$ is defined, then by definition $\varphi(x_0 + m\alpha) = \varphi(x_0)$ for every integer m.

We see that $\varphi(x)$ is defined on the set

(3.9) $$S_\alpha = \{x = m\alpha + v\beta \mid v \geqq 0, m \text{ an arbitrary integer}\}.$$

We define likewise a second function $\psi(x)$ by

(3.7′) $\psi(u\alpha) = \omega(u\alpha) \quad (u \geqq 0)$,

(3.8′) $\psi(x)$ has the period β,

so that the domain of definition of $\psi(x)$ is the set

(3.9′) $$S_\beta = \{x = u\alpha + n\beta \mid u \geqq 0, n \text{ an arbitrary integer}\}.$$

From (3.5), (3.7), (3.7′) we conclude that

(3.10) $$\omega(x) = \varphi(x) + \psi(x) \qquad (x \in S).$$

7. Properties of the functions $\varphi(x)$, $\psi(x)$

Let $t = u\alpha + n\beta$ and $s = u'\alpha + n'\beta$ be any two distinct elements of S_β. We now apply Lemma 2.2 for the pair of nonnegative integers u, u'. Since $\omega(u\alpha) = \psi(u\alpha)$, $\omega(u'\alpha) = \psi(u'\alpha)$, we obtain the inequality

$$\frac{\psi(u\alpha) - \psi(u'\alpha)}{(u - u')\alpha + h\beta} \geqq -\lambda.$$

Since h is here an arbitrary integer (restricted only by $h \neq 0$ if $u = u'$), we may set here $h = n - n'$. Since $\psi(u\alpha) = \psi(u\alpha + n\beta) = \psi(t)$ and $\psi(u'\alpha) = \psi(u'\alpha + n'\beta) = \psi(s)$, our last inequality turns into

(3.11) $(\psi(t) - \psi(s))/(t - s) \geqq -\lambda$ if $t \in S_\beta$, $s \in S_\beta$, $t \neq s$.

Similarly we can show that

(3.12) $(\varphi(y) - \varphi(x))/(y - x) \geqq -\lambda$ if $x \in S_\alpha$, $y \in S_\alpha$, $x \neq y$.

Observe that (3.12) is equivalent to the statement that $\lambda x + \varphi(x)$ is nondecreasing in S_α, and similarly (3.11) means that $\lambda x + \psi(x)$ is nondecreasing in S_β.

However, we may derive more precise information concerning the slopes of these functions in a sense already suggested by Theorem 1.2 of Part I. For this purpose we introduce the quantities

$$(3.13) \qquad -\mu = \inf_{\substack{x,y \in S_\alpha \\ x \neq y}} \frac{\varphi(y) - \varphi(x)}{y - x}, \qquad -\nu = \inf_{\substack{t,s \in S_\beta \\ t \neq s}} \frac{\psi(t) - \psi(s)}{t - s},$$

which we know to be finite by (3.11) and (3.12).

At this point let us interrupt the logical sequence of our arguments for a moment and start anew by assuming that we have a function $\varphi(x)$, defined in S_α, of period α, with $\varphi(0) = 0$, and likewise a function $\psi(x)$, defined in S_β, of period β, with $\psi(0) = 0$. Let us moreover assume that (3.13) hold, where μ and ν are finite, necessarily nonnegative. We now define

$$\omega(x) = \varphi(x) + \psi(x) \qquad (x \in S)$$

and claim that it satisfies (3.5). But this is evident, because

$$\begin{aligned} \omega(x) = \omega(u\alpha + v\beta) &= \varphi(u\alpha + v\beta) + \psi(u\alpha + v\beta) = \varphi(v\beta) + \psi(u\alpha) \\ &= \varphi(u\alpha) + \varphi(v\beta) + \psi(u\alpha) + \psi(v\beta) \\ &= \omega(u\alpha) + \omega(v\beta). \end{aligned}$$

From (3.13) we also see that nowhere in S will the slope of $\omega(x)$ fall below $-\mu - \nu$. If we select λ so that

$$(3.14) \qquad \lambda \geqq \mu + \nu,$$

then $f(x) = \lambda x + \omega(x)$ will be a function enjoying the properties (3.2) and (3.3).

8. Proof of the relation $\lambda \geqq \mu + \nu$

We return to our previous train of thought, in particular to the relations (3.13), and wish to show that the nonnegative μ and ν there defined, do satisfy the relation (3.14), where $\lambda = \lim f(x)/x$ $(x \to \infty, x \in S)$.

We need the following lemma.

LEMMA 3.1. *Let $g(x)$ be defined in an everywhere dense set E, and let ξ' and η' be points of E, and ρ a number such that*

$$(3.15) \qquad (g(\eta') - g(\xi'))/(\eta' - \xi') \leqq \rho \qquad (\xi' < \eta').$$

If δ *is such that* $0 < \delta \leqq (\eta' - \xi')/5$, *then we can find* $\xi, \eta \in E$ *such that* $\delta < \eta - \xi < 5\delta$ *while*

$$(3.16) \qquad (g(\eta) - g(\xi))/(\eta - \xi) \leqq \rho.$$

This lemma seems fairly obvious if we think of dividing the interval (ξ, η) into partial intervals of lengths close to 3δ and slightly displacing the points of division to make them belong to E. A conclusive argument runs as follows: There exists an odd integer $2r + 1$ such that $3(\eta' - \xi')/5\delta \leqq 2r + 1 < (\eta' - \xi')/\delta$; indeed $(\eta' - \xi')/\delta - 3(\eta' - \xi')/5\delta = 2(\eta' - \xi')/5\delta \geqq 2$. We now divide (ξ', η') into $2r + 1$ subintervals of equal lengths, and we select in the $(2j)^{\text{th}}$ open subinterval a point $\xi_j \in E$ $(j = 1, \cdots, r)$. Let moreover $\xi_0 = \xi'$, $\xi_{r+1} = \eta'$. Evidently, by construction

$$(\eta' - \xi')/(2r + 1) < \xi_{j+1} - \xi_j < 3(\eta' - \xi')/(2r + 1) \quad (j = 0, 1, \cdots, r).$$

From (3.15) we conclude that if we set $\xi = \xi_j$, $\eta = \xi_{j+1}$, for some appropriate j, then (3.16) will hold. On the other hand

$$\delta < (\eta' - \xi')/(2r + 1) < \eta - \xi < 3(\eta' - \xi')/(2r + 1) \leqq 5\delta,$$

which completes the proof.

We now return to the relations (3.13). Given $\varepsilon > 0$ we can certainly find numbers x, y, s, t, such that

$$(3.17) \quad (\varphi(y) - \varphi(x))/(y - x) < -\mu + \varepsilon, \qquad x \in S_\alpha, \quad y \in S_\alpha, \quad x < y,$$

$$(3.18) \quad (\psi(t) - \psi(s))/(t - s) < -\nu + \varepsilon, \qquad s \in S_\beta, \quad t \in S_\beta, \quad s < t.$$

However, Lemma 3.1 will allow us to choose intervals (x, y) and (s, t) which are nearly comparable in size. More precisely we have

LEMMA 3.2. *We may choose intervals* (x, y), (s, t) *satisfying* (3.17) *and* (3.18) *such that*

$$(3.19) \qquad y - x < t - s < 5(y - x).$$

Proof. Let (x', y') and (s', t') be two intervals satisfying the conditions (3.17) and (3.18), respectively, and let us construct the intervals (x, y) and (s, t) required by our lemma. We consider several cases:

We first assume that $y' - x' \geqq t' - s'$. We set $s = s'$, $t = t'$ and will determine (x, y) by applying Lemma 3.1 to $g = \varphi$ with $\delta = (t' - s')/5$ which is $\leqq (y' - x')/5$, as required by Lemma 3.1. There exist, therefore, acceptable $x, y \in S_\alpha$, $x < y$, such that $(t' - s')/5 < y - x < t' - s'$, i.e., $(t - s)/5 < y - x < t - s$ which is seen to agree with (3.19).

Secondly, if $(t' - s')/5 < y' - x' < t' - s'$, we set $x = x'$, $y = y'$, $s = s'$, $t = t'$ and already have what we want.

Thirdly, if $y' - x' \leqq (t' - s')/5$, we apply Lemma 3.1 to $g = \psi$ with $\delta = y' - x'$ setting $x = x'$, $y = y'$. We obtain acceptable $s, t \in S_\beta$ such that $y - x < t - s < 5(y - x)$, and our lemma is established.

We may now establish the inequality (3.14). We start with the intervals (x, y) and (s, t) satisfying the conditions (3.17), (3.18), and (3.19). The first inequality (3.19) may also be written as $y - t < x - s$. Since α/β is irrational, we can find positive integers u, v such that $y - t < v\beta - u\alpha < x - s$, which imply $s + v\beta < x + u\alpha$ and $y + u\alpha < t + v\beta$. However $x < y$, and therefore also $x + u\alpha < y + u\alpha$. The positive integers u, v may also be selected to be as we wish, and we select them so large that all four numbers

$$s + v\beta < x + u\alpha < y + u\alpha < t + v\beta \tag{3.20}$$

are elements of S. The function $\nu x + \psi(x) = \chi(x)$ being nondecreasing in S_β, hence also in S, is defined for the values (3.20) which furnish the inequality

$$\chi(y + u\alpha) - \chi(x + u\alpha) \leqq \chi(t + v\beta) - \chi(s + v\beta).$$

From this and (3.18) we obtain

$$\begin{aligned} &\nu(y - x) + \psi(y + u\alpha) - \psi(x + u\alpha) \\ &\qquad \leqq \nu(t - s) + \psi(t) - \psi(s) < \varepsilon(t - s). \end{aligned} \tag{3.21}$$

On the other hand, $\varphi(x + u\alpha) = \varphi(x)$, $\varphi(y + u\alpha) = \varphi(y)$, and therefore by (3.17)

$$\mu(y - x) + \varphi(y + u\alpha) - \varphi(x + u\alpha) < \varepsilon(y - x). \tag{3.22}$$

Adding together the extreme terms of (3.21) and (3.22) we obtain by (3.19)

$$\begin{aligned} &(\mu + \nu)(y - x) + \omega(y + u\alpha) - \omega(x + u\alpha) \\ &\qquad < \varepsilon(y - x + t - s) < 6\varepsilon(y - x). \end{aligned}$$

However, by (3.6), $\omega(y + u\alpha) - \omega(x + u\alpha) \geqq -\lambda(y - x)$, and substituting this into our last result we obtain

$$(\mu + \nu - \lambda)(y - x) < 6\varepsilon(y - x) \quad \text{or} \quad \mu + \nu - \lambda < 6\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired inequality $\mu + \nu \leqq \lambda$.

9. The main results

We summarize the situation in

THEOREM 3.1. *Let α, β be positive, α/β irrational. Let $f(x)$ be defined in the set*

$$S = \{x = u\alpha + v\beta \mid u, v \text{ nonnegative integers}\} \tag{3.1}$$

so as to satisfy the functional equation

$$f(u\alpha + v\beta) = f(u\alpha) + f(v\beta) \qquad (u \geqq 0, v \geqq 0) \tag{3.2}$$

and to be nondecreasing in S.

Then

$$\lambda = \lim f(x)/x \quad (x \to \infty,\ x \in S) \qquad (\lambda \geqq 0) \tag{2.8}$$

exists, and $f(x)$ *may be uniquely represented in the form*

$$f(x) = \lambda x + \varphi(x) + \psi(x) \qquad (x \in S), \tag{3.23}$$

where the functions $\varphi(x)$ *and* $\psi(x)$ *have the following properties:* $\varphi(x)$ *is defined in the set*

$$S_\alpha = \{x = m\alpha + v\beta \mid v \geqq 0,\ m \text{ an arbitrary integer}\} \tag{3.9}$$

where it has the period α, $\varphi(0) = 0$. *Moreover*

$$\inf_{x,y \in S_\alpha} \frac{\varphi(y) - \varphi(x)}{y - x} = -\mu \qquad (\mu \geqq 0) \tag{3.13}$$

is finite. Likewise $\psi(x)$ *is defined in the set*

$$S_\beta = \{x = u\alpha + n\beta \mid u \geqq 0,\ n \text{ an arbitrary integer}\} \tag{3.9$'$}$$

where it has the period β, $\psi(0) = 0$. *Also*

$$\inf_{s,t \in S_\beta} \frac{\psi(t) - \psi(s)}{t - s} = -\nu \qquad (\nu \geqq 0) \tag{3.13$'$}$$

is finite. Finally the inequality

$$\mu + \nu \leqq \lambda \tag{3.14}$$

holds.

Conversely, let $\varphi(x)$ *and* $\psi(x)$ *be functions of period* α *and* β *respectively,* $\varphi(0) = \psi(0) = 0$, *defined in the sets* S_α *and* S_β, *respectively, and such that* μ, ν *given by* (3.13) *and* (3.13′), *respectively, are finite. If* λ *satisfies* (3.14), *then* $f(x)$, *defined by* (3.23), *is a nondecreasing solution of the functional equation* (3.2), *and the limit relation* (2.8) *holds.*

Proof. The direct part summarizes already established results. The converse part is implied by the last paragraph of Section 7, with the exception of the very last statement that (2.8) holds. This point, however, is settled in a few words as follows: We know that $f_0(x) = \varphi(x) + \mu x$ is a nondecreasing solution of (3.2). But then, by Corollary 2.1, $\lim f_0(x)/x = \lim \varphi(x)/x + \mu$ exists as $x \to \infty$, $x \in S$; in particular $\lim \varphi(x)/x$ exists. However, since $\varphi(n\alpha) = \varphi(0) = 0$ for all integers n, we conclude that $\varphi(x) = o(x)$. Likewise $\psi(x) = o(x)$, and now (3.23) implies (2.8).

10. The extension of solutions

At this stage we can easily establish

THEOREM 3.2. *Every nondecreasing solution* $f(x)$ *of* (3.2) *can be extended to a nondecreasing solution* $F(x)$ *of the unrestricted functional equation*

$$F(m\alpha + n\beta) = F(m\alpha) + F(n\beta) \qquad (m, n \text{ arbitrary integers}). \tag{3.24}$$

Proof. This requires an extension of the definition of $f(x)$ to the entire module

$$\Sigma = \{x = m\alpha + n\beta\}.$$

This will be done by the following

Construction. We know that $\varphi(x) + \mu x$ is defined and nondecreasing in the set S_α which is everywhere dense. This implies the existence of the limits $\varphi(x - 0)$ and $\varphi(x + 0)$, for every real x, and that $\varphi(x - 0) \leqq \varphi(x + 0)$. We define an extension $\Phi(x)$, of $\varphi(x)$, throughout Σ by requiring

(i) $\Phi(x) = \varphi(x)$ if $x \in S_\alpha$.
(ii) If $0 < x < \alpha$, $x \in \Sigma$, $x \notin S_\alpha$, we assign to $\Phi(x)$ an arbitrary value subject to $\varphi(x - 0) \leqq \Phi(x) \leqq \varphi(x + 0)$.
(iii) $\Phi(x)$ has the period α.

We likewise define $\Psi(x)$ by

(i′) $\Psi(x) = \psi(x)$ if $x \in S_\beta$.
(ii′) If $0 < x < \beta$, $x \in \Sigma$, $x \notin S_\beta$, we assign to $\Psi(x)$ an arbitrary value subject to $\psi(x - 0) \leqq \Psi(x) \leqq \psi(x + 0)$.

It is now readily verified that

$$F(x) = \lambda x + \Phi(x) + \Psi(x) \qquad (x \in \Sigma) \tag{3.25}$$

is a nondecreasing solution of (3.24) which is an extension of $f(x)$.

Let us now prove the converse:

THEOREM 3.3. *The above construction of $\Phi(x)$ and $\Psi(x)$ and formula* (3.25) *furnish all nondecreasing $F(x)$, solutions of* (3.24), *which are extensions of a given nondecreasing solution $f(x)$ of* (3.2).

In particular from this theorem flows

COROLLARY 3.1. *The extension $F(x)$, of $f(x)$, is unique if and only if $\varphi(x)$ is continuous in $\Sigma - S_\alpha$ and $\psi(x)$ is continuous in $\Sigma - S_\beta$.*

A proof of Theorem 3.3 requires the following brief discussion of the monotone solutions of (3.24).

11. The monotone solutions of the unrestricted functional equation (3.24) and a proof of Theorem 3.3

We have to start from the beginning but will proceed very fast as we use only simplified versions of previous arguments. Throughout this section $F(x)$ denotes a nondecreasing solution of (3.24). Its restriction $f(x) = F(x)$ to the set S is, of course, a nondecreasing solution of (3.2), while $\varphi(x)$ and $\psi(x)$ are the old periodic functions associated with $f(x)$.

LEMMA 3.3. *The following limit exists*:

$$\lim_{x\to\pm\infty, x\in\Sigma} F(x)/x = \lambda \geqq 0. \tag{3.26}$$

Indeed, we already know that (3.26) holds if $x \to \infty$, $x \in S$. To establish (3.26) we return to Lemma 2.1 and observe that if we change its assumptions by allowing u and v to run through *all* integers, then its conclusion (2.3) may be changed to

$$\lim_{x\to\pm\infty, x\in E} F(x)/x = \lambda \geqq 0. \tag{3.27}$$

This is seen as follows: In the relations $F(u) + F(v - 1) \leqq F(u + v) \leqq F(u) + F(v + 1)$ which were there derived, u and v may now assume any integral values. If we replace u by $u - jv$ $(j = 1, \cdots, m)$ and add these relations, we obtain $F(u) - mF(v + 1) \leqq F(u - mv) \leqq F(u) - mF(v - 1)$. Let now v be positive and fixed, n arbitrary negative, and let us divide n by v, obtaining $n = u - mv$, $m > 0$, $0 \leqq u < v$, and therefore

$$\frac{F(u) - mF(v-1)}{u - mv} \leqq \frac{F(u - mv)}{u - mv} \leqq \frac{F(u) - mF(v+1)}{u - mv}.$$

As we can select v so that $F(v - 1)/v$ and $F(v + 1)/v$ are as close to λ as we wish, on letting $n \to -\infty$ we obtain (3.27), whence (3.26) follows.

We now set

$$F(x) = \lambda x + \Omega(x),$$

where $\Omega(x)$ is a solution of (3.24) and satisfies the relation

$$\lim_{x\to\pm\infty, x\in\Sigma} \Omega(x)/x = 0$$

and the inequality

$$(\Omega(y) - \Omega(x))/(y - x) \geqq -\lambda.$$

These, as in Lemma 2.2, imply the inequality

$$\frac{\Omega(m\alpha) - \Omega(m'\alpha)}{(m - m')\alpha + h\beta} \geqq -\lambda \qquad (m, m', h \text{ arbitrary integers}).$$

But then, if we decompose, as in Section 6, the solution $\Omega(x)$ into its periodic components $\Phi(x)$ and $\Psi(x)$, both now defined in Σ, we find that they satisfy the inequalities

$$\frac{\Phi(y) - \Phi(x)}{y - x} \geqq -\lambda, \qquad \frac{\Psi(y) - \Psi(x)}{y - x} \geqq -\lambda. \tag{3.28}$$

On the other hand, $\Phi(x)$ is evidently an extension of the old $\varphi(x)$ from S_α to Σ, and likewise $\Psi(x)$ extends the old $\psi(x)$ from S_β to Σ. Now (3.28) means that $\Phi(x) + \lambda x$ is nondecreasing; hence $\Phi(x \pm 0)$ exist everywhere, $\Phi(x - 0) \leqq \Phi(x + 0)$, and in particular

$$\Phi(x - 0) \leqq \Phi(x) \leqq \Phi(x + 0) \quad \text{if} \quad x \in \Sigma. \tag{3.29}$$

But S_α being everywhere dense we conclude that

$$(3.30)\quad \Phi(x-0)=\varphi(x-0),\qquad \Phi(x+0)=\varphi(x+0)\quad \text{for all } x.$$

Now (3.29) and (3.30) imply that

$$(3.31)\qquad \varphi(x-0)\leqq \Phi(x)\leqq \varphi(x+0)\quad \text{if}\quad x\in\Sigma,\quad x\notin S_\alpha,$$

and similarly we obtain

$$(3.32)\qquad \psi(x-0)\leqq \Psi(x)\leqq \psi(x+0)\quad \text{if}\quad x\in\Sigma,\quad x\notin S_\beta.$$

The inequalities (3.31) and (3.32) complete our proof of Theorem 3.3 because they show that *all* monotone solutions, extensions of $f(x)$, were obtained by the Construction used in the proof of Theorem 3.2.

Beyond the immediate objective just reached it might be worthwhile to add the following remarks concerning monotone solutions of (3.24). By (3.28) we know that

$$(3.33)\qquad -\mathrm{M}=\inf_{\Sigma}\frac{\Phi(y)-\Phi(x)}{y-x},\qquad -\mathrm{N}=\inf_{\Sigma}\frac{\Psi(y)-\Psi(x)}{y-x}$$

are finite. Beyond this we can easily show that

$$(3.34)\qquad \mathrm{M}=\mu,\qquad \mathrm{N}=\nu.$$

Indeed, clearly

$$(3.35)\qquad \mathrm{M}\geqq\mu,\qquad \mathrm{N}\geqq\nu,$$

since, by (3.33), M is defined by an *extension* Φ of φ. However, given $\varepsilon>0$ and a difference quotient $(\Phi(y)-\Phi(x))/(y-x)$ $(x<y,\ x, y\in\Sigma)$, we can evidently find, because of (3.30), numbers x', y' in S_α, $x'<y'$, such that

$$(\varphi(y')-\varphi(x'))/(y'-x')<(\Phi(y)-\Phi(x))/(y-x)+\varepsilon.$$

This and (3.13) imply that $-\mu<-\mathrm{M}+\varepsilon$, $-\mu\leqq-\mathrm{M}$, or $\mu\geqq\mathrm{M}$. Now (3.35) gives the first relation (3.34), and the second is shown in a like manner.

Observe that (3.34) and (3.14) imply that $\lambda\geqq\mathrm{M}+\mathrm{N}$. At this point we notice that we have established an exact analogue of Theorem 3.1 for the functional equation (3.24). This analogue, which we need not state explicitly, shows how to construct *ab initio* the most general nondecreasing solution (3.25) of the equation (3.24).

12. Two examples

1. Corresponding to the counterexample (2) of our Introduction we consider the function

$$(3.36)\qquad f(x)=[x/\alpha]+[x/\beta].$$

Evidently $f(x)=\lambda x+\varphi(x)+\psi(x)$, where

$$\lambda=\alpha^{-1}+\beta^{-1},\qquad \varphi(x)=[x/\alpha]-x/\alpha,\qquad \psi(x)=[x/\beta]-x/\beta,$$

and $f(x)$ satisfies (3.2), because φ and ψ have the right periods and $\varphi(0) = \psi(0) = 0$. Their discontinuities are at $x = m\alpha$ and $x = n\beta$, respectively. Since $m\alpha \in S_\alpha$ and $n\beta \in S_\beta$, we conclude by Corollary 3.1 that there is a unique monotone $F(x)$ which is an extension of $f(x)$ and satisfies (3.24). This extension is evidently given by the formula (3.36) because $f(x)$ satisfies (3.24).

2. Let α be irrational, $0 < \alpha < 1$, and let us consider the function

$$f(x) = [x/\alpha] + [x + \alpha]. \tag{3.37}$$

Here $f(x) = \lambda x + \varphi(x) + \psi(x)$, where

$$\lambda = \alpha^{-1} + 1, \qquad \varphi(x) = [x/\alpha] - x/\alpha, \qquad \psi(x) = [x + \alpha] - x.$$

Again $f(x)$ satisfies (3.2) because φ and ψ have the right periods while $\varphi(0) = \psi(0) = 0$. Notice that $\psi(x)$ is discontinuous at $x = -\alpha$ and that $-\alpha \in \Sigma - S_\beta$. Actually $\psi(-\alpha - 0) = \alpha - 1 < \psi(-\alpha + 0) = \alpha$. We know by Theorem 3.3 that there are infinitely many extensions of $f(x)$ and that if we define

$$\begin{aligned} \Psi(x) &= [x + \alpha] - x & &\text{if} \quad x \not\equiv -\alpha \pmod 1, \\ &= \gamma, \text{ where } \alpha - 1 \leqq \gamma \leqq \alpha & &\text{if} \quad x \equiv -\alpha \pmod 1, \end{aligned}$$

then

$$F(x) = [x/\alpha] + \Psi(x) + x$$

gives *all* extensions of $f(x)$.

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