ABSTRACT HOMOTOPY IN CATEGORIES OF FIBRATIONS AND THE SPECTRAL SEQUENCE OF EILENBERG AND MOORE

BY

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In previous articles [5], [6] we have examined those formal characteristics of the category of topological spaces which make possible the constructions of homotopy theory and, accordingly, of homology theory. These characteristics are shared by many other categories, which thus admit their own homotopy and homology theories: as to the latter we might say that, as "extraordinary" homology generalizes by changing coefficient objects, so a further generalization occurs in changing the domain of homology theory.

Our primary purpose here is to exhibit the existence of such structure in three cases: the category of spaces over a fixed space; the category of Hurewicz fibrations over a fixed space; the category of spaces provided with a fixed group of operators.[2] We thus justify assertions made in the references cited above. This is not however an empty generalization. We shall use it to give perspicuous derivations of two spectral sequences due to Eilenberg and Moore [4]. One of these involves the homology of the pullback of a fibration, the other the homology of the fiber bundle with a prescribed fiber associated to a principal bundle.

These spectral sequences have been derived in a number of ways. The original techniques of Eilenberg and Moore use chain-complex arguments and appear to give results only for singular homology. Rector [8] derives the pullback spectral sequence by cosimplicial methods which seem to be quite different from those advanced here. The construction of L. Smith [10], however, is quite similar to the one below, as is that of Steenrod and Rothenberg [8] for the associated bundle.

Smith (loc. cit.) remarks on the analogy between the pullback spectral sequence and the Adams spectral sequence (cf. [1]). There seems to be no doubt that they belong to a common domain whose boundaries however have not yet been completely determined. The argument below, which uses stable homotopy methods and is couched in terms of homology rather than cohomology (thus avoiding finiteness restrictions after the fashion of Eilenberg and Moore) reinforces this analogy.

In §§1, 2 below we restate the axioms for abstract homotopy theory in an

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[2] I understand that J. F. McClendon and I. M. James have investigated these categories, or near neighbors, from similar points of view in as yet unpublished studies.

Added in proof. Cf. also I. M. James, *Ex-homotopy theory I*, Illinois J. Math., vol. 15 (1971), pp. 324–337.

"unpointed" version and discuss the introduction of basepoints. These, together with §§3–7, which establish the relevant properties of the categories in question, are more or less self-contained. For the remainder of the paper, certain generalities concerning stable homotopy and homology theories in the abstract case are assumed. For these [5] and [6] may serve as reference.

In §§8–11 the "pulback" spectral sequence is derived and its convergence discussed. The "associated-bundle" spectral sequence presents many analogies, and is treated in somewhat more abbreviated fashion in §§12–14.

1. Abstract homotopy theory

In [5], [6] we have introduced the notion of an *hc*-category as a framework for abstract homotopy theory. The axioms given there were for pointed categories, i.e., categories with a 0-object. It is convenient to have an unpointed version, which we adduce here with the terminological convention that what was previously called an *hc*-category is now a *pointed hc*-category. The latter often arise from the former by a process of introducing basepoints, which we shall describe below.

A *c-category* is a category $\mathcal{C}$ with an initial object $\emptyset$ and a terminal object P, provided with a subcategory Cof $\mathcal{C}$, whose morphisms are called *cofibrations*, such that:

(C1) Cof $\mathcal{C}$ contains all isomorphisms and all morphisms $\emptyset \to A$.

(C2) If $a : A' \to A$ in Cof $\mathcal{C}$ and $f' : A' \to B'$ in $\mathcal{C}$ then the pushout diagram (called a *c-pushout*)

$$(1.1)\qquad \begin{array}{ccc} A' & \xrightarrow{a} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{b} & B \end{array}$$

exists in $\mathcal{C}$, and $b \in$ Cof $\mathcal{C}$.

We shall occasionally use the notation $B = A \cup B'$, and write morphisms $B \to C$ in the matrix form $(u \quad v)$ with $(u \quad v)f = u$, $(u \quad v)b = v$. The special cases $A' = \emptyset$, $B' = P$ give respectively the coproduct $B = A + B'$ the *cofiber* $B = A/A'$.

($\mathcal{C}$, Cof $\mathcal{C}$) is a *pointed hc*-category if $\emptyset = P$; we then write $\emptyset = P = 0$, and use the notation $A \vee B$ for $A + B$.

If ($\mathcal{C}'$, Cof $\mathcal{C}'$) is also a *c*-category a functor $\mathcal{C} \to \mathcal{C}'$ is a *c-functor* if it preserves cofibrations and *c*-pushouts. If $\mathcal{C}$, $\mathcal{C}'$ are pointed, a *pointed c*-functor preserves 0 as well.

If ($\mathcal{C}$, Cof $\mathcal{C}$) is a *c*-category a full subcategory $\mathcal{C}' \subset \mathcal{C}$ is a *c-subcategory* if ($\mathcal{C}'$, $\mathcal{C}' \cap$ Cof $\mathcal{C}$) is a *c*-category and inclusion is a *c*-functor.

The primary example we shall have in mind below is the category $\mathcal{I}$ of topological spaces which are compactly generated and in which continuous real-valued functions separate points. Morphisms are continuous maps, and cofibrations are those for which the homotopy extension condition holds. The full subcategory $\mathcal{W} \subset \mathcal{T}$ of spaces having the homotopy type of a CW-complex is easily seen to be a c-subcategory.

An hc-category is a c-category $(\mathcal{C}, \operatorname{Cof} \mathcal{C})$ provided with a congruence $\simeq$, called *homotopy*, satisfying four additional axioms. The quotient category $\mathcal{C}^{\square} = \mathcal{C}/\simeq$ is the *homotopy category*; a morphism in $\mathcal{C}$ is a *homotopy equivalence* if its image in $\mathcal{C}^{\square}$ is an isomorphism. The axioms are the following:

(HC 1) (Additivity) If $f \simeq f' : A \to X$ and $g \simeq g' : B \to X$ then

$$(f \quad g) \simeq (f' \quad g') : A + B \to X.$$

(HC 2) (Homotopy extension) If

$$\begin{array}{ccccc} A' & \xrightarrow{a} & A & \xrightarrow{\bar{a}} & A/A' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ B' & \xrightarrow{b} & B & \xrightarrow{\bar{b}} & B/B' \end{array}$$

commutes, a and b being cofibrations, and $g' \simeq f'$ then there exist $g \simeq f$, $g'' \simeq f''$ such that $ga = bg'$, $g''\bar{a} = \bar{b}g$.

(HC 3) (Deformation-retraction) If in the c-pushout (1.1) the morphism a is a homotopy equivalence then so also is b.

(HC 4) (Mapping cylinder) Any morphism in $\mathcal{C}$ has a factorization gf with g a homotopy equivalence and f a cofibration.

The fact that $\mathcal{T}$, supplied with the usual notions of cofibration and homotopy, is an hc-category is a sequence of commonplaces of homotopy theory.

The notion of an *hc-subcategory* being defined in the obvious way, it is clear that $\mathcal{W}$ is an hc-subcategory of $\mathcal{T}$.

2. Relative homotopy; introducing basepoints

If $a : A' \to A$ is a cofibration in an hc-category $\mathcal{C}$ we may construct a c-pushout

$$\begin{array}{ccc} A' & \xrightarrow{a} & A \\ {\scriptstyle a}\downarrow & & \downarrow \\ A & \longrightarrow & A \cup_a A \end{array}$$

A *relative cylinder over* a is a mapping cylinder factorization

$$A \cup_a A \xrightarrow{(i_0 \quad i_1)} Z_a A \xrightarrow{\alpha} A$$

of $(1 \quad 1) : A \cup_a A \to A$. If

$$A \cup_a \cup \xrightarrow{(i_0' \quad i_1')} Z_a' A \xrightarrow{\alpha'} A$$

is another we may construct the pushout

$$\begin{array}{ccc} A & \xrightarrow{i_1} & Z_a A \\ {\scriptstyle i_0'}\downarrow & & \downarrow{\scriptstyle u} \\ Z_a' & \xrightarrow{u'} & Z_a A \oplus Z_a' A \end{array}$$

With this notation we have the following lemma.

LEMMA 2.1. (i)

$$A \cup_a A \xrightarrow{(i_1 \quad i_0)} Z_a A \xrightarrow{\alpha} A$$

is a relative cylinder over a; (ii)

$$A \cup_a A \xrightarrow{(ui_0 \quad u'i_1')} Z_a A \oplus Z_a' A' \xrightarrow{(\alpha \quad \alpha')} A$$

is a relative cylinder over a; (iii) *if*

$$B \cup_b B \xrightarrow{j_0 \quad j_1} Z_b B \xrightarrow{\beta} B$$

is a relative cylinder over $b : B' \to B$ *and* $(f', f) : a \to b$ *then there is a*

$$z : Z_a A \to Z_b B$$

such that $z(i_0 \quad i_1) = (j_0 \quad j_1)f$.

If $f_0, f_1 : A \to X$ we say that they are homotopic (rel a, or rel A') if there is an $F : Z_a A \to X$ with $F(i_0 \quad i_1) = (f_0 \quad f_1)$; this implies of course that $f_0 a = f_1 a$. From the lemma we conclude the following.

PROPOSITION 2.2. *Homotopy* (rel a) *is an equivalence relation on* $\mathcal{C}(A, X)$. *If* $g : X \to Y$ *and* $f_0 \simeq f_1$ (rel a) *then* $gf_0 \simeq gf_1$ (rel a). *If* $b : B' \to B$ *is a cofibration and* $(g', g) : b \to a$ *then* $f_0 g \simeq f_1 g$ (rel b).

Notice that $f_0 \simeq f_1$ (rel A') implies $f_0 \simeq f_1$. Conversely $f_0 \simeq f_1$ is equivalent to $f_0 \simeq f_1$ (rel $\emptyset$).

An essential property of relative homotopy is its behavior with respect to pushouts.

PROPOSITION 2.3. *If, in the c-pushout* (1.1),

$$(u_0 \quad v),\ (u_1 \quad v) : B \to X \quad \textit{and} \quad u_0 \simeq u_1 \text{ (rel } A')$$

then $(u_0 \quad v) \simeq (u_1 \quad v)$ (rel B').

The characterization of (HC3) as a "deformation retraction" axiom is explained by the following observation. We say that $a : A' \to A$ is a *strong deformation retract* if there is a retraction $r : A \to A'$, i.e., left inverse of a, such that $ar \simeq 1_A$ (rel A').

PROPOSITION 2.4. *If $a : A' \to A$ is a cofibration and a homotopy equivalence retract.*

The proof proceeds just as in the standard topological context.

We may now "introduce basepoints" in an arbitrary *hc*-category ($\mathcal{C}$. Cof $\mathcal{C}$, $\simeq$). The objects of $\mathcal{C}^\bullet$ are the cofibrations $x_0 : P \to X$ in $\mathcal{C}$. If also $y_0 : P \to Y$ is a cofibration then $\mathcal{C}^\bullet(x_0, y_0)$ consists of $f : X \to Y$ such that $fx_0 = y_0$. Such an f is a cofibration in $\mathcal{C}^\bullet$ if it is one in $\mathcal{C}$. Homotopy in $\mathcal{C}^\bullet$ is homotopy (rel P) in $\mathcal{C}$. We write Cof $\mathcal{C}^\bullet$, $\simeq^\bullet$ for the cofibrations and homotopies thus defined.

PROPOSITION 2.5. *If* ($\mathcal{C}$, Cof $\mathcal{C}$, $\simeq$) *is an hc-category then* ($\mathcal{C}^\bullet$, Cof $\mathcal{C}^\bullet$, $\simeq^\bullet$) *is a pointed hc-category.* Moreover, the forgetful functor $\mathcal{C}^\bullet \to \mathcal{C}$ *and its coadjoint* $X \to X^+ = X + P$ *are hc-functors.*

The category $\mathcal{C}^\bullet$ may well be trivial; examples are easily supplied. However $\mathfrak{T}^\bullet$ is certainly not trivial, and we shall introduce additional nontrivial examples below. Notice that if $\mathcal{C}$ is pointed then $\mathcal{C}^\bullet = \mathcal{C}$.

If $F : \mathcal{C} \to \mathcal{C}'$ is an *hc*-functor we define $F^\bullet : \mathcal{C}^\bullet \to \mathcal{C}'^\bullet$ by means of the *c*-pushout

$$\begin{array}{ccc} FP & \xrightarrow{Fx_0} & FX \\ \downarrow & & \downarrow \\ P' & \longrightarrow & F^\bullet X \end{array}$$

PROPOSITION 2.6. *$F^\bullet$ is a pointed hc-functor.*

3. Spaces over a fixed base

By a "space over B" we mean a map $p_X : X \to B$ in $\mathfrak{T}$. These are the objects of the category $\mathfrak{T}_B$. The morphisms are maps $f : X \to Y$ with $p_Y f = p_X$; we shall call them, as usual, fiber-maps over B. $\mathfrak{T}_B$ has initial object $\emptyset \to B$ and terminal object $p_B = 1_B$.

A *fiber-homotopy over B* is a fiber-map $X \times I \to Y$, where $X \times I$ is given the structure of a space over B by $p_{X \times I} = p_X pr_X$. A morphism $a : A' \to A$ is a *cofibration over B* if it satisfies the following *fiberhomotopy-extension* condition:

$$\begin{array}{ccc} A' \times \{0\} & \xrightarrow{a \times \{0\}} & A \times \{0\} \\ \downarrow & & \downarrow \\ A' \times I & \longrightarrow & A \times I \end{array}$$

is a weak pushout in $\mathfrak{T}_B$.

Following the argument of Puppe [7] we obtain a condition for identifying cofibrations over B.

LEMMA 3.1. *$a : A' \to A$ is a cofibration over B if and only if it is a homeomorphism of A' onto a subspace of A and (making the implied identification) there is a function $\rho : A \to I$ and a fiber-homotopy $F : A \times I \to A$ stationary on A' such that*

$$\rho^{-1}(0) = A', \quad F_0 = 1, \quad F_1\rho^{-1}[0, 1) \subset A'.$$

For a is a cofibration over B if and only if

$$A \times \{0\} \cup A' \times I \to A \times I$$

has a left inverse r in $\mathfrak{I}_B$. The first statement follows at once. Now, supposing $A' \subset A$, set

$$\rho a = \max_{t \in I} (t - pr_I\, r(a, t)), \qquad F(a, t) = pr_A\, r(a, t).$$

Conversely, given ρ and F we may define the retraction r by

$$\begin{aligned} r(a, t) &= (F(a, t/\rho a), 0) \quad &&\text{for } a \in A - A', t \leq \rho a \\ &= (F(a, 1), t - \rho a) \quad &&\text{for } t \geq \rho a. \end{aligned}$$

To see that the map so defined is continuous at points of $A' \times \{0\}$ observe that if $a' = F(a', 0) \in U$, U open in A, then, since I is compact, there is an open V containing a' such that $F(V \times I) \subset U$.

Pursuing further Puppe's reasoning, we arrive at the following statements.

LEMMA 3.2. *If $A' \subset A$, $C' \subset C$ are cofibrations over B then so is*

$$A \times_B C' \cup A' \times_B C \subset A \times_B C.$$

Let ρ, F be as in 3.1 and suppose that σ, G play similar roles with respect to $C' \subset C$. Let

$$\varphi : I^3 - \{(0, 0, 1)\} \to I$$

be a continuous function such that $\varphi(r, s, 0) = \varphi(0, s, t) = 0$ and $\varphi(r, s, 1) = 1$ for $r \geq s$. Let $\tau : A \times_B C \to I$ be given by $\tau(a, c) = (\rho a)(\sigma c)$ and define $H : (A \times_B C) \times I \to A \times_B C$ by

$$\begin{aligned} H(a', c', t) &= (a', c') \quad \text{for } (a', c') \in A' \times_B C' \\ H(a, c, t) &= (F(a, \varphi(\sigma c, \rho a, t)), G(c, \varphi(\rho a, \sigma c, t))) \quad \text{for } (a, c) \notin A' \times_B C'. \end{aligned}$$

Continuity at points of $(A' \times_B C') \times I$ follows from the compactness of I as in the proof of 3.1.

Since for $A \in \mathfrak{I}_B$, $A \times I = A \times_B (B \times I)$ it follows that $A \times \dot{I} \cup A' \times I \subset A \times I$ is a cofibration over B whenever $A' \subset A$ is.

LEMMA 3.3. *If $A' \subset A$ is a cofibration over B and also a fiber-homotopy equivalence then A' is a strong fibered deformation retract of A.*

From the homotopy extension condition it follows that it is a weak fibered deformation retract and thus that $A \times \{0\} \cup A' \times I \subset A \times I^{\bullet} \cup A' \times I$ is a fiber homotopy equivalence, as must also be $A \times I^{\bullet} \cup A' \times I \subset A \times I$. Since this inclusion is a cofibration over B it has a retraction in $\mathfrak{J}_B$, from which the strong deformation retraction of A onto A' is easily manufactured.

These lemmas provide the nontrivial part of the proof of the following theorem.

THEOREM 3.4. *$(\mathfrak{J}_B, \text{Cof } \mathfrak{J}_B, \simeq_B)$ is an hc-category, where Cof $\mathfrak{J}_B$ is the class of cofibrations over B and $\simeq_B$ is fiber homotopy.*

Axiom C1 is trivial. For C2, if $A' \subset A$ is a cofibration over B and $A' \to C'$ is a fiber map then the pushout $A \cup C'$ exists in $\mathfrak{J}$ and $p_{A \cup C'} = (p_A \;\; p_{C'})$ makes it a pushout in $\mathfrak{J}_B$. Fiber homotopy extension for $C' \to (A \cup C')$ is a formal consequence of the construction.

Conditions HC1, 2 are immediate, while HC4 follows from the usual construction of the mapping cylinder, with the observation that this construction respects the structure of maps and spaces over B. Finally, HC3 follows, as in the classical case $\mathfrak{J}$, from 3.3.

As in §2 we may introduce "basepoints", in this case rather to be described as "base-sections", and thus construct the pointed *hc*-category $(\mathfrak{J}_B^{\bullet}, \text{Cof } \mathfrak{J}_B^{\bullet}, \simeq_B^{\bullet})$ whose objects are the diagrams

$$B \xrightarrow{\sigma_X} X \xrightarrow{p_X} B$$

with $p_X \sigma_X = 1_B$ and σ_X a cofibration over B.

If we write $\mathfrak{W}_B$ for the full subcategory of $p_X : X \to B$ in $\mathfrak{J}_B$ with $X \in \mathfrak{W}$, this is clearly an *hc*-subcategory provided $B \in \mathfrak{W}$. Similarly $\mathfrak{W}_B^{\bullet}$ is an *hc*-subcategory of $\mathfrak{J}_B^{\bullet}$.

4. Pullbacks and their adjoints

If $w : B \to C$ in $\mathfrak{J}$ we write

$$\begin{array}{ccc} w^!X = B \times_C X & \longrightarrow & X \\ {\scriptstyle p_{w^!X}} \downarrow & & \downarrow {\scriptstyle p_X} \\ B & \xrightarrow{\;w\;} & C \end{array}$$

for the pullback. If $f : X \to Y$ is a fiber-map over C then $(w^!f)(b, x) = (b, fx)$ defines a fiber-map over B; this makes $w^! : \mathfrak{J}_C \to \mathfrak{J}_B$ a functor.

In the other direction we have $w_! : \mathfrak{J}_B \to \mathfrak{J}_C$ given by $w_!(X, p_X) = (X, wp_Y)$.

PROPOSITION 4.1. *$w^!$ is adjoint to $w_!$.*

The adjunction is given by

$$w : \mathfrak{J}_C(w_! X, Y) \approx \mathfrak{J}_B(X, w^! Y)$$

where $wf = (fp_X)$, $w^{-1}g = pr_Y\, g$.

PROPOSITION 4.2. *$w_!$ is an hc-functor.*

As a coadjoint $w_!$ preserves pushouts, *a fortiori* c-pushouts. That it preserves fiber-homotopy is obvious. The preservation of cofibrations is an immediate consequence of 3.1.

PROPOSITION 4.3. *$w^!$ also has an adjoint.*

If $X \in \mathfrak{J}_B$ consider the set

$$E = \coprod_{c \in C} \sigma(w^{-1}c)$$

where for $B' \subset B$, $\sigma(B')$ is the set of cross-sections of X over B'. E is provided with a map $p_E : E \to B$ by sending $\sigma(w^{-1}c)$ into c. E may be topologized by the rule that if K is a compact Hausdorff space then $\varphi : K \to E$ is continuous exactly when the map $\psi : K \times_C B \to X$ with $\psi(k, b) = (\varphi k)b$ is continuous. Then $X \mapsto E$ is adjoint to $w^!$.

We can now make the appropriate assertion about $w^!$.

PROPOSITION 4.4. *If $w : B \to C$ then $w^! : \mathfrak{J}_C \to \mathfrak{J}_B$ is an hc-functor.*

Since $w^!(X \times I) = (w^!X) \times I$ it is clear that $w^!$ perserves homotopy. Lemma 3.1 provides for the preservation of cofibrations. The preservation of c-pushouts follows of course from 4.3.

Moreover $w^!$ preserves initial and (unlike $w_!$) terminal objects and thus defines $w^! : \mathfrak{J}^\bullet_C \to \mathfrak{J}^\bullet_B$, again an hc-functor. Associated to $w_!$, on the other hand we have the hc-functor $w_! : \mathfrak{J}^\bullet_B \to \mathfrak{J}^\bullet_C$, coadjoint to $w^!$.

If $w : B \to C$ in $\mathcal{W}$ then $w_!$ takes $\mathcal{W}_B$ into $\mathcal{W}_C$ and $w^\bullet_!$ takes $\mathcal{W}^\bullet_B$ into $\mathcal{W}^\bullet_C$; both restrictions are hc-functors. No similar assertion, evidently, can be made about $w^!$ (but see §6 below).

5. Fibrations over B

We shall denote by Fib_B the full subcategory of $\mathfrak{J}_B$ consisting of $p_X : X \to B$ which are fibrations. We recall that this means that

$$\begin{array}{ccc} X^I & \longrightarrow & X \\ {\scriptstyle p_X^I}\downarrow & & \downarrow{\scriptstyle p_X} \\ B^I & \longrightarrow & B \end{array}$$

is a weak pullback in $\mathfrak{J}$, where the horizontal arrows stand for evaluation at 0, or equivalently that there be a *path-lifting-function* (PLF) for X, i.e., a cross-section of the canonical map $X^I \to B^I \times_B X$.

We introduce into Fib_B the same homotopy relation as that used in $\mathfrak{J}_B$, or rather its restriction. But it is not asserted that Fib_B is a c-subcategory of $\mathfrak{J}_B$.

A morphism $a : A' \to A$ in Fib_B is a *fiber-cofibration over B* if it is a cofibration over B (so that we might as well assume it to be an inclusion) and, further, there exist PLF $\mathfrak{a}'$ for A', $\mathfrak{a}$ for A such that $\mathfrak{a}$ extends $\mathfrak{a}'$, i.e., such that

$$\begin{array}{ccc} B^I \times_B A' & \longrightarrow & B^I \times_B A \\ \downarrow{\scriptstyle \mathfrak{a}'} & & \downarrow{\scriptstyle \mathfrak{a}} \\ A'^I & \longrightarrow & A^I \end{array}$$

commutes.

It is of course intended that, provided with this notion of cofibration, Fib_B should be an *hc*-category. This is indeed the case. There is however a delicate point in the proof which we introduce here as a lemma.

LEMMA 5.1 (L. Berkhout). *Suppose*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

is a c-pushout in $\mathfrak{T}_B$, *that* A', A, X' *are in* Fib_B, *and that* $A' \to A$ *is a fiber-cofibration over* B. *Then any PLF* $\mathfrak{x}'$ *for* X' *extends to a PLF for* X.

Let $\mathfrak{a}'$ be a PLF for A' extending to a PLF $\mathfrak{a}$ for A. Suppose that

$$\rho : A \to I \quad \text{and} \quad F : A \times I \to A$$

are as in 3.1. Define $\hat{\rho} : B^I \times_B A \to [0, 2]$ by

$$\hat{\rho}(\sigma, a) = 2 \max_{t \in I} \rho(\mathfrak{a}(\sigma, a)t).$$

Then $\hat{\rho}^{-1}(0) = B^I \times_B A'$. Further define $\Phi : B^I \times_B A \to B^I$ by

$$\Phi(\sigma, a)t = \sigma(\min(1, t + \hat{\rho}(\sigma, a))).$$

We may now attempt to construct a PLF $\mathfrak{x}$ for X by setting

(2.2a) $$\mathfrak{x}(\sigma, x') = \mathfrak{x}'(\sigma, x') \quad \text{for } (\sigma, x') \in B^I \times_B X',$$

(2.2b) $$\mathfrak{x}(\sigma, fa)t = fF(\mathfrak{a}(\sigma, a)t, t/\hat{\rho}(\sigma, a))$$
$$\text{for } (\sigma, a) \in B^I \times_B (A - A'),\ t \leq \hat{\rho}(\sigma, a),$$

(2.2c) $$\mathfrak{x}(\sigma, fa)t = \mathfrak{x}'(\Phi(\sigma, a), fF(\mathfrak{a}(\sigma, a)\hat{\rho}(\sigma, a), 1))(t - \hat{\rho}(\sigma, a))$$
$$\text{for } (\sigma, a) \in B^I \times_B A,\ t \geq \hat{\rho}(\sigma, a).$$

Observe that if $\hat{\rho}(\sigma, a) \leq t$ C 1 then $\rho\mathfrak{a}(\sigma, a)\hat{\rho}(\sigma, a) \leq \frac{1}{2}$ so that

$$F(\mathfrak{a}(\sigma, a)\hat{\rho}(\sigma, a), 1) \in A'$$

and 2.2c is defined.

The following statements are easily verified: for fixed $(\sigma, x) \in B^I \times_B X$,

$$\mathfrak{x}(\sigma, x) \in X^I; \quad \mathfrak{x}(\sigma, x)0 = x; \quad p_X\, \mathfrak{x}(\sigma, x) = \sigma.$$

What remains is to show the continuity of $\mathfrak{x}$ or, what comes to the same thing, that of the adjoint map $(B^I \times_B X) \times I \to X$.

This continuity is clear on

$$(B^I \times_B (X - X')) \times I \approx (B^I \times_B (A - A')) \times I.$$

Moreover each point of $(B^I \times_B X') \times (0, 1]$ has a neighborhood on which $\mathfrak{x}$ is defined by 2.2a and 2.2c so that continuity is clear at such points as well. For points in $(B^I \times_B X') \times \{0\}$, finally, an argument analogous to that used in the proof of 3.1 is adequate.

COROLLARY 5.2. *If $A' \subset A$ is a fiber-cofibration over B then any PLF for A' extends to one for A.*

We need only take $f' = 1_{A'}$ in the lemma.

We now define Cof Fib_B to be the class of fiber-cofibrations over B.

THEOREM 5.3. $(\text{Fib}_B, \text{Cof Fib}_B, \simeq_B)$ *is an hc-category.*

Axiom C1 follows from the corollary to the lemma, which is itself, in view of 3.4, essentially C2. Conditions HC1–3 follow immediately from 3.4, while the lemma above comes into play once more to show that the mapping cylinder remains in Fib_B, thus proving HC4.

If $w : B \to C$ then $w^!$ takes Fib_C into Fib_B. Furthermore it preserves fiber-cofibrations. For if $\mathfrak{a}$ is a PLF for $A \in \text{Fib}_C$ then $\bar{\mathfrak{a}}(\sigma, b, x)t = (\sigma t, \mathfrak{a}(w\sigma, x)t)$ defines a PLF for $w^!A$, and the operation $\mathfrak{a} \to \bar{\mathfrak{a}}$ clearly preserves extension of PLF. Thus $w^! : \text{Fib}_C \to \text{Fib}_B$ is an *hc*-functor.

The coadjoint $w_!$ does not in general preserve fibrations except in the special case in which w itself is a fibration. In this case it preserves fiber-cofibrations as well, since if $\mathfrak{a} : C^I \times_C B \to B^I$ is a PLF then

$$C^I \times_C X = (C^I \times_C B) \times_B X \xrightarrow{\mathfrak{a} \times_B X} B^I \times_B X \xrightarrow{\mathfrak{x}} X^I$$

associates to a PLF $\mathfrak{x}$ for $X \in \text{Fib}_B$ a PLF for $w_! X \in \text{Fib}_C$, and this association preserves extension of PLF. Thus if $w : B \to C$ is a fibration then $w_! : \text{Fib}_B \to \text{Fib}_C$ is an *hc*-functor.

We may of course apply 2.5, 2.6 in this case, to produce pointed *hc*-categories $(\text{Fib}_B^\bullet, \text{Cof Fib}_B^\bullet, \simeq_B^\bullet)$ and *hc*-functors $w^!$ and, for w a fibration, $w_!^\bullet$.

6. $\mathfrak{W}$-fibrations

If $p : X \to B$ is a fibration, B is a CW-complex, and each fiber of p has the homotopy type of a CW-complex then it is easy to see that X also has the homotopy type of a CW-complex: this is clear when B is a cell, follows easily when B is a sphere and generalizes by induction over the skeletons of B. The same conclusion clearly holds if it is simply assumed that B has the homotopy type of a CW-complex:

PROPOSITION 6.1. *If $p : X \to B$ is a fibration such that B and each fiber is in $\mathfrak{W}$ then X is in $\mathfrak{W}$.*

The full subcategory of Fib_B consisting of these will be denoted by $\mathcal{W}\ \mathrm{fib}_B$; we call them *$\mathcal{W}$-fibrations.*

PROPOSITION 6.2. *$\mathcal{W}\ \mathrm{fib}_B$ is an hc-subcategory of* Fib_B.

It is asserted by 6.1 that $\mathcal{W}\ \mathrm{fib}_B \subset \mathcal{W}_B$; it is not of course an *hc*-subcategory. Since pullbacks preserve fibers we have the following.

PROPOSITION 6.3. *If $w : B \to C$ in $\mathcal{W}$ then*

$$w^! : \mathcal{W}\ \mathrm{fib}_C \to \mathcal{W}\ \mathrm{fib}_B \quad \textit{and} \quad w^! : \mathcal{W}\ \mathrm{fib}^\bullet_C \to \mathcal{W}\ \mathrm{fib}^\bullet_B$$

are hc-functors.

Using 6.1 once again we have

PROPOSITION 6.4. *If $w : B \to C$ is a $\mathcal{W}$-fibration then*

$$w_! : \mathcal{W}\ \mathrm{fib}_B \to \mathcal{W}\ \mathrm{fib}_C \quad \textit{and} \quad w^\bullet_! : \mathcal{W}\ \mathrm{fib}^\bullet_B \to \mathcal{W}\ \mathrm{fib}^\bullet_C$$

are hc-functors.

7. *G*-spaces

By a "topological group" we shall, for the purposes of this paper, mean a group in $\mathfrak{T}$. Since the product in $\mathfrak{T}$ is not the same as that in the category of all topological spaces this may not agree with the more usual notion.

The categories of left and of right G-spaces, for G a topological group, are defined in the usual way. We denote the former by ${}_G\mathfrak{T}$. Homotopy in ${}_G\mathfrak{T}$, denoted by ${}_G\simeq$, means equivariant homotopy. Cofibrations are defined by an equivariant homotopy extension condition: $A' \to A$ in ${}_G\mathfrak{T}$ is a *G-cofibration* if

$$\begin{array}{ccc} A' \times \{0\} & \longrightarrow & A' \times I \\ \downarrow & & \downarrow \\ A \times \{0\} & \longrightarrow & A \times I \end{array}$$

is a weak pushout in ${}_G\mathfrak{T}$, with I having, of course, the trivial operation.

In ${}_G\mathfrak{T}$ and relative to these notions of cofibration and homotopy the analogies of Lemmas 3.1–3 hold, the proofs again following the same pattern. For example:

LEMMA 7.1. *A morphism $A' \to A$ in ${}_G\mathfrak{T}$ is a G-cofibration if and only if it is a homeomorphism of A' onto a subspace of A and (making the implied identification) there is an equivariant (i.e., invariant) function $\rho : A \to I$ and an equivariant homotopy $F : A \times I \to A$, stationary on A', such that $\rho^{-1}(0) = A'$, $F_0 = 1_A$, $F_1 \rho^{-1}[0, 1) \subset A'$.*

Paralleling the argument of §3 we reach this conclusion:

THEOREM 7.2. *(${}_G\mathfrak{T}$, Cof ${}_G\mathfrak{T}$, ${}_G\simeq$) is an hc-category. Similarly (${}_G\mathfrak{T}^\bullet$, Cof ${}_G\mathfrak{T}^\bullet$, ${}_G\simeq^\bullet$) is a pointed hc-category (we should perhaps write $({}_G\mathfrak{T})^\bullet$, but omit the paren-*

theses for brevity). The forgetful functors ${}_G\mathfrak{T} \to \mathfrak{T}$, ${}_G\mathfrak{T}^\bullet \to \mathfrak{T}^\bullet$ *are hc-functors, as are their adjoints* $X \mapsto G \times X$, $X \mapsto G^+ \# X$. *The full subcategories* ${}_G\mathcal{W}$, ${}_G\mathcal{W}^\bullet$ *of G-spaces in* $\mathcal{W}$ *are hc-subcategories of* ${}_G\mathfrak{T}$, ${}_G\mathfrak{T}^\bullet$.

If W is a right G-space and Y is a left G-space we write $W \times^G Y$ for the quotient-space of $W \times Y$ with respect to the relations $(wg, y) \sim (w, gy)$, $g \in G$. Thus $W \times^G P = B$ is the orbit space of W under the operation of G. Commutativity in

$$\begin{array}{ccc} W \times Y & \xrightarrow{pr_W} & W \\ \eta \downarrow & & \downarrow \\ W \times^G Y & \xrightarrow{q_Y} & B \end{array},$$

where the vertical arrows denote the identification maps, defines $q_Y : W \times^G Y \to B$.

For the purposes of this paper we shall say that W is a *right principal G-bundle* if it is a principal bundle in the usual sense, i.e. has free G-operation and local product structure in the neighborhood of each point of B and in addition has an equivariant PLF $\mathfrak{w} : B^I \times_B W \to W^I$. We remark that this additional condition is redundant if the local product structure is numerable (Dold [2]).

PROPOSITION 7.3. *If W is a right principal G-bundle then*

$$Y \to (q_Y : W \times^G Y \to B)$$

is an hc-functor

$$W \times^G - : {}_G\mathfrak{T} \to \mathrm{Fib}_B .$$

For $\mathfrak{y}(\sigma, \eta(w, y))t = \eta(\mathfrak{w}(\sigma, w), y)$ defines a PLF for $W \times^G Y$, and this construction respects extensions of PLF. The fiber-homotopy extension condition for $W \times^G A' \subset W \times^G A$, where $A' \subset A$ is a G-cofibration, follows from 3.1, 7.1. For if $\rho : A \to I$, $F : A \times I \to A$ are as in 7.1 we may define

$$\bar{\rho} : W \times^G A \to I \quad \text{by} \quad \bar{\rho}\eta(w, a) = \rho a$$

and

$$\bar{F} : (W \times^G A) \times I \to W \times^G A \quad \text{by} \quad \bar{F}(\eta(w, a), t) = \eta(w, F(a, t)).$$

It remains only to show that $W \times^G -$ preserves c-pushouts. A square in $\mathfrak{T}_B$ (*a fortiori* in Fib_B) is a pushout whenever for each set $u : U \subset B$ of some open covering of B, $u^!$ applied to the square is a pushout. But $u^!\ (W \times^G -)$, when W splits over U, is isomorphic to the product with U.

The functor $W \times^G -$ preserves terminal objects and thus defines

$$W \times^G - : {}_G\mathfrak{T}^\bullet \to \mathrm{Fib}_B^\bullet ,$$

again an hc-functor. If $B \in \mathcal{W}$ then also $W \times^G -$ takes ${}_G\mathcal{W}$ (${}_G\mathcal{W}^\bullet$) into $\mathcal{W}\ \mathrm{fib}_B$ ($\mathcal{W}\ \mathrm{fib}_B^\bullet$).

8. Homology in $\mathcal{W}^\bullet_B$

We shall suppose from now on that we are supplied with a homology theory h on $\mathcal{W}^\bullet$ (a *reduced* homology theory) with values in $\mathfrak{Ab}^\infty$ and satisfying $h(VX_\alpha) = \coprod_\alpha hX_\alpha$. If $h \to h'$ is a morphism of such holomogy theories which is an isomorphism at S^0 then it is an isomorphism at all X (Dold [3]).

We shall suppose further that h is multiplicative, i.e. that it is provided with a natural transformation $hX \otimes hY \to h(X \# Y)$ which gives to $\Lambda = hS^0$ the structure of a graded ring with unit via $hS^0 \otimes hS^0 \to h(S^0 \# S^0) = hS^0$ and to each hX the structure of a unitary Λ-module via $hS^0 \otimes hX \to h(S^0 \# X) = hX$, so that h factors canonically through mod (Λ). For simplicity (though this is not essential) we shall suppose that the multiplication is commutative, i.e. that

$$\begin{array}{ccc} hX \otimes hY & \longrightarrow & hY \otimes hX \\ \downarrow & & \downarrow \\ h(X \# Y) & \longrightarrow & h(Y \# X) \end{array}$$

commutes, where the top row is $x \otimes y \to (-1)^{\deg x \deg y} y \otimes x$ and the bottom is the value of h on the transposition isomorphism $X \# Y \approx Y \# X$.

For such a multiplicative homology theory $hX \otimes hY \to h(X \# Y)$ factors canonically as

$$hX \otimes hY \to hX \otimes_\Lambda hY \to h(X \# Y),$$

and we have the small Künneth theorem.

PROPOSITION 8.1. *If hX is a flat Λ-module then $hX \otimes_\Lambda hY \to h(X \# Y)$ is an isomorphism.*

For $Y \to hX \otimes_\Lambda hY$ and $Y \to h(X \# Y)$ are both homology theories.

Thus if $X \in \mathcal{W}$ and hX is Λ-flat, and δ_X is the diagonal map of X then

$$\delta_X^+ : X^+ \to (X \times X)^+ = X^+ \# X^+$$

gives to hX^+ the structure of a commutative coalgebra over Λ. If $f : Y \to X$ in $\mathcal{W}$ the composition

$$Y^+ \xrightarrow{\delta_Y^+} Y^+ \# Y^+ \xrightarrow{1 \# f^+} Y^+ \# X^+$$

gives to hY^+ the structure of a comodule over hX^+.

Now if $B \in \mathcal{W}$ and $s : B \to P$ the adjunction $s_!^\bullet \dashv s^!$ gives a natural transformation $\theta : 1 \to s^! s_!^\bullet$; for $X \in \mathcal{W}^\bullet_B$ this is just the quotient of $x \to (p_X x, x)$. If further hB^+ is Λ-flat then $s_!^\bullet \theta_X : s_!^\bullet X \to s_!^\bullet s^! s_!^\bullet X = B^+ \# s_!^\bullet X$ gives to $hs_!^\bullet X$ the structure of a hB^+-comodule.

LEMMA 8.2. *If $B \in \mathcal{W}$ and hB^+ is Λ-flat, and $s : B \to P$, then $hs_!^\bullet$ factors canonically as $\mathcal{W}^\bullet_B \xrightarrow{h_B}$ comod $(hB^+) \xrightarrow{\nu} \mathfrak{Ab}^\infty$ where h_B is a homology theory on $\mathcal{W}^\bullet_B$ and ν is the forgetful functor.*

We recall [5] that a homology theory on a pointed hc-category factors canonically through the stable homotopy category. We shall again denote by h_B the resulting functor $h_B : \mathrm{Stab}_\Sigma\,(\mathcal{W}^{\bullet}_B)^{\square} \to \mathrm{comod}\ (hB^+)$.

Now comod (hB^+) has a relative abelian structure (cf. for example [4]) whose proper exact sequences are those which are split exact as sequences of Λ-modules. The corresponding relative injectives are the extended comodules $hb^+ \otimes_\Lambda M$, $M \in \mathrm{mod}\ (\Lambda)$ and their retracts.

Thus if $Y \in \mathcal{W}^{\bullet}$ then $h_B\, s^! Y = h s^{\bullet}_!\, s^! Y = h(B^+ \# Y) = hB^+ \otimes_\Lambda hY$ is a relative injective.

If $X \in \mathcal{W}^{\bullet}_B$ the adjunction $s^{\bullet}_! \dashv s^!$ gives in addition to

$$\theta_X : X \to s^! s^{\bullet}_!\, X$$

a morphism $\mu : s^{\bullet}_!\, s^! s^{\bullet}_!\, X \to s^{\bullet}_!\, X$ such that $\mu(s^{\bullet}_!\, \theta_X) = 1$.

LEMMA 8.3. *If*

$$\Sigma^{-1} s^! s^{\bullet}_!\, X \to X' \xrightarrow{\ x\ } X \to s^! s^{\bullet}_!\, X$$

is a cofibration triangle in $\mathrm{Stab}_\Sigma\,(\mathcal{W}^{\bullet}_B)^{\square}$ *then* $h_B\, x = 0$ *and*

$$0 \to h_B\, x \to h_B\, s^! s^{\bullet}_!\, X \to h_B\, \Sigma X' \to 0$$

is a proper short exact sequence for the relative abelian structure of comod (hB^+).

9. Homology of pullbacks: 0^{th} approximation

Let us suppose that $w : B \to C$ in $\mathcal{W}$ and write $s : B \to P$, $t : C \to P$ so that $tw = s$. If $X \in \mathcal{W}_C$ then $\theta_X : X \to t^! t^{\bullet}_!\, X$ so that

$$s^{\bullet}_!\, w^! \theta_X : s^{\bullet}_!\, w^! X \to s^{\bullet}_!\, w^! t^! t^{\bullet}_!\, X = s^{\bullet}_!\, s^! t^{\bullet}_!\, X = B^+ \# t^{\bullet}_!\, X.$$

Up to identifications this map is essentially the inclusion of $w^! X = B \times_C X$ in $B \times C$. Thus it is easy to see that the compositions of $s^{\bullet}_!\, w^! \theta_x$ with the maps

$$\alpha = (B \times w)^+ \delta^+_B \# t^{\bullet}_!\, X,$$

$$\beta = B^+ \# t^{\bullet}_!\, \theta_X : B^+ \# t^{\bullet}_!\, X \to B^+ \# C^+ \# t^{\bullet}_!\, X$$

are equal.

If hC^+ is Λ-flat then $(B \times w)^+ \delta^+_B$ and $t^{\bullet}_!\, \theta_X$ are the maps which give to hB^+ and $h_C\, X$ their structure as hC^+-comodules. If hB^+ is also Λ-flat we may accordingly construct a commutative diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
h s^{\bullet}_!\, w!\, X = h_B\, w!\, X & \xrightarrow{\ \varphi_x\ } & hB^+ \,\square_{hC^+}\, h_C\, X \\
{\scriptstyle h s^{\bullet}_!\, w!\, \theta_X}\downarrow & & \downarrow \\
h(B^+ \# t^{\bullet}_!\, X) & \longrightarrow & hB^+ \otimes_\Lambda h_C\, X \\
{\scriptstyle h\alpha - h\beta}\downarrow & & \downarrow \\
h(B^+ \# C^+ \# t^{\bullet}_!\, X) & \longrightarrow & hB^+ \otimes_\Lambda hC^+ \otimes_\Lambda h_C\, X
\end{array}$$

in which $\square$ denotes the cotensor product, so that the right-hand column is exact, and in which the unmarked horizontal arrows represent the inverses of the Künneth isomorphisms.

We shall refer to the state of affairs just described, *viz.* $w : B \to C$ in $\mathcal{W}$, $s : B \to P$, $t : B \to C$, hB^+ and hC^+ both Λ-flat, as the *standard pullback situation.* Inasmuch as all the functors and natural transformations we have used are stable with respect to X we arrive at the following conclusion.

LEMMA 9.1. *In the standard pullback situation φ is a natural transformation of the functors*

$$h_B\, w^!,\ hB^+ \,\square_{hC^+}\, h_C : \mathrm{Stab}_\Sigma\ (\mathcal{W}^\bullet_C)^\square \to \mathrm{comod}\ (hB^+).$$

If $X = t^!Y$, $Y \in \mathrm{Stab}_\Sigma\ \mathcal{W}^{\bullet\square}$ then φ_X is an isomorphism.

It is sufficient to prove the latter assertion in the unstable case. If $Y \in \mathcal{W}^\bullet$ then $X = C \times Y$, $t^\bullet_! X = C^+ \# Y$, $w^!X = B \times Y$, $s^\bullet_!\, w^!X = B^+ \# Y$.

10. The pullback spectral sequence

We suppose ourselves in the standard pullback situation. If

$$X \in \mathrm{Stab}_\Sigma\ (\mathcal{W}^\bullet_C)^\square$$

we may construct a diagram

$$\begin{array}{ccccccccc}
 & & \Sigma^{-1}Y_0 & & \Sigma^{-1}Y_1 & & \Sigma^{-1}Y_2 & & \\
 & & \downarrow{\scriptstyle \tilde{x}_0} & & \downarrow{\scriptstyle \tilde{x}_1} & & \downarrow{\scriptstyle \tilde{x}_2} & & \\
\cdots & \longrightarrow & X_{-1} & \xrightarrow{x_0} & X_0 & \xrightarrow{x_1} & X_1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \hat{x}_{-1}} & & \downarrow{\scriptstyle \hat{x}_0} & & \downarrow{\scriptstyle \hat{x}_1} & & \\
 & & Y_{-1} & & Y_0 & & Y_1 & &
\end{array} \tag{10.1}$$

with the following properties:

(i) $X_p = X,\ p \geq 0;\ x_p = 1_X,\ p \geq 1;$

(ii) $Y_p = t^! t^\bullet_!\, X_p,\ p \leq 0;\ \hat{x}_p = \theta_X,\ p \leq 0;$

(iii) $$\Sigma^{-1}Y_p \xrightarrow{\tilde{x}_p} -X_{p-1} \xrightarrow{x_p} X_p \xrightarrow{\hat{x}_p} Y_p$$

is a cofibration triangle, all p.

For $p < 0$, of course, this is done inductively.

If we set $C^1_p = h\Sigma^{-p}X_p$, $E^1_p = h_C\,\Sigma^{-p}$ and provide these comodules with the homomorphisms coming in the obvious way from 10.1 they constitute an exact couple in comod (hC^+). If this is regarded as an exact couple of bigraded abelian groups (so that $C^1_{pq} = h_{p+q} X_p$, $E^1_{pq} = h_{p+q} Y_p$) it has morphisms of the usual bidegrees.

From 8.3 we conclude that $C^2_p = 0$, $p \leq 0$, while $C^2_p = h_C \Sigma^{-p} X$ for $p > 0$. Since in the first derived couple

$$C^2_p \to C^2_{p+1} \to E^2_p \to C^2_{p-1} \to C^2_p$$

is exact it follows that $E^2_p = 0$, $p \neq 0$ while $E^2_0 = h_C X$. Thus

$$0 \to C^1_0 \to E^1_0 \to E^1_{-1} \to E^1_{-2} \to \cdots$$

is exact. But, again by 8.3, it is split exact as a sequence of Λ-modules and, according to 8.1, E^1_p is a relative injective.

Lemma 10.2. *(E^1, d^1) is a relative injective resolution of $h_C X$.*

If we apply the functor $w^!$ to 10.1 we get a similar diagram in $\mathrm{Stab}_\Sigma (\mathcal{W}^\bullet_B)^\square$, to which we may apply the same construction, getting an exact couple in comod (hB^+) which we denote by ${}^wC^1_p = h_B \Sigma^{-p} w^! X_p$, ${}^wE^1_p = h_B \Sigma^{-p} w^! Y_p$. This is the *pullback exact couple* and its associated spectral sequence is the *pullback spectral sequence* of Eilenberg and Moore.

The pullback exact couple depends on a sequence of choices and is not functorial. Its first derived couple however is.

Proposition 10.3.

$$X \mapsto ({}^wC^2 \to {}^wC^2 \to {}^wE^2 \to {}^wC^2)$$

is a functor from $\mathrm{Stab}_\Sigma (\mathcal{W}^\bullet_C)^\square$ *to the category of exact couples in* comod (hB^+).

This results from a standard argument which need not be repeated *in extenso* here. In outline, if $f : X \to \bar{X}$ in $\mathrm{Stab}_\Sigma (\mathcal{W}^\bullet_C)^\square$ there is a morphism of the corresponding diagrams 10.1 which at X_0 is just f; this in turn leads to a morphism of the exact couples. It suffices to show that if $f = 0$ then the first derived morphism is 0.

Corollary 10.4. *The filtration of $h_B w^! X$ given by*

$$\Phi^p (h_B w^! X) = \mathrm{im}\ (h_B w^! X_p \to h_B w^! X)$$

is functorial on $\mathrm{Stab}_\Sigma (\mathcal{W})^{\bullet\square}_C$.

The term ${}^wE^2$ of the pullback spectral sequence may be computed as follows.

Proposition 10.5. ${}^wE^2_{pq} = \mathrm{Cotor}^{hC^+}_{pq} (hB^+, h_C X)$.

This is an immediate consequence of 9.1, 10.2. The "cotor" which appears here is *a priori* the relative derived functor of the cotensor product. However, since we have assumed that hB^+ is Λ-flat this is isomorphic to the absolute derived functor.

11. Convergence in the pullback spectral sequence

In the presence of certain additional conditions we shall make an assertion about convergence in the pullback spectral sequence of §10. We shall need first some data about the reduced integral singular homology H of fibrations.

If $B \in \mathfrak{J}$ then to each $X \in \mathrm{Fib}_B^\bullet$ is associated the local coefficient system on B defined by the homology groups of the fibers. We shall assume that B is arcwise connected so that these are all isomorphic, X will be called *simple* if the local coefficient system is constant, so that they are canonically isomorphic. In this case we denote the constant value by HFX. The property of being simple is invariant under fiber-homotopy equivalence, so that it may be asserted in $(\mathrm{Fib}_B^\bullet)^\square$, and also under suspension in this homotopy category, so that it is, finally, a property of objects in $\mathrm{Stab}_\Sigma\ (\mathrm{Fib}_B^\bullet)^\square$. The notation HFX may also be used, then, for X in this stable homotopy category whenever X is simple.

If B is simply connected then of course every X is simple. Further, any pullback of a simple X is simple.

If $X \in \mathrm{Stab}_\Sigma\ (\mathrm{Fib}_B^\bullet)^\square$ is simple the Serre spectral sequence for X (with respect to reduced homology) begins with $E_{pq}^2 = H_p(B^+, H_q F)$ and converges strongly to $Hs_!^\bullet X$, where $s : B \to P$ (this is more familiar perhaps for $X \in \mathrm{Fib}_B^\bullet$, but the generalization is trivial).

LEMMA 11.1 *If $X \in \mathrm{Stab}_\Sigma\ (\mathrm{Fib}_B^\bullet)^\square$ is simple then $H_q\ s_!^\bullet X = 0$ for $q < q_0$ if and only if $H_q FX = 0$ for $q < q_0$.*

We now suppose

$$B \xrightarrow{\ w\ } C \xrightarrow{\ t\ } P,$$

$s = tw$ in $\mathcal{W}$, and observe that if $X \in \mathrm{Stab}_\Sigma\ (\mathcal{W}\ \mathrm{fib}_C^\bullet)^\square$ then the diagram 10.1 may be constructed entirely within that category.

LEMMA 11.2 *If C is arcwise connected and simply connected and $H_q\ t_!^\bullet X = 0$, $q < q_0$ then for $q < q_0 - n$,* (i) $H_q t_!^\bullet X_n = 0$, (ii) $H_q FX_n = 0$, (iii) $H_q s_!^\bullet s^! X_n = 0$.

The first assertion follows inductively from the exactness of the H-homology sequence of

$$t_!^\bullet\, \Sigma^{-1} Y_h \to t_!^\bullet\, X_{n-1} \to t_!^\bullet\, X_n \to t_!^\bullet\, Y_n$$

together with the Künneth theorem for $t_!^\bullet\, Y_n = B^+ \# \ t_!^\bullet\, X_n$, which shows that $t_!^\bullet\, X_n$ and $t_!^\bullet\, Y_n$ have the same homology in the two lowest degrees in which it does not vanish. The remaining two assertions are immediate consequences of 11.1.

Now suppose that h is, once more, a multiplicative homology theory on $\mathcal{W}^\bullet$ such that hB^+, hC^+ are $\Lambda = hS^0$ flat, so that we are in the standard pullback situation. We shall say that h is *connective* if $h_q\, S^0 = 0$, $q < 0$ (the generalization $q < q_0$ is empty). This implies of course $h_q\, X = 0$, $q < 0$ for all $X \in \mathcal{W}^\bullet$.

THEOREM 11.3. *In the standard pullback situation let h be connective and suppose that C is simply connected. Then, for any $X \in \mathrm{Stab}_\Sigma\ (\mathcal{W}\ \mathrm{fib}^\bullet_C)^\square$,*

(i) *the filtration Φ of $h_n\ s^\bullet_!\ w^!X$ is finite for any n,*

(ii) *for each p, q and sufficiently large r,*

$$ {}^wE^r_{pq} = {}^wE^{r+1}_{pq} = \cdots = {}^wE^\infty_{pq}, $$

(iii) $\Phi^p(h_{p+q}\ s^\bullet_!\ w^!X)/\Phi^{p-q}(h_{p+q}\ s^\bullet_!\ w^!X) \approx {}^wE^\infty_{pq}$.

Quite generally, of course, $\Phi^0 h_n\ s^\bullet_!\ w^!X = h_n\ s^\bullet_!\ w^!X$; in virtue of Lemma 11.2 and the connectivity of h we conclude that $\Phi^{-p}h_n\ s^\bullet_!\ w^!X = 0$ for large p. Assertion (ii) results from the fact that if $h_q\ t^\bullet_!\ X = 0$, $q < q_0$ then $E^1_{pq} = 0$ except for $-q/2 + q_0 \leq p \leq 0$, which is again a consequence of 11.2, since

$$ h_{p+q}\ s^\bullet_!\ w^!X_p = C^1_{pq} \to E^1_{pq} \to C^1_p = h_{p+q-1}\ s^\bullet_!\ w^!X_{p-1} $$

is exact.

12. Homology in ${}_G\mathcal{W}^\bullet$

If G is a group in $\mathcal{W}$ with unit $u : P \to G$ and multiplication

$$ \mu : G \times G \to G $$

then for any multiplicative homology theory h, $hu^+ : hS^0 \to hG^+$ and the composition

$$ hG^+ \otimes_\Lambda hG^+ \to h(G^+ \# G^+) \xrightarrow{h\mu^+} hG^+ $$

give to hG^+ the structure of a Λ-algebra. If Y is a right G-space with operation $\eta : Y \times G \to Y$ then

$$ hY^+ \otimes hG^+ \to h(Y^+ \# G^+) \xrightarrow{h\eta^+} hY^+ $$

makes hY^+ a right hG^+-module.

If $Y \in {}_G\mathcal{W}^\bullet$ the adjunction of the forgetful functor with $G^+ \# -$ gives

$$ \gamma_Y : G^+ \# Y \to Y; $$

this map is also characterized by the fact that its composition with $G \times Y \to G^+ \# Y$ is the operation of G on Y. Thus we have

$$ hG^+ \otimes_\Lambda hY \to h(G^+ \# Y) \to hY, $$

giving hY the structure of a left hG^+-module. In view of the evident behavior of this structure with respect to homotopy and suspension we conclude the following.

LEMMA 12.1. *The composition*

$$ \mathrm{Stab}_\Sigma\ ({}_G\mathcal{W}^\bullet)^\square \to \mathcal{W}^\bullet \xrightarrow{h} \mathcal{A}b^\infty $$

factors canonically as

$$\mathrm{Stab}_\Sigma\ ({}_G\mathcal{W}^{\bullet})^{\square} \xrightarrow{{}^G h} \mathrm{Mod}\ (hG^+) \rightarrow \mathfrak{A}b^\infty,$$

where the unmarked arrows stand for forgetful functors and ${}^G h$ *is a homology theory.*

Now mod (hG^+) has a relative abelian structure in which the proper exact sequences are those which are split exact as sequences of Λ-modules. The corresponding projectives are the extended modules $HG^+ \otimes_\Lambda M$ and their retracts.

LEMMA 12.2. *If* hG^+ *is flat as a* Λ*-module and*

$$G^+ \# Y \rightarrow Y \xrightarrow{y} Y' \rightarrow \Sigma(G^+ \# Y)$$

is a cofibration triangle in $\mathrm{Stab}_\Sigma\ ({}_G\mathcal{W}^{\bullet})^{\square}$ *then* ${}^G h(G^+ \# Y)$ *is a relative projective,* ${}^G hy = 0$ *and* $0 \rightarrow {}^G h\Sigma^{-1}Y' \rightarrow {}^G h(G^+ \# Y) \rightarrow {}^G hY \rightarrow 0$ *is a relative short exact sequence in* $\mathrm{Mod}\ (hG^+)$.

(Compare 8.2.)

If $W \rightarrow B$ is a right principal G-bundle with $B \in \mathcal{W}$ and $Y \in {}_G\mathcal{W}^{\bullet}$ then the two compositions

$$W^+ \# G^+ \# Y \underset{W^+ \# \gamma_Y}{\overset{w^+ \# Y}{\rightrightarrows}} W^+ \# Y \xrightarrow{\lambda} s_!^{\bullet}\ (W \times^G Y),$$

where w is the operation of G on W, $s : B \rightarrow P$ and λ is the identification map are easily seen to be equal. Thus we may construct a commutative diagram

$$\begin{array}{ccc}
hE^+ \otimes_\Lambda hG^+ \otimes_\Lambda G_{hY} & \longrightarrow & h(E^+ \# G^+ \# Y) \\
\downarrow & & \downarrow{\scriptstyle h(w^+ \# Y) - h(W^+ \# \gamma_Y)} \\
hE^+ \otimes_\Lambda Gh_Y & \longrightarrow & h(E^+ \# Y) \\
\downarrow & & \downarrow{\scriptstyle h\lambda} \\
hE^+ \underset{hG^+}{\otimes} G_{hY} & \xrightarrow{\psi_Y} & hs_!^{\bullet}(W \times^G Y) \\
\downarrow & & \\
0 & &
\end{array}$$

LEMMA 12.3. ψ *is a natural transformation of the functors*

$$hE^+ \otimes_{hG^+} {}^G h\ -,\ hs_!^{\bullet}(W \times^G -) : \mathrm{Stab}_\Sigma\ ({}_G\mathcal{W}^{\bullet})^{\square} \rightarrow \mathfrak{A}b^\infty.$$

If hG^+ *is* Λ*-flat and* $Y = G^+ \# Z$ *then* ψ_Y *is an isomorphism.*

(Compare 9.1.)

13. The associated-bundle spectral sequence

We assume that W is a principal right G-bundle and that h is a multiplicative homology theory such that hG^+ is Λ-flat. If $Y \in \mathrm{Stab}_\Sigma\,({}_G\mathcal{W}^\bullet)^\square$ we may construct a diagram

$$
(13.1)\qquad
\begin{array}{ccccccccc}
 & & Z_{-1} & & Z_0 & & Z_1 & & \\
 & & \downarrow{\scriptstyle \hat{y}_{-1}} & & \downarrow{\scriptstyle \hat{y}_0} & & \downarrow{\scriptstyle \hat{y}_1} & & \\
\cdots & \longrightarrow & Y_{-1} & \xrightarrow{y_{-1}} & Y_0 & \xrightarrow{y_0} & Y_1 & \xrightarrow{y_1} & \cdots \\
 & & \downarrow{\scriptstyle \tilde{y}_{-2}} & & \downarrow{\scriptstyle \tilde{y}_{-1}} & & \downarrow{\scriptstyle \tilde{y}_0} & & \\
 & & \Sigma Z_{-2} & & \Sigma Z_{-1} & & \Sigma Z_0 & &
\end{array}
$$

with the following properties:

(i) $Y_p = Y,\ p \le 0;\ y_p = 1_Y,\ p \le -1;$
(ii) $Z_p = G^+ \# Y_p,\ \hat{y}_p = \gamma_{Y_p},\ p \ge 0;$
(iii) $Z_p \to Y_p \to Y_{p+1} \to \Sigma Z_p$ is a cofibration triangle, all p.

We set $C_p^1 = {}^G h \Sigma^{-(p+1)} Y_{p+1}$, $E_p^1 = {}^G h\Sigma^{-p} Z_p$ and provide these modules with the morphisms coming from 13.1; they constitute an exact couple in Mod (hG^+). From 12.2 we see that $C_p^2 = {}^G h\Sigma^{-(p+1)} Y$, $p < 0$, $C_p^2 = 0$, $p \ge 0$ so that $E_p^2 = 0$, $p \ne 0$, $E_0^2 = {}^G hY$.

LEMMA 13.1. *(E^1, d^1) is a relative projective resolution of ${}^G hY$.*

If we apply the functor $s_!^\bullet(W^r \times^G -)$ to the diagram 13.1 we get a similar diagram in $\mathrm{Stab}_\Sigma\,(\mathcal{W}^\bullet)^\square$ to which we may apply the same construction; the result is the *associated-bundle exact couple.* We denote its terms by

$$ {}^W C_{pq}^1 = h_{p+q+1}\, s_!^\bullet(W \times^G Y_{p+1}),\quad {}^W E_{pq}^1 = h_{p+q}\, s_!^\bullet(W \times^G Z_p). $$

LEMMA 13.2.

$$ Y \to ({}^W C^2 \to {}^W C^2 \to {}^W E^2 \to {}^W C^2) $$

is a functor from $\mathrm{Stab}_\Sigma\,({}_G\mathcal{W}^\bullet)^\square$ *to the category of exact couples of graded abelian groups.*

COROLLARY 13.3. *The filtration of $hs_!^\bullet(W \times^G Y)$ given by*

$$ \Psi^p(hs_!^\bullet(W \times^G Y)) = \ker\,(hs_!^\bullet(W \times^G Y) \to hs_!^\bullet(W \times^G Y_{p+1})) $$

is functorial.

The term ${}^W E^2$ of the associated bundle spectral sequence is easily computed.

PROPOSITION 13.4. ${}^W E_{pq}^2 = \mathrm{Tor}_{pq}^{hG+}(hW^+, {}_G hY)$.

This follows at once from 12.3, 13.1. The "Tor" which appears here is the relative derived functor of the tensor product $hW^+ \otimes_{hG^+} -$.

14. Convergence in the associated-bundle spectral sequence

In the situation of §13 we may, without further hypothesis, make the following assertion regarding convergence of the associated-bundle spectral sequence $\{{}^W E^r\}$.

THEOREM 14.1.

$$ {}^W E^\infty_{pq} = \operatorname{colim}_r E^r_{pq} \approx \Psi^p(hs^\bullet_!(W \times^G Y))/\Psi^{p-1}(hs^\bullet_!(W \times^G Y)); $$

$$ \Psi^{-1}(hs^\bullet_!(W \times^G Y)) = 0; \qquad U_p \Psi^p(hs^\bullet_!(W \times^G Y)) = hs^\bullet_!(W \times^G Y)). $$

For if we use the Atiyah-Hirzebruch-Serre spectral sequence to compute the homomorphism $hs^\bullet_!(W \times^G Y_p) \to hs^\bullet_!(W \times^G Y_{p+1})$ we observe that (by 12.2) the homomorphism of the homology of the fibers is 0, so that the homomorphism of spectral sequences is 0 at E^2. Thus $hs^\bullet_!(W \times^G Y_p) \to hs^\bullet_!(W \times^G Y_{p+r})$ decreases the Serre filtration by r, and colim $({}^W C^c_{pq} \to {}^W C^r_{p+1,q-1} \to \cdots) = 0$. The theorem then follows by purely formal arguments.

But we may further observe, by applying integral singular homology to 13.1, that if $H_q Y = 0$, $q < q_0$ then $H_q Y_p = 0$, $q < q_0 + p$. From the Atiyah-Hirzebruch-Serre spectral sequence it follows then that $H_q s^\bullet_!(W \times^G Y_p) = 0$, $q < q_0 + p$.

THEOREM 14.2. *If h is a connective homology theory then $\{{}^W E^r\}$ is a "first quadrant" spectral sequence, i.e. $E^r_{pq} = 0$ for $p < 0$ or $q < q_0$, and the canonical filtration of each $h_n s^\bullet_! (W \times^G Y)$ is finite.*

BIBLIOGRAPHY

1. J. F. ADAMS, *Lectures on generalized homology theories*, Lecture Notes in Mathematics, Springer-Verlag, New York, 1969.
2. A. DOLD, *Partitions of unity in the theory of fibrations*, Ann. of Math., vol. 78(1963), pp. 223–255.
3. ———, *Halbexakte Homotopiefunktoren*, Lecture Notes in Mathematics 12, Springer-Verlag, New York, 1966.
4. S. EILENBERG AND J. C. MOORE, *Homology and fibrations I, II*, Comment. Math. Helv., vol. 40(1966), pp. 199–236.
5. A. HELLER, *Stable homotopy categories*, Bull. Am. Math. Soc., vol. 74(1968), pp. 28–63.
6. ———, *Completions in abstract homotopy theory*, Trans. Amer. Math. Soc., vol. 147 (1970), pp. 573–602.
7. D. PUPPE, *Bemerkungen uber die Erweiterungen von Homotopien*, Arch. Math., vol. 18(1967), pp. 81–88.
8. D. RECTOR, *Steenrod operations for Eilenberg-Moore spectral sequences*, Comm. Math. Helv., vol. 45 (1970), pp. 540–552.
9. M. ROTHENBERG AND N. E. STEENROD, *Cohomology of classifying spaces of H-spaces*, Bull. Amer. Math. Soc., vol. 71 (1965), pp. 872–875.
10. L. SMITH, *On the construction of the Eilenberg-Moore spectral sequence*, Bull. Amer. Math. Soc., vol. 75(1969), pp. 873–878.

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