A CHARACTERIZATION OF SOME SPECTRAL MANIFOLDS FOR A CLASS OF OPERATORS[1]

BY

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Introduction

In this paper we shall characterize certain spectral manifolds for a class of bounded linear operators acting on a complex Banach space. Each operator T of the class has a real spectrum $\sigma(T)$ and its resolvent operator $R(\zeta; T) = (\zeta I - T)^{-1}$ satisfies an *n-th* order rate of growth (G_n) near $\sigma(T)$ in the sense that

$$(G_n) \qquad |\operatorname{Im} \zeta|^n \| R(\zeta; T) \| \leq K \quad \text{for} \quad 0 < |\operatorname{Im} \zeta| < 1,$$

$$|\operatorname{Im} \zeta| \| R(\zeta; T) \| \leq K \quad \text{for} \quad 1 \leq |\operatorname{Im} \zeta|.$$

This characterization will be as the null spaces (kernels) of certain bounded operators constructed from T by means of contour integrals. Bounded operators satisfying (G_n) were studied by R. G. Bartle [1], [2] and, independently, unbounded operators satisfying this condition were studied by the author [6]. Under additional assumptions, each operator of the class has a spectral decomposition similar to that of a self-adjoint transformation (cf. [2] or [6]).

For each bounded operator T satisfying the condition (G_n) and for each closed subset $F \subset R$ let $\mathbf{X}(F)$ denote the closed linear manifold of all vectors x whose local spectra relative to T lie in F. In §1 we review properties of operators $K(a, b)$ studied in [6] and introduced by E. R. Lorch [7] for self-adjoint operators. In §2 we introduce for each $t \in R$ operators $H_-(t)$ and $H_+(t)$ and derive their basic properties. We shall prove that $\mathbf{X}((-\infty, t])$ is the kernel of $H_+(t)$, that $\mathbf{X}([t, +\infty))$ is the kernel of $H_-(t)$, and that $\mathbf{X}([a, b])$ is the kernel of $(T - aI)^n(T - bI)^n - K(a, b)$. This characterizes $X(F)$ for any closed interval F for each T of the class. These results strengthen similar results in [6] where the author assumed that T lacked a point spectrum and then, at a later stage, assumed that T had a purely continuous spectrum. In §3, with additional hypotheses we shall obtain a spectral decomposition of T in terms of the kernel of $H_+(t)$ and the closure of its range. These manifolds yield a closed resolution of the identity for T in the sense of F. J. Murray [8]. The section concludes with a generalization of a form of the spectral theorem for self-adjoint transformations which is applied to obtain the classical integral representation for bounded self-adjoint operators.

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1. Preliminaries

Let $\mathbf{X} \neq \{0\}$ denote a complex Banach space and let $B(\mathbf{X})$ denote the algebra of all bounded linear transformations from $\mathbf{X}$ to $\mathbf{X}$. For a linear transformation T in $\mathbf{X}$ we write $\sigma(T)$ for its *spectrum* and $\rho(T)$ for its *resolvent set*. For each $\lambda \in \rho(T)$ the *resolvent operator* of T at λ is

$$R(\lambda; T) = (\lambda I - T)^{-1}.$$

If $x \in \mathbf{X}$ then by *an analytic extension* of $R(\cdot; T)x$ we shall mean an $\mathbf{X}$-valued function f analytic on an open set $D(f)$ containing $\rho(T)$ such that

$$(\lambda I - T)f(\lambda) = x$$

holds for all $\lambda \in D(f)$. T is said to have the *single-valued extension property* if for all $x \in \mathbf{X}$ and any analytic extensions f and g of $R(\cdot; T)x$ we have $f(\lambda) = g(\lambda)$ for $\lambda \in D(f) \cap D(g)$. If T has the single-valued extension property then for each $x \in \mathbf{X}$ there is a maximal analytic extension $x^\wedge(\cdot)$ of $R(\cdot; T)x$. The domain $\rho(x)$ of $x^\wedge(\cdot)$ is called the (local) *resolvent set* of x relative to T and $\sigma(x) = C - \rho(x)$ is called the (local) *spectrum* of x relative to T.

1.1. THEOREM. *Let $T \in B(\mathbf{X})$ have the single-valued extension property. If x and y are in $\mathbf{X}$ and if $\lambda \neq 0$ is a scalar, then:*

(1.1a) *$\sigma(x)$ is a compact subset of $\sigma(T)$.*

(1.1b) $\sigma(\lambda x) = \sigma(x)$.

(1.1c) *$\sigma(x)$ is empty if and only if $x = 0$.*

(1.1d) $\sigma(x + y) \subset \sigma(x) \cup \sigma(y)$.

(1.1e) *$\sigma(Ax) \subset \sigma(x)$ for each $A \in B(X)$ which commutes with T.*

(1.1f) *$\sigma(x^\wedge(\zeta)) = \sigma(x)$ for each $\zeta \in \rho(x)$.*

Parts (a) and (b) are clear. The proofs of the remaining parts can be found in [3; pp. 1–3].

We shall assume throughout that T has a real spectrum and that the resolvent operator $R(\cdot; T)$ satisfies the condition

$$(G_n) \qquad |\operatorname{Im} \zeta|^n \| R(\zeta; T) \| \leq K \quad \text{if} \quad 0 < |\operatorname{Im} \zeta| < 1,$$
$$|\operatorname{Im} \zeta| \| R(\zeta; T) \| \leq K \quad \text{if} \quad 1 \leq |\operatorname{Im} \zeta|$$

for a fixed $n \in N$ and a fixed $K > 0$. The notation $T \in (G_n)$ will be used to indicate that $T \in B(X)$ and satisfies the condition (G_n). Such operators T have the single-valued extension property.

1.2. THEOREM. *Let $T \in (G_n)$, $x \in \mathbf{X}$ and let $t \in R$. For each closed subset $F \subset R$, let $\mathbf{X}(F) = \{x \in \mathbf{X}: \sigma(x) \subset F\}$. Then:*

(1.2a) *$\sigma(x) \subset \{t\}$ if and only if $(T - tI)^n x = 0$.*

(1.2b) *$\mathbf{X}(F)$ is a closed linear manifold in $\mathbf{X}$.*

(1.2c) *For each $\epsilon > 0$ there is a $\delta > 0$ such that if J is any closed interval* *of length less than δ, then $\| (T - tI)^n x \| \leq \epsilon \| x \|$ holds for all $x \in \mathbf{X}(J)$ and all $t \in J$.*

(1.2d) *If $\sigma(x) \subset (-\infty, t]$ then there is a $c > 0$ such that*

$$| \zeta - t |^n \| x^\wedge(\zeta) \| \leq 2K \| x \|$$

holds for all ζ in the right-hand half-neighborhood of t given by

$$0 < | \zeta - t | < c, \quad \operatorname{Re} \zeta > t.$$

Proofs are given in [1; pp. 266–267]. If $\sigma(x) \subset [t, +\infty)$ then a result similar to (1.2d) holds for ζ in a left-hand half-neighborhood of t.

Let $T \in (G_n)$ and let a and b be real numbers with $a < b$. By an *admissible contour* $C(a, b)$ we shall mean a piece-wise smooth positively oriented Jordan curve which meets the real axis only at the points a and b and at nonzero angles. For each such pair a, b consider the operator given by

$$K(a, b) = (2\pi i)^{-1} \int_{C(a,b)} (\zeta - a)^n (\zeta - b)^n R(\zeta; T) \, d\zeta$$

for any admissible contour $C(a, b)$. It follows from the growth condition (G_n) that the integrand is bounded near $\sigma(T)$; hence, the integral exists in $B(\mathbf{X})$ as an improper Riemann integral. Since $R(\cdot\,; T)$ is analytic on $C - R$, it follows that the integral does not depend on the particular contour $C(a, b)$ chosen. The operators $K(a, b)$ were introduced by E. R. Lorch [7] for self-adjoint transformations and later used by N. Dunford [4], [5] in his spectral theory.

1.3. THEOREM. *Let $T \in (G_n)$ and let a and b be real numbers with $a < b$. Then $\sigma(K(a, b)x) \subset \sigma(x) \cap [a, b]$ holds for each $x \in X$.*

Proof. Let $x \in \mathbf{X}$. Choose any admissible contour $C(a, b)$ and consider the integral

$$f(\lambda) = (2\pi i)^{-1} \int_{C(a,b)} (\lambda - \zeta)^{-1} (\zeta - a)^n (\zeta - b)^n R(\zeta; T) x \, d\zeta$$

for λ in the exterior of $C(a, b)$. Clearly, f is analytic in the exterior of $C(a, b)$. Using the continuity of T, we have

$$\begin{aligned} Tf(\lambda) &= (2\pi i)^{-1} \int_{C(a,b)} (\lambda - \zeta)^{-1} (\zeta - a)^n (\zeta - b)^n T R(\zeta; T) x \, d\zeta \\ &= (2\pi i)^{-1} \int_{C(a,b)} (\lambda - \zeta)^{-1} (\zeta - a)^n (\zeta - b)^n [\zeta R(\zeta; T) x - x] \, d\zeta \\ &= (2\pi i)^{-1} \int_{C(a,b)} (\lambda - \zeta)^{-1} \zeta (\zeta - a)^n (\zeta - b)^n R(\zeta; T) x \, d\zeta. \end{aligned}$$

Hence,

$$(\lambda I - T)f(\lambda) = (2\pi i)^{-1} \int_{C(a,b)} (\zeta - a)^n(\zeta - b)^n R(\zeta; T)x \, d\zeta = K(a, b)x.$$

It follows that f is an analytic extension of $R(\cdot; T)K(a, b)x$ to the exterior of $C(a, b)$ and so $\rho(K(a, b)x)$ contains the exterior of $C(a, b)$. Since this is true for each admissible contour $C(a, b)$ it follows that $\sigma(K(a, b)x) \subset [a, b]$. The assertion now follows from (1.1e).

The author used the operators $K(a, b)$ in [6] to study spectral manifolds for unbounded operators which satisfied the growth condition (G_n) and which lacked point spectrum. In the next section we shall study these manifolds for bounded $T \in (G_n)$ by introducing new operators $H_-(t)$ and $H_+(t)$ and without the need of further assumptions concerning $\sigma(T)$.

2. Manifolds corresponding to intervals

Let $T \in (G_n)$ and let t, a and b be real numbers with $a \leq b$. We shall consider the following closed linear manifolds in $\mathbf{X}$:

$$\mathbf{X}((-\infty, t]) = \{x \in \mathbf{X} : \sigma(x) \subset (-\infty, t]\}, \quad \mathbf{X}([a, b]) = \{x : \sigma(x) \subset [a, b]\}$$

and

$$\mathbf{X}([t, +\infty)) = \{x : \sigma(x) \subset [t, +\infty)\}.$$

By (1.2a), $\mathbf{X}([a, a])$ is the kernel of the operator $(T - aI)^n$. We shall characterize the other manifolds in terms of the operators $K(a, b)$ and operators $H_-(t)$ and $H_+(t)$ which we now define.

Let $T \in (G_n)$ and let $p = \min \sigma(T)$ and $q = \max \sigma(T)$. For $t \in R$ define $H_-(t)$ and $H_+(t)$ to be the integrals

$$H_-(t) = (2\pi i)^{-1} \int_{C(s,t)} (\zeta - t)^n R(\zeta; T) \, d\zeta, \quad s < p,$$

$$H_+(t) = (2\pi i)^{-1} \int_{C(t,s)} (\zeta - t)^n R(\zeta; T) \, d\zeta, \quad s > q.$$

It follows from the growth condition (G_n) that $H_-(t)$ and $H_+(t)$ exist in $B(\mathbf{X})$. They are independent of the choice of s by the analyticity of $R(\cdot; T)$ off $[p, q]$.

2.1. Theorem. *Let $T \in (Gn)$, let $x \in \mathbf{X}$ and let t, a and b be real with $a < b$.*

(2.1a) $(T - sI)^n H_-(t) = K(s, t)$ *for* $s < p = \min \sigma(T)$,

$(T - sI)^n H_+(t) = K(t, s)$ *for* $s > q = \max \sigma(T)$.

(2.1b) $(T - tI)^n = H_-(t) + H_+(t)$.

(2.1c) $(T - aI)^n(T - bI)^n$

$$= (T - bI)^n H_-(a) + K(a, b) + (T - aI)^n H_+(b).$$

(2.1d) $\sigma(H_-(t)x) \subset \sigma(x) \cap (-\infty, t]$

and $\sigma(H_+(t)x) \subset \sigma(x) \cap [t, +\infty)$.

Proof. The first three parts are proved readily by the use of contour integrals. From part (a) it then follows that

$$H_-(t) = (-1)^n R(s; T)^n K(s, t) \quad \text{for } s < p,$$

$$H_+(t) = (-1)^n R(s; T)^n K(t, s) \quad \text{for } s > q.$$

Part (d) now follows from (1.1e) and Theorem 1.3.

From (2.1b) and (2.1d) it follows that each vector $y \in (T - tI)^n \mathbf{X}$, $t \in R$, can be written as a sum $y = y' + y''$ with $\sigma(y') \subset (-\infty, t]$ and $\sigma(y'') \subset [t, +\infty)$. This decomposition was basic in the developments [2] and [6] of a spectral decomposition for $T \in (G_n)$. The next result characterizes the manifolds $\mathbf{X}((-\infty, t])$ and $\mathbf{X}([t, +\infty))$ in terms of the operators $H_+(t)$ and $H_-(t)$.

2.2. THEOREM. *Let $T \in (G_n)$ and let $t \in R$.*

(2.2a) $\mathbf{X}((-\infty, t])$ *is the kernel of* $H_+(t)$.

(2.2b) $\mathbf{X}([t, +\infty))$ *is the kernel of* $H_-(t)$.

Proof. We shall prove (a); the proof of (b) is similar.

First, suppose $x \in \mathbf{X}$ is such that $H_+(t)x = 0$. Choose any $u > t$ and then choose $v > \max\{u, \max \sigma(T)\}$. For any admissible contour $C(t, v)$ we have

$$0 = H_+(t)x = (2\pi i)^{-1} \int_{C(t,v)} (\zeta - t)^n R(\zeta; T)x \, d\zeta.$$

Let

$$g(\lambda) = (\lambda - t)^{-n}(2\pi i)^{-1} \int_{C(t,v)} (\zeta - \lambda)^{-1}(\zeta - t)^n R(\zeta; T)x \, d\zeta.$$

Clearly, g is analytic if λ is in the interior of $C(t, v)$. By the continuity of T, we have

$$\begin{aligned} Tg(\lambda) &= (\lambda - t)^{-n}(2\pi i)^{-1} \int_{C(t,v)} (\zeta - \lambda)^{-1}(\zeta - t)^n TR(\zeta; T)x \, d\zeta \\ &= (\lambda - t)^{-n}(2\pi i)^{-1} \int_{C(t,v)} (\zeta - \lambda)^{-1}(\zeta - t)^n [\zeta R(\zeta; T)x - x] \, d\zeta \\ &= (\lambda - t)^{-n}(2\pi i)^{-1} \int_{C(t,v)} (\zeta - \lambda)^{-1}(\zeta - t)^n \zeta R(\zeta; T)x \, d\zeta - x. \end{aligned}$$

Hence, $(\lambda I - T)g(\lambda) = -(\lambda - t)^{-n} H_+(t)x + x = x$ for each λ that is in the interior of $C(t, v)$. Thus $(t, v) \subset \rho(x)$ and, in particular, $u \in \rho(x)$. Since this holds for each $u > t$ we have $(t, +\infty) \subset \rho(x)$ so that $(-\infty, t] \supset \sigma(x)$.

Conversely, suppose $\sigma(x) \subset (-\infty, t]$. By (1.2d) there is a $c > 0$ such that if $0 < |\zeta - t| < c$ and $\operatorname{Re} \zeta > t$, then $|\zeta - t|^n \| x^\wedge(\zeta) \| \leq 2K \| x \|$ Choose $u > t$ with u in this right-hand half-neighborhood of t and choose for $C(t, u)$ the circle with center $(\frac{1}{2}(u + t), 0)$ and radius $r = \frac{1}{2}(u - t)$. Using

$$H_+(t)x = (2\pi i)^{-1} \int_{C(t,u)} (\zeta - t)^n R(\zeta; T)x \, d\zeta$$

we obtain $\| H_+(t)x \| \leq (2\pi)^{-1}(2K\| x \|)(2\pi r) = 2Kr\| x \|$. If we let $u \to t+$, we obtain $H_+(t)x = 0$. This completes the proof of (a).

2.3. THEOREM. *Let $T \in (G_n)$ and let $a < b$. Then $\mathbf{X}([a, b])$ consists precisely of those vectors x such that*

$$(T - aI)^n(T - bI)^n x = K(a, b)x.$$

Proof. First, suppose $\sigma(x) \subset [a, b]$. Then

$$\sigma(x) \subset (-\infty, b] \cap [a, +\infty)$$

so that $H_-(a)x = H_+(b)x = 0$ by Theorem 2.2. The relation

$$(T - aI)^n(T - bI)^n x = K(a, b)x$$

now follows from (2.1c).

Conversely, suppose that $x \in \mathbf{X}$ is such that

$$(T - aI)^n(T - bI)^n x = K(a, b)x.$$

By (2.1c) we have $(T - aI)^n H_+(b)x = -(T - bI)^n H_-(a)x$. Using (2.1d) and (1.1e), we infer that this vector has spectrum in

$$(-\infty, a] \cap [b, +\infty) = \emptyset$$

and so is the zero vector by (1.1c). It then follows from (1.2a) and (2.1d) that $H_+(b)x = H_-(a)x = 0$. Hence, $\sigma(x) \subset (-\infty, b] \cap [a, +\infty) = [a, b]$ by Theorem 2.2. This completes the proof.

Theorems 2.2 and 2.3 show that the manifolds

$$\mathbf{X}((-\infty, t]), \quad \mathbf{X}([t, +\infty)) \quad \text{and} \quad \mathbf{X}([a, b])$$

are the kernels of the operators

$$H_+(t), \quad H_-(t) \quad \text{and} \quad K(a, b) - (T - aI)^n(T - bI)^n,$$

respectively. This characterizes the manifolds $\mathbf{X}(J)$ for each closed interval J. By means of arguments similar to those in the proofs of Theorems 2.2 and 2.3 one can show that the sum $\mathbf{X}((-\infty, a]) + \mathbf{X}([b, +\infty))$ is the kernel of $K(a, b)$. We will not prove this result as it is not needed in the sequel. We remark that this last assertion as well as Theorem 2.3 can be proved for unbounded transformations which satisfy the growth condition (G_n) which are closed and have dense domains.

With additional hypotheses the results of this section will be used in the next section to obtain reducing manifolds for the operators $T \in (G_n)$ in terms of the kernels and ranges of the operators $H_+(t)$.

3. Resolving manifolds

If T is a spectral operator in the sense of N. Dunford [5], [4], then for each closed set F the manifold $\mathbf{X}(F)$ has a (full) complement in $\mathbf{X}$. In general, for $T \in (G_n)$ we do not expect a complement for $\mathbf{X}(J)$, J a closed interval. In this section we shall impose further conditions on T and $\mathbf{X}$ in order to obtain a quasi-complement for $\mathbf{X}((-\infty, t])$ for each $t \in R$. According to F. J. Murray [8], two closed linear manifolds $\mathbf{M}$ and $\mathbf{N}$ in $\mathbf{X}$ are *quasi-complements* in $\mathbf{X}$ if $\mathbf{M} \cap \mathbf{N} = \{0\}$ and $\mathbf{M} + \mathbf{N}$ is dense in $\mathbf{X}$. This is equivalent to the existence of a closed projection (that is, a closed densely-defined idempotent) in $\mathbf{X}$ which has $\mathbf{M}$ as its range and $\mathbf{N}$ as its null space (kernel). In the sequel we shall denote the closure of $\mathbf{M} + \mathbf{N}$ by $\mathbf{M} \vee \mathbf{N}$. If $\{\mathbf{M}_1, \cdots, \mathbf{M}_k\}$ is any finite collection of subspaces $\mathbf{M}_j$ of $\mathbf{X}$, then $\bigvee_{j=1}^{k} \mathbf{M}_j$ or $\mathbf{M}_1 \vee \cdots \vee \mathbf{M}_k$ shall denote the smallest closed subspace of $\mathbf{X}$ that contains all the $\mathbf{M}_j$.

If $T \in (G_n)$ and if $t \in R$ we shall denote the kernel of $H_+(t)$ by $\mathbf{M}(t)$ and the closure of $H_+(t)\mathbf{X}$ by $\mathbf{N}(t)$. Note that $\mathbf{M}(t) = \mathbf{X}((-\infty, t])$. We shall indicate two cases (Theorems 3.2 and 3.3) in which $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are quasi-complements in $\mathbf{X}$ for each $t \in R$. In the first of the two cases we shall require the following Lemma which is due to N. Dunford ([4; Lemma 13, pp. 261–262] or [5; Lemma 13, pp. 2159–2160]). We shall refer to it as "Dunford's Lemma 13."

3.1. LEMMA. *If $\mathbf{X}$ is reflexive and if $T \in (G_1)$, then for each $t \in R$ the manifold $(T - tI)\mathbf{X} + \{x \in \mathbf{X} : (T - tI)x = 0\}$ is dense in $\mathbf{X}$.*

3.2. THEOREM. *If $\mathbf{X}$ is reflexive and if $T \in (G_1)$, then $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are quasi-complements in $\mathbf{X}$ for each real t.*

Proof. Let $t \in R$. We infer from Theorem 2.1(b, d) and Theorem 2.2 that

$$(T - tI)\mathbf{X} = [H_-(t) + H_+(t)]\mathbf{X} \subset H_-(t)\mathbf{X} + H_+(t)\mathbf{X} \subset \mathbf{M}(t) + \mathbf{N}(t).$$

Also,

$$\{x \in \mathbf{X} : (T - tI)x = 0\} \subset \mathbf{X}([t, t]) \subset \mathbf{M}(t)$$

by Theorem 2.2. Hence,

$$(T - tI)\mathbf{X} + \{x : (T - tI)x = 0\} \subset \mathbf{M}(t) + \mathbf{N}(t)$$

and so the conclusion $\mathbf{X} = \mathbf{M}(t) \vee \mathbf{N}(t)$ follows from Lemma 3.1. To show that $\mathbf{M}(t) \cap \mathbf{N}(t) = \{0\}$, we let T^* be the conjugate to the operator T and we let $K_+(t)$ be the operator in $B(\mathbf{X}^*)$ given by

$$K_+(t) = (2\pi i)^{-1} \int_{C(t,s)} (\zeta - t)R(\zeta; T^*)\, d\zeta$$

where $s > \max \sigma(T) = \max \sigma(T^*)$.

We note that $T \in (G_1)$ implies that $T^* \in (G_1)$ since $R(\zeta; T^*) = R(\zeta; T)^*$ for all $\zeta \in \rho(T^*) = \rho(T)$. If we let $\mathbf{M}(t)^*$ and $\mathbf{N}(t)^*$ denote the kernel and the closure of the range of $K_+(t)$, respectively, then we infer

$$\mathbf{X}^* = \mathbf{M}(t)^* \vee \mathbf{N}(t)^*$$

by the argument above. Taking ortho-complements in $\mathbf{X}$ and using the fact that $K_+(t)$ is the conjugate to $H_+(t)$ we infer that $\{0\} = \mathbf{M}(t) \cap \mathbf{N}(t)$. This completes the proof that $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are quasi-complements.

We remark that a result similar to 3.2 was obtained by Bartle [2; Lemma 3.6] and, independently, by W. R. Parzynski [9; Theorem 2.12] for unbounded operators. However, they used

$$\mathbf{X}((-\infty, t]) \cap \mathrm{Cl}\ \mathrm{Range}\ (T - tI) \quad \text{and} \quad \mathbf{X}([t, +\infty))$$

in the place of $\mathbf{M}(t)$ and $\mathbf{N}(t)$, respectively.

3.3. THEOREM. *If $T \in (G_n)$ and if T has a purely continuous spectrum, then the manifolds $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are quasi-complements in $\mathbf{X}$ for each real t.*

Proof. Let $t \in R$. By the density of $(T - tI)\mathbf{X}$ in $\mathbf{X}$ and induction it follows that $(T - tI)^n\mathbf{X}$ is dense in $\mathbf{X}$. Using (2.1b) we infer that

$$\mathbf{M}(t) \vee \mathbf{N}(t) = \mathbf{X},$$

as in the proof of 3.2. Let $x \in \mathbf{M}(t) \cap \mathbf{N}(t)$. Then $H_-(t)x = H_+(t)x = 0$ and by Theorem 2.2 we have

$$\sigma(x) \subset (-\infty, t] \cap [t, +\infty) = \{t\}.$$

Thus $(T - tI)^n x = 0$ by (1.2a), and it follows from the invertibility of $T - tI$ that $x = 0$. Hence, $\mathbf{M}(t) \cap \mathbf{N}(t) = \{0\}$ and this completes the proof.

3.4. COROLLARY. *If $T \in (G_n)$ and if T has a purely continuous spectrum, then the manifolds $\mathbf{X}((-\infty, t])$ and $\mathbf{X}([t, +\infty))$ are quasi-complements in $\mathbf{X}$ for each real t.*

Proof. Using (2.1d) and (2.2b) we infer that $\mathbf{N}(t) = \mathrm{Cl}\ (H_+(t)\mathbf{X})$ is contained in $\mathbf{X}([t, +\infty))$. Also, $\mathbf{M}(t) = \mathbf{X}((-\infty, t])$. Hence, by 3.3 we have

$$\mathbf{X}((-\infty, t]) \vee \mathbf{X}([t, +\infty)) \supset \mathbf{M}(t) \vee \mathbf{N}(t) = \mathbf{X}$$

and so $\mathbf{X} = \mathbf{X}((-\infty, t]) \vee \mathbf{X}([t, +\infty))$. If $x \in \mathbf{X}((-\infty, t]) \cap \mathbf{X}([t, +\infty))$ then $\sigma(x) \subset \{t\}$. From (1.2a) it follows that $(T - tI)^n x = 0$, whence, $x = 0$ by the invertibility of $T - tI$. Thus the manifolds $\mathbf{X}((-\infty, t])$ and $\mathbf{X}([t, +\infty))$ have only the zero vector in common and this completes the proof that they are quasi-complements.

The result 3.4 was obtained by the author [6; Theorem 3.3] for unbounded operators satisfying the growth condition (G_n) and, independently, by Bartle [2; Theorem 2.5] for bounded operators.

We abstract the situations in 3.2 and 3.3 and *assume* in the next theorem that $\mathbf{X}$ and T are such that for each real t the manifolds $\mathbf{M}(t)$ (the null space of $H_+(t)$) and $\mathbf{N}(t)$ (the closure of the range of $H_+(t)$) are quasi-complements in $\mathbf{X}$. If this is the case, then for each $t \in R$ there is a closed projection $E(t)$ having $\mathbf{M}(t)$ as range and $\mathbf{N}(t)$ as null space. The family

$$\mathcal{E} = \{E(t) : t \in R\}$$

is a closed resolution of the identity for T in the sense of [8].

3.5. Theorem. *Let $T \in (G_n)$ and suppose that $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are quasi-complements in $\mathbf{X}$ for each $t \in R$. Let $E(t)$ be the closed projection having$\mathbf{M}(t)$ as range and $\mathbf{N}(t)$ as null space.*

(3.5a) *$E(t) = 0$ for $t < \min \sigma(T)$, and $E(t) = I$ for $t > \max \sigma(T)$.*

(3.5b) *Each $E(t)$ commutes with all $A \in B(X)$ which commute with T.*

(3.5c) *If $s < t$, then $E(s) \leq E(t)$ in the sense that $\mathbf{M}(s) \subset \mathbf{M}(t)$ and $\mathbf{N}(s) \supset \mathbf{N}(t)$.*

Proof. Parts (a) and (b) are clear. For (c) we note that

$$\mathbf{M}(s) = \mathbf{X}((-\infty, s]) \subset \mathbf{X}((-\infty, t]) = \mathbf{M}(t) \quad \text{whenever } s < t.$$

We shall assume that $\min \sigma(T) \leq s < t \leq \max \sigma(T)$ in proving that $\mathbf{N}(s) \supset \mathbf{N}(t)$ if $s < t$; the assertion is clear otherwise. For this, let $y = H_+(t)x \in H_+(t)\mathbf{X}$. Then $\sigma(y) \subset [t, +\infty)$ by (2.1d). Since $s < t$ we have $s \in \rho(y)$ so that $(sI - T)y^\wedge(s) = y$. If we let $y_1 = -y^\wedge(s)$, then $y = (T - sI)y_1$ and $\sigma(y_1) = \sigma(y)$ by (1.1f). Thus $s \in \rho(y_1)$ and this argument implies the existence of a $y_2 \in \mathbf{X}$ such that

$$y_1 = (T - sI)y_2 \text{ with } \sigma(y_2) = \sigma(y_1).$$

Continuing in this manner we obtain $y_n \in \mathbf{X}$ such that $y = (T - sI)^n y_n$ with $\sigma(y_n) = \sigma(y)$. By Theorem 2.2 we have $H_-(s)y_n = 0$; hence, by (2.1b), we infer

$$y = (T - sI)^n y_n = H_+(s)y_n.$$

Thus $y \in H_+(s)\mathbf{X}$. This shows that $H_+(t)\mathbf{X} \subset H_+(s)\mathbf{X}$ and so $\mathbf{N}(t) \subset \mathbf{N}(s)$.

The hypotheses of the next theorem are satisfied in the case where $\mathbf{X}$ is reflexive and T satisfies a first order rate of growth (G_1) as well as in the case where $\mathbf{X}$ is arbitrary and T has a purely continuous spectrum.

3.6. Theorem. *Let $T \in (Gn)$ and suppose that for each $t \in R$ the manifold $(T - tI)\mathbf{X} + \{x \in \mathbf{X}: (T - tI)x = 0\}$ is dense in $\mathbf{X}$. If a, b and t are real with $a < b$, then*

(3.6a) $\mathbf{X} = \mathbf{X}((-\infty, t]) \vee \mathbf{X}([t, +\infty))$ *and*

(3.6b) $\mathbf{X} = \mathbf{X}((-\infty, a]) \vee \mathbf{X}([a, b]) \vee \mathbf{X}([b, +\infty))$.

Proof. For (a). We obtain by induction the density in $\mathbf{X}$ of

$$(T - tI)^n\mathbf{X} + \{x \in \mathbf{X} : (T - tI)^n x = 0\}.$$

From (2.1b), (2.1d) and Theorem 2.2 it follows as in the proof of 3.2 that this sum is contained in $\mathbf{X}((-\infty, t]) + \mathbf{X}([t, +\infty))$. Assertion (a) follows.

For (b). We denote the manifold $\{x \in \mathbf{X}\colon (T - tI)^n x = 0\}$ by $\mathbf{X}(t)$. The density of $(T - bI)^n\mathbf{X} + \mathbf{X}(b)$ in $\mathbf{X}$ implies the density of

$$(T - aI)^n(T - bI)^n\mathbf{X} + (T - aI)^n\mathbf{X}(b)$$

in $(T - aI)^n\mathbf{X}$ and, hence, the density of

$$(T - aI)^n(T - bI)^n\mathbf{X} + (T - aI)^n\mathbf{X}(b) + \mathbf{X}(a)$$

in $\mathbf{X}$. Clearly, we have $(T - aI)^n\mathbf{X}(b) + \mathbf{X}(a) \subset \mathbf{X}([a, b])$. By (2.1c) and (1.3), we have

$$(T - aI)^n(T - bI)^n\mathbf{X} \subset \mathbf{X}((-\infty, a]) + \mathbf{X}([a, b]) + \mathbf{X}([b, +\infty)).$$

Hence, this last sum is dense in $\mathbf{X}$ which proves (b).

3.7. *Remark.* Under the hypothesis of 3.6 let (a, b) be any open interval which contains $\sigma(T)$ and let $\pi = \{t_0, t_1, \cdots, t_k\}$ be any partition of $[a, b]$. Then $\mathbf{X} = \bigvee_{j=1}^{k}\mathbf{X}([t_{j-1}, t_j])$.

The proof of this assertion consists of an iteration of the method used in the proof of (3.6b) above, together with the identity

$$\prod_{j=0}^{k} (T - t_jI)^n = \sum_{j=1}^{k} \left(\prod_{i\neq j-1,j} (T - t_iI)^n\right)K(t_{j-1}, t_j),$$

which can be proved by the use of contour integrals.

This last result and the "Lorch approximation theorem" (1.2c) for $T \in (G_1)$ now yield the following generalization of a form of the spectral theorem for self-adjoint operators. By Dunford's Lemma 13, the hypothesis holds if $\mathbf{X}$ is reflexive.

3.8. THEOREM. *Let* $\mathbf{X}$ *and* $T \in (G_1)$ *be such that for each* $t \in R$ *the set*

$$(T - tI)\mathbf{X} + \{x \in \mathbf{X} : (T - tI)x = 0\}$$

is dense in $\mathbf{X}$. *Let* (a, b) *be any open interval that contains* $\sigma(T)$. *For each* $\epsilon > 0$ *there is a* $\delta(\epsilon) > 0$ *such that if* $\pi = \{t_0, t_1, \cdots, t_k\}$ *is any partition of* $[a, b]$ *of norm less than* $\delta(\epsilon)$, *then*

(3.8a) $\mathbf{X} = \bigvee_{j=1}^{k} \mathbf{X}([t_{j-1}, t_j])$, *and*

(3.8b) $\| Tx - tx \| \leq \epsilon \| x \|$ *holds for all* $x \in \mathbf{X}([t_{j-1}, t_j])$,

$$t \in [t_{j-1}, t_j], \qquad j = 1, \cdots, k.$$

3.9. *Self-adjoint operators.* We conclude with a short proof of the integral representation of a bounded self-adjoint operator using the machinery developed above.

Let $\mathbf{H}$ be a Hilbert space and let $T = T^* \in B(\mathbf{H})$. Then $T \in (G_1)$ with $K = 1$. Since $T = T^*$ and $R(\zeta; T^*) = R(\bar{\zeta}; T)$ it follows that

$$H_+(t) = (2\pi i)^{-1} \int_{C(t,s)} (\zeta - t)(R(\zeta; T)\, d\zeta$$

is self-adjoint (choose $C(t, s)$ symmetric about the real axis). Hence, for each $t \in R$, the manifolds $\mathbf{M}(t)$, the null space of $H_+(t)$, and $\mathbf{N}(t)$, the closure of the range of $H_+(t)$, are orthogonal complements in $\mathbf{H}$ and so the projection $E(t)$ is self-adjoint. Let (a, b), ϵ, $\delta(\epsilon)$ and π be as in Theorem 3.8. The manifolds

$$\mathbf{H}([t_{j-1}, t_j]) = \text{Range}\,[E(t_j) - E(t_{j-1})], \quad j = 1, \cdots, k,$$

are then mutually orthogonal. For any $x \in \mathbf{H}$ we may write

$$x = \textstyle\sum_{j=1}^{k} x_j \quad \text{where } x_j = [E(t_j) - E(t_{j-1})]x \in \mathbf{H}([t_{j-1}, t_j]).$$

Then

$$Tx = \textstyle\sum_{j=1}^{k} Tx_j \quad \text{and} \quad Tx_j \in \mathbf{H}([t_{j-1}, t_j])$$

since each $E(t)$ commutes with T by (3.5b). Choosing $\lambda_j \in [t_{j-1}, t_j]$, $j = 1, \cdots, k$, and using $\| Tx_j - \lambda_j x_j \| \leq \varepsilon \| x_j \|$ for each j, we obtain

$$\begin{aligned}
\| Tx - \textstyle\sum_{j=1}^{k} \lambda_j [E(t_j) - E(t_{j-1})]x \|^2 \\
&= \| \textstyle\sum_j (Tx_j - \lambda_j x_j) \|^2 \\
&= \textstyle\sum_j \| Tx_j - \lambda_j x_j \|^2 \\
&\leq \textstyle\sum_j \varepsilon^2 \| x_j \|^2 \\
&= \varepsilon^2 \| x \|^2.
\end{aligned}$$

This shows that $\| T - \sum_{j=1}^{k} \lambda_j [E(t_j) - E(t_{j-1})] \| \leq \varepsilon$. Hence, the Riemann-Stieltjes integral $\int_a^b t\, dE(t)$ exists in $B(\mathbf{H})$ and equals T.

References

1. R. G. Bartle, *Spectral localization of operators in Banach spaces*, Math. Ann., vol. 153 (1964), pp. 261–269.
2. ———, *Spectral decomposition of operators in Banach spaces*, Proc. London Math. Soc. (3), vol. 20 (1970), pp. 438–450.
3. I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.
4. N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc., vol. 64 (1958), pp. 217–274.
5. N. Dunford and J. T. Schwartz, *Linear operators, Part III: Spectral operators*, Wiley-Interscience, New York, 1971.
6. D. Kocan, *Spectral manifolds for a class of operators*, Illinois J. Math., vol. 10 (1966), pp. 605–622.
7. E. R. Lorch, *Return to the self-adjoint transformation*, Acta Sci. Math. Szeged, vol. 12B (1950), pp. 137–144.
8. F. J. Murray, *The analysis of linear transformations*, Bull. Amer. Math. Soc., vol. 48 (1942), pp. 76–93.
9. W. R. Parzynski, *A spectral decomposition for certain operators in a Banach space*, Dissertation, Stevens Institute of Technology, 1968.

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