A SIMILARITY BETWEEN HYPONORMAL AND NORMAL SPECTRA

BY
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1. Introduction

A bounded operator T on a Hilbert space $\mathfrak{H}$ is said to be hyponormal if

$$T^*T - TT^* = D \geqq 0. \tag{1.1}$$

Let T have the Cartesian representation

$$T = H + iJ, \qquad H = \int \lambda \, dE_\lambda, \tag{1.2}$$

and, for any open interval Δ and corresponding projection $E(\Delta)$, consider the operator $T_\Delta = E(\Delta)TE(\Delta)$ on the Hilbert space $E(\Delta)\mathfrak{H}$. More generally, for any bounded operator A on $\mathfrak{H}$, define $A_\Delta = E(\Delta)AE(\Delta)$ on $E(\Delta)\mathfrak{H}$. Since

$$HJ - JH = -iC, \qquad D = 2C, \tag{1.3}$$

then

$$H_\Delta J_\Delta - J_\Delta H_\Delta = -iC_\Delta, \tag{1.4}$$

and hence T_Δ is hyponormal on $E(\Delta)\mathfrak{H}$.

It was recently shown in Putnam [3] that

$$\text{sp}\ (T_\Delta) \subset \text{sp}\ (T). \tag{1.5}$$

(Relation (1.5) was proved in the special case in which D is completely continuous by Clancey [1].) It will be shown below that the inclusion (1.5) can be sharpened as follows:

THEOREM. *If T is hyponormal and if Δ is any open interval and T_Δ is defined as above, then*

$$\text{sp}\ (T_\Delta) \cap \{z : \text{Re}\ (z) \in \Delta\} = \text{sp}\ (T) \cap \{z : \text{Re}\ (z) \in \Delta\}. \tag{1.6}$$

Thus, those parts of the spectra of T and of T_Δ which lie over the open interval Δ must coincide. Relation (1.6) is of course a well-known property of normal operators. Since the spectra of the real and imaginary parts of a hyponormal operator T are the projections of the set sp (T) onto the coordinate axes (see [2, p. 46]) it is clear that sp (T_Δ), T_Δ always regarded as an operator on $E(\Delta)\mathfrak{H}$, lies in the closure of the strip $\{z : \text{Re}\ (z) \in \Delta\}$.

An immediate consequence of the theorem is the following

COROLLARY. *If T is hyponormal with the representation* (1.2) *and if t is*

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real and Δ *is any open interval containing* t, *then*

$$\text{(1.7)}\quad \operatorname{Im}[\operatorname{sp}(T) \cap \{z : \operatorname{Re}(z) = t\}] = \bigcap_{\Delta}\{\operatorname{sp}(E(\Delta)JE(\Delta))\}, \quad t \in \Delta.$$

2. Some lemmas

Lemma 1. *Let* T *be hyponormal and let* $\Delta = (a, b)$ *and define* T_Δ *as above. Let* $a < \operatorname{Re}(z) < b$. *If* $(T - zI)x_n \to 0$ *holds for a sequence of unit vectors* x_n *in* $\mathfrak{H}$ *then* $(T_\Delta - zI)y_n \to 0$ *holds for a sequence of unit vectors* y_n *in* $E(\Delta)\mathfrak{H}$, *and conversely.*

Proof. First, suppose that $(T - zI)x_n \to 0$, $\|x_n\| = 1$, x_n in $\mathfrak{H}$. If $z = t + is$, then

$$(T - zI)^*(T - zI) = (H - tI)^2 + (J - sI)^2 + C$$

and, since $C \geqq 0$, one has

$$(H - tI)x_n \to 0 \quad \text{and} \quad (J - sI)x_n \to 0.$$

But $(H - tI)x_n \to 0$ and $a < t = \operatorname{Re}(z) < b$ imply that $x_n - E(\Delta)x_n \to 0$. Hence $(T_\Delta - zI)y_n \to 0$ clearly holds for $y_n = E(\Delta)x_n/\|E(\Delta)x_n\|$.

Next, suppose that $(T_\Delta - zI)y_n \to 0$ where $y_n = E(\Delta)y_n$ and $\|y_n\| = 1$. Then (cf. above),

$(H - tI)y_n = (H_\Delta - tI)y_n \to 0$ and $E(\Delta)(J - sI)y_n = (J_\Delta - sI)y_n \to 0$.

Now (1.3) holds if H and J are replaced by $H - tI$ and $J - sI$, so that

$$\text{(2.1)}\qquad (H - tI)(J - sI) - (J - sI)(H - tI) = -iC.$$

On taking inner products in (2.1) and using $(H - tI)y_n \to 0$, one obtains

$$(Cy_n, y_n) = \|C^{1/2}y_n\|^2 \to 0,$$

hence $Cy_n \to 0$, and hence $(H - tI)(J - sI)y_n \to 0$. Since $a < t < b$, this fact and $E(\Delta)(J - sI)y_n \to 0$ yield $(J - sI)y_n \to 0$. Thus $(T - zI)y_n \to 0$ and the proof is complete.

The above argument is essentially that used in the proof of Lemma 3 of [3].

LEMMA 2. *Let* T *be an arbitrary non-singular bounded operator on* $\mathfrak{H}$. *Then* T^{-1} *is the uniform limit of a sequence of polynomials in* T *and* T^*.

Proof. Let T have the polar form $T = PU$, where P is positive definite and U is unitary. Then $T^{-1} = U^*P^{-1}$ and $TT^* = PU(U^*P) = P^2$. Let P have the spectral resolution $P = \int_a^b \lambda\, dG_\lambda$, where $0 < a < b$, so that

$$P^2 = \int_a^b \lambda^2\, dG_\lambda \text{ and } P^{-1} = \int_a^b \lambda^{-1}\, dG_\lambda .$$

Since $0 < a < b$, it is clear from the Weierstrass approximation theorem that λ^{-1} is the uniform limit on $[a, b]$ of polynomials in λ^2 and hence P^{-1} is the uniform limit of polynomials in TT^*, hence in T and T^*. Since $T^* = U^*P$, then $U^* = T^*P^{-1}$ and so U^* is also the uniform limit of polynomials in T and T^*. The same must also hold for $T^{-1} = U^*P^{-1}$ and the proof is complete.

LEMMA 3. *Let T be hyponormal with the representation* (1.1) *and* (1.2). *Let $\Delta = (a, b)$ and $\delta = (c, d)$ denote disjoint open intervals at a distance r apart. Then*

$$\| E(\Delta)JE(\delta) \| \leqq | \delta |^{1/2} \| J \| / r^{1/2}. \tag{2.2}$$

Proof. First it will be convenient to obtain an estimate for $C^{1/2}E(\Delta)$; cf. [2, p. 20]. Multiplications of (1.3) on the left and right by $E(\Delta)$ yield

$$\int_\Delta (\lambda - \lambda_0)\, dE\, JE(\Delta) - E(\Delta)J \int_\Delta (\lambda - \lambda_0)\, dE = -iE(\Delta)CE(\Delta),$$

where λ_0 is an arbitrary constant. If λ_0 is chosen to be the midpoint of Δ, then, on taking inner products, one obtains

$$\| C^{1/2}E(\Delta)x \|^2 \leqq 2(\tfrac{1}{2}) | \Delta | \, \| E(\Delta)JE(\Delta)x \| \, \| x \| \leqq | \Delta | \, \| J \| \, \| x \|^2$$

and hence

$$\| C^{1/2}E(\Delta) \| = \| E(\Delta)C^{1/2} \| \leqq | \Delta |^{1/2} \| J \|^{1/2}. \tag{2.3}$$

Next, multiply (1.3) on the left by $E(\Delta)$ and on the right by $E(\delta)$. Then, for arbitrary constants λ_1 and λ_2,

$$\begin{aligned} &\int_\Delta (\lambda - \lambda_1)\, dEJE(\delta) - E(\Delta)J \int_\delta (\lambda - \lambda_2)\, dE \\ &\qquad = -iE(\Delta)CE(\delta) + (\lambda_2 - \lambda_1)E(\Delta)JE(\delta). \end{aligned} \tag{2.4}$$

If λ_1 and λ_2 are taken to be the midpoints of Δ, δ respectively, it is seen that the norms of the two operators on the left of (2.4) are majorized by $\tfrac{1}{2}| \Delta | \, \| E(\Delta)JE(\delta) \|$ and $\tfrac{1}{2}| \delta | \, \| E(\Delta)JE(\delta) \|$. Since

$$| \lambda_2 - \lambda_1 | = r + \tfrac{1}{2}(| \Delta | + | \delta |)$$

it follows from (2.4) that $\| E(\Delta)JE(\delta) \| \leqq \| E(\Delta)CE(\delta) \| / r$ and hence, by (2.3) and a similar relation with Δ replaced by δ, that

$$\| E(\Delta)JE(\delta) \| \leqq | \Delta |^{1/2} | \delta |^{1/2} \| J \| / r.$$

If Δ is expressed as the union of disjoint intervals (open or half-open) $\Delta_1, \Delta_2, \cdots$, and if r_j denotes the distance from Δ_j to δ, one obtains

$$\| E(\Delta)JE(\delta)x \|^2 = \textstyle\sum_j \| E(\Delta_j)JE(\delta)x \|^2 \leqq | \delta | \, \|J \|^2 \, \| x \|^2 \sum_j | \Delta_j | / r_j^2,$$

and hence

$$\| E(\Delta)JE(\delta) \| \leqq | \delta |^{1/2} \| J \| \left(\int_\Delta q^{-2}\, dq\right)^{1/2}, \tag{2.5}$$

where q denotes the distance from δ to a point of Δ. Thus, if Δ is to the right of

δ, $q = x - d$ and

$$\int_\Delta q^{-2}\, dq = \int_a^b (x - d)^{-2}\, dx \leqq \int_a^\infty (x - d)^{-2}\, dx = (a - d)^{-1} = r^{-1}.$$

A similar result holds if Δ is to the left of δ and (2.2) now follows from (2.5).

3. Proof of the theorem

It is clear from (1.5) that the set on the left side of (1.6) is contained in that on the right side. Thus it is necessary to prove the reverse inclusion. Define T_t by

$$(3.1) \qquad T_t = E((-\infty, t))TE((-\infty, t)) \quad \text{on } E((-\infty, t))\mathfrak{H}.$$

It is sufficient to show that for every real b, if $z \in \operatorname{sp}(T)$ and if $\operatorname{Re}(z) < b$ then $z \in \operatorname{sp}(T_b)$. In fact, if this has been established, it will be clear from the proof given below that a similar argument applied to

$$T_\Delta = E((a, \infty)T_b E((a, \infty))$$

then yields (1.6) for any open interval $\Delta = (a, b)$. Consequently, suppose that

$$(3.2) \qquad z \in \operatorname{sp}(T) \quad \text{and} \quad \operatorname{Re}(z) < b.$$

The theorem will then be proved if it is shown that

$$(3.3) \qquad z \in \operatorname{sp}(T_b).$$

Next, in case there exists a sequence $\{x_n\}$ of unit vectors in $\mathfrak{H}$ satisfying $(T - zI)x_n \to 0$ then, by the second relation of (3.2) and Lemma 1, one has $(T_b - zI)y_n \to 0$ for a sequence $\{y_n\}$ of unit vectors in the Hilbert space $E((-\infty, b))\mathfrak{H}$. Thus, in particular, (3.3) holds.

Hence, it remains to be shown that (3.2) implies (3.3) in case we assume the following relation:

(3.4) there does not exist a sequence $\{x_n\}$, $\| x_n \| = 1$, for which $(T - zI)x_n \to 0$.

Suppose, if possible, that (3.3) is false, so that

$$(3.5) \qquad z \notin \operatorname{sp}(T_b).$$

It will be shown that (3.2), (3.4) and (3.5), which are now being assumed, yield a contradiction.

In order to prove this, it will first be shown that (assuming (3.2), (3.4) and (3.5)) there exists some $\beta \geqq b$ such that

$$(3.6) \qquad z \in \operatorname{sp}(T_c) \text{ for } c > \beta \quad \text{and} \quad z \notin \operatorname{sp}(T_\beta).$$

To see this, define β by

$$(3.7) \qquad \beta = \sup \{t : t \geqq b, z \notin \operatorname{sp}(T_t).$$

In view of (3.5), the set $\{\cdots\}$ of (3.7) is not empty and, in view of (3.2), β exists as a finite number, thus $b \leqq \beta < \infty$.

Since, by (1.5), sp $(T_t) \subset$ sp (T_s) for $t < s$, it is clear from (3.5) that

$$(3.8) \qquad z \notin \text{sp}\,(T_t) \quad \text{for } t < \beta,$$

and from (3.7) that the first relation of (3.6) holds. Suppose, if possible, that (3.6) fails to hold, so that, in addition to (3.8),

$$(3.9) \qquad z \in \text{sp}\,(T_\beta).$$

Next, note that there do not exist positive numbers δ, η with the property that Re $(z) + \delta < \beta - \eta$ and $\{w : |w - z| < \delta\} \cap \text{sp}\,(T_t) = \emptyset$ for $\beta - \eta \leqq t < \beta$. For, otherwise,

$$\|(T_t - zI)x\| \geqq \|(T_t - zI)^*x\| \geqq \delta\|E((-\infty, t)x\| \quad \text{for all } x \text{ in } \mathfrak{H}.$$

Use is made here of the basic property of hyponormal operators T:

$$(3.10) \qquad \|Tx\| \geqq \|T^*x\| \geqq \text{dist}\,(0, \text{sp}\,(T))\|x\|.$$

On letting $t \to \beta - 0$, one obtains

$$\|(T_\beta - zI)x\| \geqq \|(T_\beta - zI)^*x\| \geqq \delta\|E((-\infty, \beta)x\|,$$

so that $z \notin$ sp (T_β), in contradiction to (3.9).

Further, there cannot exist a pair of positive numbers δ, η with the property that

$$\{w : |w - z| < \delta\} \cap \text{sp}\,(T_{\beta-\eta}) = \{w : |w - z| < \delta\}.$$

For, otherwise, $z \in$ sp $(T_{\beta-\eta})$, in violation of (3.8). Consequently, for every δ satisfying Re $(z) + \delta < \beta$, one can choose an arbitrarily small $\eta > 0$ so that Re $(z) + \delta < \beta - \eta$ and $\{w : |w - z| < \delta\} \cap$ sp $(T_{\beta-\eta})$ is a *proper*, non-empty subset of $\{w : |w - z| < \delta\}$.

Consequently, there exists a boundary point q of sp $(T_{\beta-\eta})$ in the disk $\{w : |w - z| < \delta\}$. Hence there exists a sequence of unit vectors

$$y_n = E((-\infty, \beta - \eta))y_n$$

for which $(T_{\beta-\eta} - qI)y_n \to 0$ as $n \to \infty$. Since Re $(q) < \beta - \eta$, it follows from Lemma 1 that there exists a sequence $\{X_n\}$ of unit vectors in $\mathfrak{H}$ satisfying $(T - qI)X_n \to 0$. On choosing $\delta = \delta_k \to 0$ one can obtain a sequence of numbers $\{q_k\}$ satisfying $q_k \to z$ and corresponding sequences of unit vectors $\{X_n^k\}$ such that $(T - q_kI)X_n^k \to 0$ as $n \to \infty$ for each fixed k. If $k = k_n$ is chosen so that $\|(T - q_{k_n}I)X_n^{k_n}\| < 1/n$ and $k_n \to \infty$ then clearly $(T - zI)x_n \to 0$ for $x_n = X_n^{k_n}$, in contradiction to (3.4).

So far, it has been established that (3.2), (3.4) and (3.5) lead to (3.6). In order to complete the proof of the Theorem it will be shown that (3.6) is impossible.

4. Impossibility of (3.6)

By considering translations of T, one can suppose that $z = 0$ in (3.6) and hence, in particular, $\beta > 0$. Thus, relation (3.6) becomes

$$0 \in \mathrm{sp}\,(T_c) \text{ for } 0 < \beta < c \quad \text{and} \quad 0 \notin \mathrm{sp}\,(T_\beta). \tag{4.1}$$

It will be shown that (4.1) leads to a contradiction. In view of the first part of (4.1), for every $c > \beta$ there exists a sequence of unit vectors $\{y_n\}$, where where $y_n = E((-\infty, c))y_n$, satisfying

$$T_c^* y_n = E((-\infty, c))T^* y_n \to 0;$$

cf. (3.10). By choosing a sequence $c_n \to \beta + 0$ it is clear that one can find unit vectors $x_n = E((-\infty, c_n))x_n$ satisfying

$$E((-\infty, c_n))T^*E((-\infty, c_n))x_n \to 0 \quad \text{as } c_n \to \beta + 0.$$

Let $\Delta = (-\infty, \beta)$ and $\delta_n = [\beta, c_n)$. Then one has

$$E(\Delta \cup \delta_n)T^*E(\Delta \cup \delta_n)x_n \to 0, \tag{4.2}$$

where

$$x_n = E(\Delta \cup \delta_n)x_n, \qquad \| x_n \| = 1. \tag{4.3}$$

A multiplication of (4.2) on the left by $E(\Delta)$ yields

$$E(\Delta)T^*E(\Delta)x_n + E(\Delta)T^*E(\delta_n)x_n \to 0. \tag{4.4}$$

Since $E(\Delta)T^*E(\delta_n) = -iE(\Delta)JE(\delta_n)$ and $|\delta_n| \to 0$, it follows from Lemma 3 that for any fixed $\varepsilon > 0$ one has

$$\| E((-\infty, \beta - \varepsilon))JE(\delta_n) \| \to 0 \quad \text{as } n \to \infty$$

and hence (note that $\Delta = (-\infty, \beta)$)

$$(H - \beta I)y_n \to 0, \qquad y_n = E(\Delta)T^*E(\delta_n)x_n. \tag{4.5}$$

Since $T_\beta = T_\Delta = E(\Delta)TE(\Delta)$ is, by the second relation of (4.1), non-singular (on $E(\Delta)\mathfrak{H}$), then (4.4) becomes

$$E(\Delta)x_n + (T_\Delta^*)^{-1}y_n \to 0. \tag{4.6}$$

Next, it will be shown that

$$(H - \beta I)E(\Delta)x_n \to 0, \tag{4.7}$$

where x_n is given by (4.3). In order to see this, let T_Δ^* and $E(\Delta)\mathfrak{H}$ be identified with T and $\mathfrak{H}$ of Lemma 2. Then, for any $\epsilon > 0$, there exists a polynomial in T_Δ and T_Δ^*, hence also a polynomial in H_Δ and J_Δ (where $H_\Delta = E(\Delta)H = \int_\Delta \lambda\, dE$ and $J_\Delta = E(\Delta)JE(\Delta)$), say $p(H_\Delta, J_\Delta)$, such that

$$\| (T_\Delta^*)^{-1} - p(H_\Delta, J_\Delta) \| < \varepsilon. \tag{4.8}$$

In view of (4.5) it follows from the relation (1.4) that if $q(J_\Delta)$ denotes any polynomial in J_Δ then $(H - \beta I)q(J_\Delta)y_n = (H_\Delta - \beta I)q(J_\Delta)y_n \rightarrow 0$. (Cf. [2, p. 46] for a similar argument.) Consequently,

$$(H - \beta I)p(H_\Delta, J_\Delta)y_n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and relation (4.7) now follows from (4.6) and (4.8).

On forming the inner product of the vector in (4.2) with x_n and taking the real part, one obtains

$$(Hx_n, x_n) = (HE(\Delta)x_n, E(\Delta)x_n) + (HE(\delta_n)x_n, E(\delta_n)x_n) \rightarrow 0.$$

Hence, on using (4.7) and noting that $\Delta = (-\infty, \beta)$ and $\delta_n = [\beta, c_n)$ with $c_n \rightarrow \beta + 0$, one obtains

$$\beta(\| E(\Delta)x_n \|^2 + \| E(\delta_n)x_n \|^2) = \beta \| x_n \|^2 \rightarrow 0.$$

Since $\beta > 0$, then $x_n \rightarrow 0$, in contradiction to $\| x_n \| = 1$ of (4.3). As noted above, this shows that (4.1) is impossible and the proof of the theorem is now complete.

5. Remarks

The relation (1.5) holds if Δ is any Borel set of the real line. To see this, note that the same proof of (1.5) in [3], for the case in which Δ is an open interval, also holds when Δ is any open set. Further, if Δ is an arbitrary Borel set it is sufficient to prove (1.5) for the case in which T is completely hyponormal, that is, T has no non-trivial reducing subspaces on which it is normal. (In fact, if $\mathfrak{M}$ is a normal reducing subspace of T then, since (1.5) surely holds for normal T and for any Borel set Δ, clearly

$$\text{sp}\ (T_\Delta / E(\Delta)\mathfrak{M}) \subset \text{sp}\ (T/\mathfrak{M});$$

cf. [3, beginning of Section 3].) In this case, $H = \text{Re}\ (T)$ is absolutely continuous; see [2, p. 42]. Next, still assuming that Δ is any Borel set, choose open sets $\Delta_n \supset \Delta$ satisfying $\text{meas}_1\ (\Delta_n - \Delta) \rightarrow 0$, where meas_1 denotes ordinary Lebesgue measure on the real line.

Next, suppose that $z \in \text{sp}\ (T_\Delta)$. It will be shown that $z \in \text{sp}\ (T)$. First, note that there exist $z_n \in \text{sp}\ (T_n)$, where $T_n = T_{\Delta_n}$, for which $z_n \rightarrow z$. In fact, otherwise, there would exist a $\delta > 0$ and a sequence of positive integers $n_1 < n_2 < \cdots$ such that

$$\text{sp}\ (T_{n_k}) \cap \{w : |w - z| < \delta\} = \emptyset \quad \text{for } k = 1, 2, \cdots.$$

Hence

$$\| (T_{n_k} - zI)E(\Delta_{n_k})x \| \geqq \| (T_{n_k} - zI)^* E(\Delta_{n_k})x \| \geqq \delta \| E(\Delta_{n_k})x \|.$$

Since E_λ is absolutely continuous, $E(\Delta_{n_k}) \rightarrow E(\Delta)$ (strongly) as $k \rightarrow \infty$, and hence

$$\| (T_\Delta - zI)E(\Delta)x \| \geqq \| (T_\Delta - zI)^* E(\Delta)x \| \geqq \delta \| E(\Delta)x \|,$$

so that $z \notin \text{sp}\ (T_\Delta)$, a contradiction. Thus, there exist $z_n \in \text{sp}\ (T_n)$ satisfying $z_n \to z$. As noted above, (1.5) certainly holds with Δ replaced by any of the (open) sets Δ_n, so that $\text{sp}\ (T_n) \subset \text{sp}\ (T)$. Thus, $z_n \in \text{sp}\ (T)$, hence $z \in \text{sp}\ (T)$, as was to be shown.

It is clear from the proof given in the present paper that (1.6) holds if Δ is any open set. On the other hand, it is easily shown that relation (1.6) need not hold if Δ is an arbitrary Borel set, in fact, not even if Δ is a closed interval and T is normal. To see this, let T be any normal operator for which

$$\text{sp}\ (T) = \alpha \cup \{z : 1 \leqq |\text{Re}\ (z)| \leqq 2 \text{ and } |\text{Im}\ (z)| \leqq 2\},$$

where $\alpha = \{z : |\text{Re}\ (z)| \leqq 1 \text{ and } 1 \leqq |\text{Im}\ (z)| \leqq 2$, and with the further property that ± 1 are not in the point spectrum of $H = \text{Re}\ (T)$. Then if $\Delta = [-1, 1]$, the left side of (1.6) is the set α, while the right side consists of α together with the segments $\{z : |\text{Re}\ (z)| = 1 \text{ and } |\text{Im}\ (z)| \leqq 1\}$.

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