WILD DISKS IN E^n THAT CAN BE SQUEEZED ONLY TO TAME ARCS

BY

ROBERT J. DAVERMAN[1]

1. Introduction

For each 2-cell D in Euclidean n-space E^n $(n \geq 3)$, no matter how wildly embedded, there exist many essentially distinct maps f of E^n to itself such that each f is a homeomorphism of $E^n - D$ onto $E^n - f(D)$, $f(D)$ is an arc, and $f \mid D$ is related to a canonical projection map [7], [8], [15]. One might expect the image arc to be quite tangled, for the wildness spread over the disk must be compressed into a 1-dimensional set, but the wildness of the image may turn out to be less complicated than that of the disk. In fact, in [7, p. 371] and [8], examples are provided of a wild disk D in E^n ($n = 3$ and $n \geq 5$) and a map of E^n to itself squeezing D to a tame arc. Theorem 3 of [15] indicates that similar examples can be found in E^4.

Let Δ_2 denote the 2-cell

$$\{(x, y) \in E^2 \mid x^2 + y^2 \leq 1\},$$

Δ_1 the 1-cell

$$\{(x, 0) \in \Delta_2 \mid -1 \leq x \leq 1\},$$

and π the projection map of Δ_2 onto Δ_1 sending (x, y) to $(x, 0)$. Suppose D is a disk in the interior of an n-manifold M. A map f of M to itself is said to *squeeze D to an arc A* if and only if there exist homeomorphisms g_2 of Δ_2 onto D and g_1 of Δ_1 onto $A = f(D)$ such that $fg_2 = g_1 \pi$ and f is a homeomorphism of $M - D$ onto $M - A$. Observe that these conditions force $f^{-1}(x)$ to be an arc if $x \in \operatorname{Int} A$; otherwise $f^{-1}(x)$ is a singleton.

As another important property of the examples mentioned above, there also exists a map of E^n to itself squeezing D to a wild arc; hence, these examples indicate that the arcs associated with any disk via squeezing maps need not be equivalently embedded. One might ask whether the embedding of a disk could be determined by studying its various images under such maps. The purpose of this paper is to indicate why the question has a negative answer: in case $n = 3$ or $n \geq 5$ we give an example of a wild disk D in E^n such that for every map f of E^n to itself squeezing D to an arc, $f(D)$ is tame.

A map of E^n to itself that squeezes a disk to an arc satisfies the less restrictive definition of a cell-like (or UV^∞) mapping. A map g of X onto Y is *cell-like* if and only if each point preimage $g^{-1}(y)$ can be embedded in some Euclidean space as a cellular subset.

Received June 29, 1970.

[1] Research supported in part by a National Science Foundation grant.

Suppose X is a topological space, A a subset of X, and B a subset of A. Then $X - A$ has *property* 1-UV *at* B if and only if for each open set U containing B, there is an open set V containing B such that (1) $V \subset U$ and (2) each loop in $V - A$ is null homotopic in $U - A$. When B is a singleton set, the assertion that $X - A$ has property 1-UV at B is equivalent to the assertion that $X - A$ is locally simply connected (1-LC) at B.

If C is a cell, then Int C and BdC denote the interior and boundary of C, respectively. The symbol Cl denotes the topological closure operator; the symbol I, the unit interval. If X is a topological space and A a closed subset of X, then X/A denotes the decomposition space associated with the upper semicontinuous decomposition of X whose only nondegenerate element is A.

2. Characterizations of disks that squeeze only to tame arcs

PROPOSITION 2.1. *Suppose D is a disk in E^n ($n \geq 5$) and f a map of E^n to itself that squeezes D to an arc. The following statements are equivalent:*

(1) *For each $p \in f(D)$, $E^n - D$ has property* 1-UV *at $f^{-1}(p)$.*

(2) *For each $p \in f(D)$, $E^n - f(D)$ is* 1-LC *at p.*

(3) *$f(D)$ is tame.*

Proof. The equivalence of (1) and (2) follows from the continuity of f and the requirement that f carry $E^n - D$ homeomorphically onto $E^n - f(D)$. That (2) implies (3) has been proved by Bryant and Seebeck [6, Th. 4.2]. Obviously, (3) implies (2).

THEOREM 2.2. *Let D be a disk in E^n ($n \geq 5$). The following statements are equivalent:*

(1) *For each arc β spanning D, $E^n - D$ has property* 1-UV *at β.*

(2) *For each map f of E^n to itself squeezing D to an arc, $E^n - f(D)$ is* 1-ULC.

(3) *For each map f of E^n to itself squeezing D to an arc, $f(D)$ is tame.*

Since (2) and (3) are equivalent [6, Th. 4.2], Theorem 2.2 is a consequence of the following result.

THEOREM 2.3. *Suppose D is a disk in E^n ($n \geq 4$). The following statements are equivalent:*

(1) *For each arc β spanning D, $E^n - D$ has property* 1-UV *at β.*

(2) *For each map f of E^n to itself squeezing D to an arc, $E^n - f(D)$ is* 1-ULC.

Proof. Assume (1) and let f be a map of E^n to itself squeezing D to an arc. The argument given for Proposition 2.1 indicates that $E^n - f(D)$ is 1-LC at each point of Int $f(D)$, and one can easily show from this that $E^n - f(D)$ is 1-LC at the endpoints of $f(D)$. Thus, $E^n - f(D)$ is 1-ULC.

Assume (2), and let β denote a spanning arc of D and U a neighborhood of β. According to Theorem 3 of [14] in case $n \geq 5$ and Theorem 2 of [15] in case $n = 4$, there exist tame arcs α_1 and α_2 in $D - \beta$ spanning D and such that the closure F of the component of $D - (\alpha_1 \cup \alpha_2)$ containing β is a subset of U. By shrinking α_1 and α_2 to points and applying [8, Th. 2] or [15, Th. 3] to the image of D, we obtain a map f of E^n onto itself squeezing D to an arc such that, for $i = 1, 2, f(\alpha_i)$ is a point. From the hypothesis that $E^n - f(D)$ is 1-ULC one can establish easily the existence of a neighborhood V' of Int $f(F)$ such that $f^{-1}(V') \subset U$ and each loop in $V' - f(D)$ is contractible in $f(U) - f(D)$. Clearly, $V = f^{-1}(V')$ is the neighborhood required to show that $E^n - D$ has property 1-UV at β.

PROPOSITION 2.4. *Suppose D and D^* denote disks in E^3 such that $D \subset$ Int D^* and D^* is locally tame modulo D, and suppose f is a map of E^3 to itself squeezing D to an arc. The following statements are equivalent:*
(1) *For each $p \in f(D)$, $E^3 - D^*$ has property* 1-UV at $f^{-1}(p)$.
(2) *For each $p \in f(D)$, $E^3 - f(D^*)$ is* 1-LC *at p.*
(3) *$f(D^*)$ is tame.*
(4) *$f(D)$ is tame.*

Proof. The equivalence of (1) and (2) follows as in Proposition 2.1. Since D^* is locally tame at points of $D^* - D$, $E^3 - f(D^*)$ is 1-LC at each point of $f(D^* = D)$. Thus (2) gives that $E^3 - f(D^*)$ is 1-ULC, so (2) implies (3) [4, Th. 8]. Obviously (3) implies both (2) and (4). Finally, (4) gives that $f(D^*)$ is locally tame modulo a tame arc, which means that $f(D^*)$ is tame [9, Th. 1].

THEOREM 2.5. *Suppose D and D^* are disks in E^3 such that $D \subset$ Int D^* and D^* is locally tame modulo D. The following statements are equivalent:*
(1) *For each arc β spanning D, $E^3 - D^*$ has property* 1-UV *at β.*
(2) *For each map f of E^3 to itself squeezing D to an arc, $f(D)$ is tame.*

Proof. With [7, Th. 2′] and Proposition 2.4 in place of [8, Th. 2] and Proposition 2.1, the argument almost parallels the one given for Theorem 2.2, except that we must separately establish the following: if statement 1 holds, then $E^3 - D^*$ is 1-LC at each point p of BdD. To do this, let U denote a neighborhood of p. Choose an arc β spanning D such that $p \in \beta$ and $\beta \subset U$. From (1) we see that p has a neighborhood V (V also contains β) such that $V \subset U$ and each loop in $V - D^*$ is nullhomotopic $U - D^*$.

3. Wild disks in E^n $(n \geq 5)$

Given $Z \subset Y \subset X \subset E^n$ we say that *loops near Y can be pushed towards Z through $E^n - X$* if and only if for each neighborhood U of Y and W of Z there exists a neighborhood V of Y such that to each loop J in $V - X$ there corresponds a map h of a disk with holes H into $U - X$ such that for one component S of BdH, $h|S$ defines J and $h(\text{Bd}H - S)$ is contained in $W - X$.

Suppose B is an arc in E^{n-1} $(n \geq 4)$ such that $E^{n-1} - B$ is 1-LC at the endpoints of B and, for each subarc B' of B and endpoint q of B', loops near

B' can be pushed towards q through $E^{n-1} - B$. Consider the disk $D = B \times I$ in $E^{n-1} \times E^1 = E^n$.

PROPOSITION 3.1. *$E^n - D$ is 1-LC at each point of* BdD.

Proof. The proof is routine; at points of Bd$B \times$ Int I this result depends on the assumption that $E^{n-1} - B$ be 1-LC at the endpoints of B.

An arc α is D is said to be *vertical* if $\alpha = p \times \alpha'$ for some $p \in B$ and $\alpha' \subset I$; similarly, an arc α in D is said to be *horizontal* if $\alpha = \alpha' \times t$ for some $\alpha' \subset B$ and $t \in I$. The following proposition is of an elementary nature, and we leave its proof to the reader.

PROPOSITION 3.2. *If α is a horizontal or vertical arc in D and p is an endpoint of α, then loops near α can be pushed towards p through $E^n - D$.*

PROPOSITION 3.3. *If α is a spanning arc of D that is the finite union of horizontal and vertical subarcs, then $E^n - D$ has property* 1-UV *at α.*

Proof. Suppose $\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k$, where each α_i is either horizontal or vertical in D, and $\alpha_i \cap \alpha_{i+1} = p_i$, an endpoint of each $(i = 1, \cdots, k-1)$. Let p_k denote Bd$\alpha_k - p_{k-1}$. Let U be a neighborhood of α.

According to Proposition 3.1, p_k has a neighborhood N_k such that each loop in $N_k - D$ is null homotopic in $U - D$. By Proposition 3.2 there exist neighborhoods N_i of p_i $(i = 1, \cdots, k-1)$ and V_i of α_i $(i = 1, \cdots, k)$, with $N_i \subset V_{i+1}$, such that each loop in $V_i - D$ can be deformed in $U - D$ to the union of loops in $N_i - D$ $(i = 1, \cdots, k)$. Let $V = \cup V_i$. We assume that $V_i \cap V_j \neq \emptyset$ *iff* $|i - j| \leq 1$ and that in this case $V_i \cap V_j$ is connected. Then each loop L in $V - D$ is homotopic (as in the definition at the beginning of this section) in $V - D$ to the finite union of loops L_j such that each L_j is contained in some V_i $(i = 1, \cdots, k)$. But the L_j's can be pushed through $U - D$ into $N_i - D$, then into $N_{i+1} - D$, and ultimately into $N_k - D$, at which spot each of the resulting collection of loops can be contracted in $U - D$. Hence, $E^n - D$ has property 1-UV at α.

PROPOSITION 3.4. *For each arc β spanning D, $E^n - D$ has property* 1-UV *at β.*

Proof. Let β denote a spanning arc of D and U a neighborhood of β. There exist arcs α_1 and α_2 spanning D such that α_i is the finite union of horizontal and vertical subarcs $(i = 1, 2)$, $(\alpha_1 \cup \alpha_2) \cap \beta = \emptyset$, and Cl$F \subset U$, where F denotes the component of $D - (\alpha_1 \cup \alpha_2)$ containing β. Observe that D is cellular, since $E^n - D$ is homeomorphic to $E^n - (B \times 0)$.

It follows from Proposition 3.3 that there exist neighborhoods W_i of α_1 such that any loop in $W_i - D$ is null homotopic in $U - D$ $(i = 1, 2)$. Assume $W_1 \cap W_2 = \emptyset$. Let W_0 denote a neighborhood of F such that $W_0 \cap D = F$ and $W_0 \subset U$. The cellularity of D implies ClF satisfies the cellularity criterion [13, Th. 1], so there exists another neighborhood V of $\mathrm{Cl}F$, with $V \subset W_0 \cup W_1 \cup W_2$, such that each loop L in $V - D$ is null homotopic in $(W_0 \cup W_1 \cup W_2) - \mathrm{Cl}F$.

Unfortunately, the image of such a contraction may intersect D in a subset of $W_1 \cup W_2$. To avoid this problem, from the domain of the contraction of L we extract a disk with holes S such that the contraction h restricted to one component J of $\mathrm{Bd}S$ defines the loop L, $h(S) \cap D = \emptyset$, and $h(\mathrm{Bd}S) - J) \subset W_1 \cup W_2$. By the construction of the W_i's, each loop of $h(\mathrm{Bd}S - J)$ is null homotopic in $U - D$, and we can readily piece together the desired contraction of L.

THEOREM 3.5. *There exists an everywhere wild disk D in E^n $(n \geq 5)$ such that, for each map f of E^n onto itself squeezing D to an arc, $f(D)$ is tamely embedded.*

Proof. To make use of the construction given at the beginning of this section, we must find a wildly embedded arc B in E^{n-1} such that $E^{n-1} - B$ is 1-LC at the endpoints of B and, for each subarc B' of B and endpoint q of B', loops near B' can be pushed towards q through $E^{n-1} - B$. A method for describing such arcs was first given by Brown [5]. The idea is to take a non-cellular arc A in E^{n-2} and define B as

$$\langle A \rangle \times I \subset E^{n-2}/A \times E^1,$$

where $\langle A \rangle$ denotes the point corresponding to A in E^{n-2}/A. It follows from [2] that $E^{n-2}/A \times E^1$ is homeomorphic to E^{n-1}.

It is easy to show that $E^{n-1} - B$ fails to be 1-LC at each point b of Int B, and, therefore, $E^n - (B \times I)$ fails to be 1-LC at each point (b, t) of $B \times I$, where $b \in \mathrm{Int}\, B$ and $t \in \mathrm{Int}\, I$. Consequently, the disk $D = B \times I$ is everywhere wild. It is easy to show that B satisfies the properties mentioned in the preceding paragraph. Hence, Proposition 3.4 and Theorem 2.2 imply that for each map f of E^n to itself squeezing D to an arc, $f(D)$ is tamely embedded.

We conclude this section with the observation that these techniques can be reapplied to prove the result stated below. Even in the case that $g(D)$ is an arc, Theorem 3.6 differs from Theorem 3.5 in that, unlike the squeezing maps, no relation between $g|D$ and the projection map $\pi : \Delta_2 \to \Delta_1$ is required

THEOREM 3.6. *Suppose D is a disk in E^n $(n \geq 5)$ such that each map squeezing D to an arc yields a tame arc, and suppose g is a cell-like map of E^n to itself such that g is a homeomorphism of $E^n - D$ onto $E^n - g(D)$ and $g(D)$ is a finite graph. Then $g(D)$ is tamely embedded.*

4. A wild disk in E^3

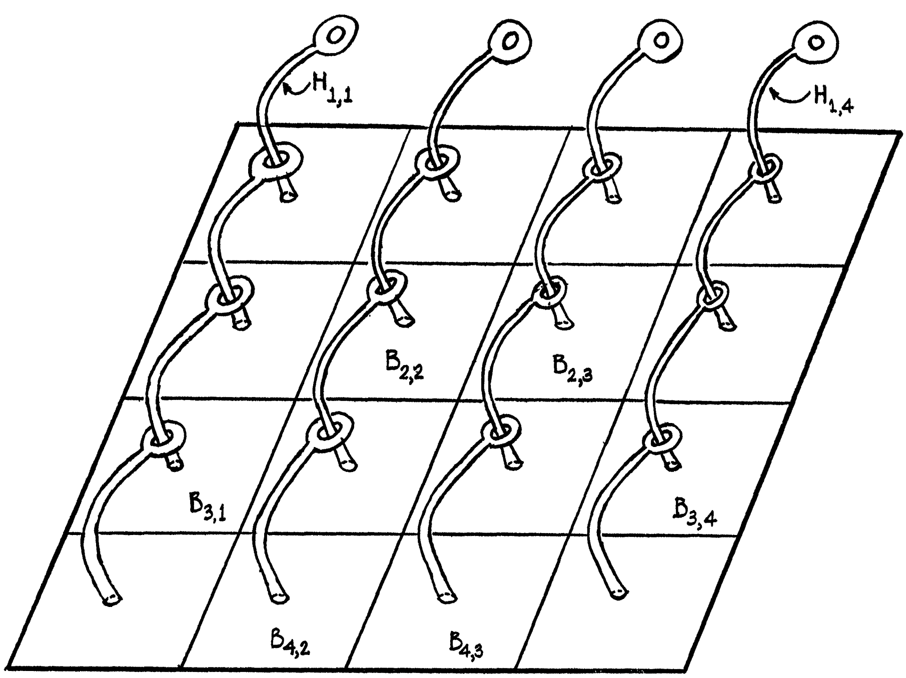
The purpose of this section is to establish the analogue of Theorem 3.5 for $n = 3$. We make extensive use of the construction methods first described by Bing [3] and later altered by Gillman [10] and Alford [1]. None of their examples has the desired properties, so we must describe a new scheme for hooking eyebolts.

THEOREM 4.1. *There exists an everywhere wild disk D in E^3 such that for each map f of E^3 to itself squeezing D to an arc, $f(D)$ is tamely embedded.*

Proof. Part I. *The example.* Let D_1 denote a planar square in E^3. Partition D_1 into subsquares $\{B_{ij}\}$ using horizontal and vertical line segments that span D_1, and enumerating as with matrix notation, so that B_{ij} denotes the square in the i^{th} row and j^{th} column of this partition.

From the center of each B_{ij} erect a solid feeler with a solid loop at the far end, as suggested in [1, 3, 10]. Denote this bent eyebolt emanating from B_{ij} as H_{ij}, and place these eyebolts so that the loop of $H_{i+1,\,j}$ encircles the stem of H_{ij}. (Of course, it is understood that the eyebolts associated with the top row of the partition, which have nothing to encircle, are allowed simply to dangle in space.) See the figure. In referring to such a configuration, we say the eyebolts link vertically pointing upward.

At the next stage we shall construct a configuration in which the eyebolts link horizontally pointing to the right; at the third stage, the eyebolts will link vertically pointing down; and at the fourth stage, horizontally pointing to the left. In the remainder of the construction we shall build configurations of eyebolts such that the linking pattern at the n^{th} stage will coincide with that



of the j^{th} stage above $(1 \leq j \leq 4)$, where $j \equiv n$ (modulo 4). We provide the details for constructing the second stage.

As in [1], [3], [10], thicken B_{ij} slightly and let $T_{ij} = B_{ij} \cup H_{ij}$. The disk D will be contained in $\bigcup T_{ij}$. A slice is removed from the loop of each T_{ij} to form a topological cube K_{ij}. Note that $\mathrm{Bd}B_{ij}$ separates $\mathrm{Bd}K_{ij}$ into two pieces, one of which has a hook in it. The interior of this hooked disk is pushed slightly into Int K_{ij}, so as to form a disk C_{ij} which lies, except for its boundary $\mathrm{Bd}C_{ij} = \mathrm{Bd}B_{ij}$, in Int T_{ij}. The disk $D_2 = \bigcup C_{ij}$ is another approximation to our disk D.

Each of the C_{ij}'s is partitioned into a subdivision of 13^2 disks, topologically like the subdivision of the original square D_1 in that rows and columns of disks are employed, and such that along the boundaries of the C_{ij}'s these subdivisions match up. The number 13 is chosen for the convenient correspondence with the notation of [1]. Enumerate the disks $\{E_{km}\}$ in the subdivision of each C_{ij} as before, such that those disks corresponding to the ends of the slice removed from T_{ij} are the disks of the subdivision denoted as $E_{7,5}$ and $E_{7,9}$.

From the center of each E_{km} erect an eyebolt L_{km} such that the loop of L_{km} encircles the stem of $L_{k,m+1}$ and that, furthermore, the loop of $L_{k,13}$ encircles the stem of the $L_{k,1}$ associated with the disk to the right of $E_{k,13}$. (Of course, if no such disk exists, $L_{k,13}$ is not required to encircle anything.) Only one other type of entangling, but of crucial importance for the wildness of the limit, is required: the eyebolts $L_{7,5}$ and $L_{7,9}$ (associated with each C_{ij}) are entangled in the same manner as (in the notation of [1]) the eyebolts T_{i5} and T_{i9} are entangled, which is depicted in [1, Figure 4]. The configuration in any one of the C_{ij}'s consists of one row, the middle one, which looks just like the string of eyebolts of [1, Figure 4] and twelve other rows which run essentially parallel to the middle row, but for which there is no entangling in the slice removed from the outer eyebolt; instead, these twelve rows appear as strings of horizontal eyebolts similar to the strings of vertical eyebolts pictured in the figure, with some additional bending resulting from the hook in C_{ij}.

The eyebolts run so close to C_{ij} that

$$\bigcup_{m=1}^{12} L_{km} \subset T_{ij} \quad \text{and} \quad L_{k,13} \subset T_{ij} \cup T_{i,j+1}.$$

For the next stage, we repeat the construction outlined in the four preceding paragraphs to obtain a configuration of eyebolts that link vertically pointing downward. The special entangling is most easily handled by defining subdivisions so that all those disks corresponding to the ends of the removed slices lie in the seventh columns and in either the fifth or ninth rows of the subdivisions.

To be sure, we must make other restrictions to guarantee that the limit of $D_1, D_2, \cdots$ is a disk D. Such restrictions are discussed extensively in [3], [10], and we omit these details, being content with the mention of the unusual aspects of our construction. Note that each stage in this process can be placed essentially above the plane of D_1; consequently, we may assume that D is contained in the interior of a disk D^* which is locally tame modulo D.

Part II. *Properties of D.* With the techniques of [3, Section 7] one can give a straightforward proof that $E^3 - D$ fails to be 1-LC at each point of Int D. Hence, D is wildly embedded.

We call an arc A in D *vertical* (*horizontal*) if, for each positive integer i, there exist two columns (rows) of the partition of D_i such that A is contained in the union of all those thickened disks, together with their associated eyebolts, in the subdivision from the two columns (rows).

Under this interpretation of vertical and horizontal, we can obtain the analogue of Proposition 3.2 for this example by simply looking at the appropriate subsequence of defining stages. For instance, if A is a horizontal subarc of D and p is the right endpoint of A, then we look at stages $2, 6, \cdots, 4n+2, \cdots$ to see that loops near A can be pushed towards p through $E^3 - D^*$. Similarly, we look at the appropriate subsequences to show that $E^3 - D^*$ is 1-LC at points of BdD.

The rest of the proof of Theorem 4.1 proceeds through analogues of Propositions 3.3 and 3.4, exactly as in Section 3, and ends with an appeal to Theorem 2.5.

THEOREM 4.2. *Suppose D is a disk in E^3 such that each map of E^3 to itself squeezing D to an arc yields a tame arc, and suppose g is a cell-like mapping of E^3 to itself such that g is a homeomorphism of $E^3 - D$ onto $E^3 - g(D)$ and $g(D)$ is a finite graph. Then $g(D)$ is tamely embedded.*

Provided the usual disk D^* exists, this result, the obvious parallel to Theorem 3.6, has a relatively easy proof. Without this hypothesis, however, the proof involves more technicalities than we wish to describe here. The key to the argument is Theorem 1 of [12]. Methods of this paper can be used to prove that $E^3 - g(\mathrm{D})$ satisfies the 1-FLG property described in [12], and some of the messier 3-space techniques, combined with [7, Th. 2′], can be used to show that $g(D)$ pierces disks at each point of a subset dense in $g(D)$.

5. Generalizations to k-cells in E^n

A natural question begs some consideration: *for which combinations of positive integers $j < k \leq n$ can we find a wildly embedded k-cell K in E^n such that every map of E^n to itself squeezing K to a j-cell yields a tame j-cell?* (We leave the definition of such a map to the reader.) With techniques very similar to those used in this paper, one can prove that this question has an affirmative answer if $1 \leq j < k < n$ and $n \geq 5$. On the other hand, Propositions 2.1 and 3.1 carry the warning not to expect the same answer when $k = n$, and the situation when $k = n = 3$ bears this out.

THEOREM 5.1. *A necessary and sufficient condition for a 3-cell K in E^3 to be tame is that for each map f of E^3 to itself squeezing K to an arc, $f(K)$ is tame.*

Proof. As remarked in [7, p. 371], the necessity follows from known results. To prove sufficiency, one must observe that, although it is not stated in the hypothesis of Theorem 3 of [7], for each $p \in \text{Bd}K$ there exists a map f squeezing K to an arc such that $f(p)$ is an endpoint of $f(K)$. Clearly, the tameness of $f(K)$ implies that $E^3 - K$ is 1-LC at p. It follows from [4, Th. 2] that the condition is sufficient.

Remark. The result above also holds for maps squeezing the 3-cell K to 2-cells. For purposes of comparison it may be of some value to observe that, according to Bing's 1-ULC condition, if there exists one map f of E^3 to itself squeezing K to a tame 2-cell, then K is tame.

References

1. W. R. ALFORD, "Some 'nice' wild 2-spheres in E^3," *Topology of 3-manifolds and Related Topics*, Prentice-Hall, New York, 1962, pp. 29–33.
2. J. J. ANDREWS AND M. L. CURTIS, *n-space modulo an arc*, Ann. of Math. (2), vol. 75 (1962), pp. 1–7.
3. R. H. BING, *A wild surface each of whose arcs is tame*, Duke J. Math., vol. 28 (1961), pp. 1–16.
4. ———, *A surface is tame if its complement is 1-ULC*, Trans. Amer. Math. Soc., vol. 101 (1961), pp. 294–305.
5. MORTON BROWN, *Wild cells and spheres in higher dimensional spaces*, Michigan Math. J., vol. 14 (1967), pp. 219–224.
6. J. L. BRYANT AND C. L. SEEBECK, III, *Locally nice embeddings of polyhedra*, Quart. J. Math. Oxford (2), vol. 19 (1968), pp. 257–274.
7. R. J. DAVERMAN AND W. T. EATON, *An equivalence for the embeddings of cells in a 3-manifold*, Trans. Amer. Math. Soc., vol. 145 (1969), pp. 369–381.
8. ———, *Each disk in E^n can be sequeezed to an arc*,
9. P. H. DOYLE AND J. G. HOCKING, *Some results on tame disks and spheres in E^3*, Proc. Amer. Math. Soc., vol. 11 (1960), pp. 832–836.
10. D. S. GILLMAN, *Note concerning a wild sphere of Bing*, Duke Math. J., vol. 31 (1964), pp. 247–254.
11. D. R. MCMILLAN, JR., *A criterion for cellularity in a manifold*, Ann. of Math. (2), vol. 79 (1964), pp. 327–337.
12. ———, *Local properties of the embedding of a graph in a 3-manifold*, Canadian J. Math., vol. 18 (1966), pp. 517–528.
13. ———, *Piercing a disk along a cellular set*, Proc. Amer Math. Soc., vol. 19 (1968), pp. 153–157.
14. CHARLES L. SEEBECK, III, *Tame arcs on wild cells*, Proc. Amer. Math. Soc., vol. 29 (1971), pp. 197–201.
15. R. B. SHER, *Tame polyhedra in wild cells and spheres*, Proc. Amer. Math. Soc., vol. 30 (1971), pp. 169–174.

THE UNIVERSITY OF TENNESSEE
KNOXVILLE, TENNESSEE