WEAKLY SEMI-COMPLETELY CONTINUOUS A^*-ALGEBRAS

BY

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1. Introduction

A Banach algebra A is called weakly semi-completely continuous (w.s.c.c.), if, for every element $a \in A$, the mapping $T_a : x \to axa$ $(x \in A)$ is a weakly completely continuous operator on A, i.e., if T_a maps bounded sets into sets which are relatively compact in the weak topology $\sigma(A, A')$, where A' is the conjugate space of A. Ogasawara and Yoshinaga [8] have studied weakly completely continuous (w.c.c.) Banach*-algebras and Alexander [1] developed a theory of compact Banach algebras (which we call semi-completely continuous or briefly s.c.c. algebras). It is thus natural to have a look at w.s.c.c. Banach algebras and to see how they are related to s.c.c. and w.c.c. Banach algebras. We confine our study of w.s.c.c. Banach algebras to A^*-algebras with the k-property, i.e. A^*-algebras A for which there exists a constant k such that $\| xy \| \leq k \| x \| \, | y |$ for all $x, y \in A$.

In §4 we show that a w.s.c.c. A^*-algebra with the k-property and an identity element is finite dimensional. Using this fact we prove that an A^*-algebra with the k-property which contains non-zero w.s.c.c. elements contains minimal idempotents. In §5 we study the relationship between s.c.c. and w.s.c.c. A^*-algebras with the k-property. A B^*-algebra is s.c.c. if and only if it is w.s.c.c. If A is an A^*-algebra with the k-property and $\mathfrak{A}$ is its completion than A is w.s.c.c. if and only if $\mathfrak{A}$ is w.s.c.c. If A is a commutative A^*-algebra with the k-property then A is s.c.c. if and only if it is w.s.c.c.

Section 6 is devoted to the study of modular annihilator Banach algebras from the point of view of s.c.c. and w.s.c.c. Banach algebras. For example we show that if A is a semi-simple Banach algebra, then A is modular annihilator if and only if for every maximal modular left (right) ideal M there exists a right (left) identity u for A modulo M such that u is an s.c.c. element of A (Theorem 6.2). Thus, in particular, every s.c.c. Banach algebra is modular annihilator. If A is an A^*-algebra with the k-property then A is modular annihilator if and only if A is w.s.c.c. (Theorem 6.7). We also show that an A^*-algebra A is modular annihilator if and only if every maximal commutative $*$-subalgebra of A is modular annihilator.

2. Preliminaries

All algebras and vector spaces under consideration are over the complex field C. A Banach algebra with an involution $x \to x^*$ is called a Banach $*$-algebra. A Banach $*$-algebra A is a B^*-algebra if the norm and the involution satisfy the condition $\| x^*x \| = \| x \|^2$, $x \in A$. If A is a Banach $*$-algebra

Received June 18, 1970.

on which there is defined a second norm $|\cdot|$ which satisfies, in addition to the multiplicative condition $|xy| \geqq |x||y|$, the B^*-algebra condition $|x^*x| = |x|^2$, then $A$is called an A^*-algebra. The norm $|\cdot|$ is called an auxiliary norm on A, and $|\cdot| \leq \beta \|\cdot\|$ for some constant $\beta > 0$ [9; p. 187]. An element x of an A^*-algebra is called normal if $x^*x = xx^*$.

Let A be an A^*-algebra with the k-property. Than A has a unique auxiliary norm topology [8; p. 18, Theorem 3] and hence can be embedded as a dense subalgebra in a unique (up to $*$-isomorphism) B^*-algebra $\mathfrak{A}$. We refer to the algebra $\mathfrak{A}$ as the *completion* of A. It follows that A is a dense two-sided ideal of $\mathfrak{A}$ [8; p. 17, Lemma 3] and $\|xy\| \leq k\|x\| |y|$ for all $x \in A$, $y \in \mathfrak{A}$. Conversely, if A is an A^*-algebra which is a dense two-sided ideal of the B^*-algebra $\mathfrak{A}$ then A has the k-property [8; p. 18, Lemma 4]. Thus the k-property characterizes those A^*-algebras which are dense two-sided ideals of B^*-algebras.

Let A be a Banach algebra. An element $a \in A$ is called completely continuous (c.c.) if the mappings $x \to ax$ and $x \to xa$ are completely continuous operators on A. An element $a \in A$ is called semi-completely continuous (s.c.c.) if the mapping $x \to axa$ is a completely continuous operator on A. (In [1] such an element is called compact.) It is clear that if a is c.c. then it is s.c.c., but the converse is not true as is shown in [1]. An element $a \in A$ is called weakly completely continuous (w.c.c.) if the mappings $x \to ax$ and $x \to xa$ are weakly completely continuous operators on A. An element $a \in A$ is called weakly semi-completely continuous (w.s.c.c.) if the mapping $x \to axa$ is a weakly completely continuous operator on A. If every element of a Banach algebra A is c.c. (resp. s.c.c., w.c.c. or w.s.c.c.) we say that A is a c.c. (resp. s.c.c., w.c.c. or w.s.c.c.) algebra.

Since every norm-closed subspace of a Banach space is weakly closed [7; p. 422, Theorem 13], it follows that every closed left (right) ideal of a Banach algebra is weakly closed.

For any subset S of an algebra A, let $l_A(S)$ and $r_A(S)$ be respectively the left and right annihilators of S in A. An algebra A is modular annihilator if every maximal modular left (right) ideal of A has a non-zero right (left) annihilator. A Banach algebra A is an annihilator algebra if for every closed left ideal J and for every closed right ideal R we have $r_A(J) = (0)$ if and only if $J = A$ and $l_A(R) = (0)$ if and only if $R = A$. It is a dual algebra if $l_A(r_A(J)) = J$ and $r_A(l_A(R) = R$ for every closed left ideal J and for every closed right ideal R of A.

If S is a subset of a Banach algebra A, $\mathrm{cl}_A(S)$ will denote the closure of S in A. For all other concepts used in this paper see [9].

3. Some lemmas

LEMMA 3.1. *Let A be a w.s.c.c. Banach algebra. Then every closed subalgebra B of A is w.s.c.c. If I is a closed two-sided ideal of A, then A/I is a w.s.c.c. Banach algebra.*

Proof. Let $x \in B$ and let $\{x_n\}$ be a bounded sequence in B. Since x is a w.s.c.c. element of A, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $y \in A$ such that $\{x x_{n_k} x\}$ converges weakly to y. But B is weakly closed and every continuous linear functional on B has a continuous linear extension to A. Hence $y \in B$, and so B is w.s.c.c.

Now let I be a closed two-sided ideal of A, $[x]$ an element of A/I and $\{[x_n]\}$ a bounded sequence in A/I, say $\| [x_n] \| \leq k$ $(n = 1, 2, \cdots)$. We can clearly choose a representative element x_n of $[x_n]$ such that $\| x_n \| \leq 2k$ $(n = 1, 2, \cdots)$. Let x be any representative of $[x]$. Since x is w.s.c.c., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x x_{n_k} x\}$ converges weakly to an element y in A. Since the conjugate space of A/I is isometrically isomorphic to

$$I^0 = \{f \in A' : f(x) = 0 \quad \text{for all} \quad x \in I\},$$

$\{[x][x_{n_k}][x]\}$ converges weakly to $[y]$ in A/I. Hence A/I is w.s.c.c.

LEMMA 3.2. *Let A be a semi-simple Banach algebra. Then every element of the socle of A is w.s.c.c. In particular, if A has dense socle, then A is a w.s.c.c. algebra.*

Proof. Since every s.c.c. element of A is w.s.c.c., it follows from [1; p. 14, Theorem 7.2] that the socle S_A of A consists of w.s.c.c. elements. If $cl_A(S_A) = A$, then A is s.c.c. by [1; p. 15, Theorem 7.3] and so w.s.c.c.

4. Existence of minimal idempotents

LEMMA 4.1. *Let A be an A^*-algebra with the k-property. Then every closed left (right) ideal of A which contains a non-zero w.s.c.c. element contains a w.s.c.c. idempotent.*

Proof. Let J be a closed left ideal of A which contains a non-zero w.s.c.c. element. Then J clearly contains a self-adjoint w.s.c.c. element, say a, such that $| a | = 1$. Then

$$\| a^{2^n} \| \leq k \| a^2 \| \, | a^{2^n - 1} | \leq k \| a^2 \| \qquad (n = 1, 2, \cdots).$$

Let $S = \{a^4, a^8, a^{16}, \cdots\}$ and let $G(a)$ be the set of all weak adherent points of S, i.e., the set of points z such that every weak neighborhood of each z contains some a^{2^n} for arbitrarily large n. Since S is contained in the set $\{axa : x \in A \text{ and } \| x \| \leq k\}$ whose weak closure is compact, by [7; p. 430, Theorem 1], $G(a)$ is not empty and every subsequence of S contains a subsequence which converges weakly to an element of $G(a)$. Moreover, it is easy to see that, for every $z \in G(a)$, there is a subsequence of S which converges weakly to z. (See the proof of [10, Lemma 3.1].) We show now that $G(a)$ contains non-zero elements. Let B be the closed $*$-subalgebra of A generated by a and let $\mathfrak{B}$ be the completion of B in the norm $| \cdot |$. It is clear that $\mathfrak{B}$ is a commutative B^*-algebra and that B is dense in $\mathfrak{B}$. Since $| a | = 1$, it is easy to see that there exists a multiplicative linear functional f on $\mathfrak{B}$ such that $|f(a)| = 1$. Let $f' = f \mid B$, the restriction of f to B. Then f' is a multiplicative linear functional on B and hence continuous. Let $g \in A'$ be an extension of f' to all of A with $\| g \| = \| f' \|$. Then $| g(a^{2^n}) | = 1$ for all $n = 0, 1, 2, \cdots$. Thus $G(a)$ contains non-zero elements. By the argument given in the proof of [5; p. 180, Theorem 4], $G(a)$ is a group. Let u be the identity of $G(a)$. Then $u \neq 0$, $u^2 = u$, and since $a^* = a$, $u^* = u$. Since J is weakly closed, $u \in J$.

THEOREM 4.2 *Let A be an A^*-algebra with the k-property and an identity element. If A is w.s.c.c. then A is finite dimensional.*

Proof. Let B be a maximal commutative $*$-subalgebra of A. By Lemma 3.2, B is w.s.c.c. Let M be a maximal closed ideal of B and let $\{u_\alpha\}$ be a maximal orthogonal family of non-zero self-adjoint idempotents in M; $\{u_\alpha\}$ is not empty by Lemma 4.1. Let Q be the set of all elements $u \in B$ which are finite sums of elements from $\{u_\alpha\}$. Let e denote the identity of A; clearly $e \in B$. Since $eu_\alpha = u_\alpha$ for all α, we have

$$\| u_{\alpha_1} + \cdots + u_{\alpha_n} \| \leq k \| e \| \, | u_{\alpha_1} + \cdots + u_{\alpha_n} | \leq k \| e \|.$$

Thus Q is bounded and since e is w.s.c.c., Q has a weak adherent point, say q. It is easy to see that q is the only weak adherent point of Q, $q \neq 0$, $q^2 = q$ and $q \in M$. Moreover, $u_\alpha q = u_\alpha$ for all α so that $u_\alpha(e - q) = 0$ for all α. Since $e \notin M$, $e - q$ is a non-zero self-adjoint idempotent which is orthogonal to all u_α. We claim that $M \cap B(e - q) = (0)$. In fact, let $I = M \cap B(e - q)$ and suppose that $I \neq (0)$. Then, by Lemma 4.1, I contains a non-zero self-adjoint idempotent, say v. Since $v = v(e - q)$, we have $vu_\alpha = 0$ for all α. As $v \in M$, this shows that $\{u_\alpha\}$ is not a maximal orthogonal family of self-adjoint idempotents in M; a contradiction. Hence $I = (0)$ and consequently $e - q$ is a minimal idempotent of B. Since $B(e - q) + Bq = B$ and $Bq \subset M$, we have

$$M = Bq = \{x - x(e - q) : x \in B\}.$$

Thus every maximal closed ideal M of B is an annihilator ideal and consequently the carrier space Ω of B is discrete. Since B has an identity element, Ω is compact and therefore a finite set. Hence B is finite dimensional. Let $\{e_1, e_2, \cdots, e_n\}$ be the set of all self-adjoint minimal idempotents in B. It is easy to see that $\{e_1, e_2, \cdots, e_n\}$ is a maximal orthogonal family of self-adjoint minimal idempotents in A and $e = e_1 + \cdots + e_n$. Hence $A = \sum_{i,j=1}^{n} e_i A e_j$ and, since $e_i A e_j$ is one dimensional for all $i, j = 1, 2, \cdots, n$ [1; p. 13, Lemma 7.1], it follows that A is finite dimensional.

COROLLARY 4.3. *A w.s.c.c. B^*-algebra with identity is finite dimensional.*

COROLLARY 4.4. *Let A be an A^*-algebra with the k-property. Then every closed left (right) ideal of A which contains a non-zero w.s.c.c. element contains a minimal idempotent (which is w.s.c.c.).*

Proof. Let J be a closed left ideal of A which contains a non-zero w.s.c.c. element. Then, by Lemma 4.1, J contains a w.s.c.c. self-adjoint idempotent $u \neq 0$. Since $B = uAu$ is a w.s.c.c. A^*-algebra with k-property and an identity element u, by Theorem 4.2, B contains a self-adjoint minimal idempotent, say e. Since $eAe = euAue = eBe$, e is also a minimal idempotent of A. Clearly $e \in J$. A similar proof holds for a closed right ideal of A which contains a non-zero w.s.c.c. element.

5. w.s.c.c. A^*-algebras

LEMMA 5.1. *Let A be an A^*-algebra with the k-property and let M be a maximal modular left ideal of A. Then $r_A(M) \neq (0)$ if and only if there exists a right identity u for A modulo M which is a normal w.s.c.c. element of A.*

Proof. Suppose that u is a normal w.s.c.c. right identity modulo M. Let B be a maximal commutative $*$-subalgebra of A containing u. Since B has the k-property and u is a w.s.c.c. element of B, by Corollary 4.4, B contains self-adjoint minimal idempotents. We claim that there exists a self-adjoint minimal idempotent in B which does not belong to M. Suppose that this is not true. Then $B \cap M$ is a non-zero modular ideal of B. Let M' be a maximal modular ideal of B containing $M \cap B$. Then M' contains all the self-adjoint minimal idempotents of B. Let $\{e_\alpha\}$ be a maximal orthogonal family of self-adjoint minimal idempotents in M', and let Q be the set of all elements of B which are finite sums of elements from $\{e_\alpha\}$. Then uQu is a bounded net and, since u is w.s.c.c., u^2Qu^2 converges weakly to a unique element v', say. Let $v = u^4$. It is clear that v is an identity modulo M' and that $v - v' \neq 0$ since $v \notin M'$ and $v' \in M'$. Moreover, it is easy to see that $(v - v')e_\alpha = 0$ for all α. Let J be the closure of $B(v - v')$ in B. Then $Je_\alpha = (0)$ for all α. Hence if $J \cap M' \neq (0)$ then there would exist a self-adjoint minimal idempotent in $J \cap M'$ which would be orthogonal to all e_α, contradicting the maximality of the family $\{e_\alpha\}$ in M'. Thus $J \cap M' = (0)$ and, since $J \neq (0)$, this shows that there exists a self-adjoint minimal idempotent e in B which does not belong to M' and consequently does not belong to M. Since e is also a minimal idempotent of A, we have $M \cap Ae = (0)$ and, since M is a maximal left ideal of A, we see that $M + Ae = A$. It now follows that $M = \{x - xe : x \in A\}$ and $r_A(M) = eA$. (See the proof of [12; p. 38, Lemma 3.3].)

Now suppose that M is a maximal modular left ideal for which $r_A(M) \neq (0)$, and let $R = r_A(M)$. Then $R^* \cap M = (0)$; for if $x \in R^* \cap M$, then $x^* \in R$ and $xx^* = 0$ which implies that $x = 0$. Since M is maximal, we have $M + R^* = A$. Thus R^* is a minimal left ideal and therefore of the form Ae, where e is a self-adjoint minimal idempotent. Thus $R = eA$ and $M = \{x - xe : x \in A\}$, where e is a normal w.s.c.c. element of A.

THEOREM 5.2. *Let A be a B^*-algebra. Then A is w.s.c.c. if and only if A is dual.*

Proof. Suppose A is w.s.c.c. Let M be a maximal modular left ideal of A

and u a right identity for A modulo M. Since $u + u^*(1 - u)$ is also a right identity for A modulo M [9; p. 42] which is self-adjoint and w.s.c.c., by Lemma 5.1, $r_A(M) \neq (0)$. Applying the continuity of the involution, we see that A is modular annihilator and therefore, by [12; p. 42, Theorem 4.1], A is dual. (Duality of A also follows from [6; p. 48, Théorème (2.9.5)] since every maximal modular left (right) ideal of A is an annihilator ideal.) Conversely, if A is dual then it has dense socle and therefore is w.s.c.c. by Lemma 3.2.

COROLLARY 5.3. *A B^*-algebra A is s.c.c. if and only if it is w.s.c.c.*

Proof. Clearly if A is s.c.c. then it is w.s.c.c. The converse follows from Theorem 5.2 and the fact that a dual B^*-algebra is s.c.c.

Let A be an A^*-algebra with the k-property and $\mathfrak{A}$ the completion of A; A is a dense two-sided ideal of $\mathfrak{A}$. For each $x \in A$ and $f \in A'$, let $x \circ f$ and $f \circ x$ be the linear functionals on $\mathfrak{A}$ defined by $(x \circ f)y = f(yx)$ and $(f \circ x)y = f(xy)$ for all $y \in \mathfrak{A}$. Since $\| xy \| \leq k \| x \| \mid y \mid$ for all $x \in A$ and $y \in \mathfrak{A}$, they are continuous linear functionals on $\mathfrak{A}$. Similarly, for $x \in \mathfrak{A}$ and $F \in \mathfrak{A}'$, we define $x \circ F$ and $F \circ x$, which are clearly continuous linear functionals on $\mathfrak{A}$. Their restrictions $(x \circ F)_A$ and $(F \circ x)_A$ to A are also continuous linear functionals on A. In fact, if $y \in A$, then

$$\mid (F \circ x)_A y \mid = \mid F(xy) \mid \leq \mid F \mid \mid xy \mid \leq \beta \mid F \mid \| xy \| \leq k\beta \mid F \mid \mid x \mid \| y \|$$

where $\mid F \mid$ denotes the bound of F in $\mathfrak{A}$. Similarly we can show that $(x \circ F)_A$ is continuous on A with respect to the norm $\| \cdot \|$.

THEOREM 5.4. *Let A be an A^*-algebra with the k-property and $\mathfrak{A}$ the completion of A. Then A is w.s.c.c. if and only if $\mathfrak{A}$ is w.s.c.c.*

Proof. Suppose that A is w.s.c.c. Let $\{x_n\}$ be a bounded sequence in $\mathfrak{A}$ and let $x \in A$. Since $\| xx_n \| \leq k \| x \| \mid x_n \mid$, $\{xx_n\}$ is a bounded sequence in A and, since A is w.s.c.c., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $z \in A$ such that, for all $f \in A'$,

$$f(x^2 x_{n_k} x) \rightarrow f(z).$$

Since $(x \circ F)_A \in A'$, for all $F \in \mathfrak{A}'$, we have

$$F(x^2 x_{n_k} x^2) = (x \circ F)_A(x^2 x_{n_k} x) \rightarrow (x \circ F)_A(z) = F(zx).$$

Thus x^2 is a w.s.c.c. element of $\mathfrak{A}$. Now every self-adjoint element of $\mathfrak{A}$ is the limit of a sequence of self-adjoint elements of A. Since every positive element a of $\mathfrak{A}$ is of the form $a = b^2$, where b is a self-adjoint element of $\mathfrak{A}$, it follows that every positive element of $\mathfrak{A}$ is w.s.c.c. In fact, let a be a positive element of $\mathfrak{A}$, $\{a_n\}$ a sequence of positive elements in A such that $\mid a_n - a \mid \rightarrow 0$, $T_a : x \rightarrow axa$ and $T_{a_n} : x \rightarrow a_n x a_n$ $(x \in A)$. Then the operator bound

$$\mid T_a - T_{a_n} \mid \leq \mid a - a_n \mid [\mid a_n \mid + \mid a \mid];$$

so that $T_{a_n} \rightarrow T_a$. Since each a_n is a w.s.c.c. element of A, it follows from

[7; p. 483, Corollary 4] that a is w.s.c.c. Now let $\mathfrak{M}$ be a maximal modular left ideal of $\mathfrak{A}$ and let u be a right identity for $\mathfrak{A}$ modulo $\mathfrak{M}$. We may assume that $u^* = u$; otherwise we take $u + u^*(1 - u)$ for a right identity modulo $\mathfrak{M}$. Then u^2 is positive and a right identity modulo $\mathfrak{M}$. Since u^2 is w.s.c.c., by Lemma 5.1, $r_{\mathfrak{A}}(\mathfrak{M}) \neq (0)$. Thus $\mathfrak{A}$ is modular annihilator and hence dual. Therefore, by Theorem 5.2, $\mathfrak{A}$ is w.s.c.c.

Conversely, suppose that $\mathfrak{A}$ is w.s.c.c. By Theorem 5.2, $\mathfrak{A}$ is dual and hence w.c.c. [8; 21 Theorem 6]. Let $x \in A$ and let $\{x_n\}$ be a bounded sequence in A. Then $\{x_n\}$ is also a bounded sequence in $\mathfrak{A}$ and hence, since $\mathfrak{A}$ is w.c.c., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $z \in \mathfrak{A}$ such that $F(x_{n_k} x) \rightarrow F(z)$ for all $F \in \mathfrak{A}'$. Now, for all $f \in A'$, $f \circ x \in \mathfrak{A}'$ and so

$$f(xx_{n_k} x) = (f \circ x)(x_{n_k} x) \longrightarrow (f \circ x)(z) = f(xz),$$

for all $f \in A'$. Since $x \in A$, $xz \in A$ so that $\{xx_{n_k} x\}$ converges weakly to an element in A. Thus A is w.s.c.c.

THEOREM 5.5. *Let A be a w.s.c.c. A^*-algebra with the k-property. If, for every $x \in A$, x belongs to the closure of Ax, then A is dual.*

Proof. Let $\mathfrak{A}$ be the completion of A. Then, by Theorems 5.2 and 5.4, $\mathfrak{A}$ is dual. Since A is a dense two-sided ideal of $\mathfrak{A}$, [8; p. 28, Lemma 8] shows that A is dual.

THEOREM 5.6. *Let A be a commutative A^*-algebra with the k-property. Then A is s.c.c. if and only if A is w.s.c.c.*

Proof. If A is s.c.c. then it is clearly w.s.c.c. So suppose now that A is w.s.c.c. Then the completion $\mathfrak{A}$ of A is a dual commutative B^*-algebra and hence c.c. Let $\{x_n\}$ be a bounded sequence in A and let $x \in A$. Since $\{x_n\}$ is bounded in the norm $| \cdot |$ and since $\mathfrak{A}$ is c.c., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{xx_{n_k}\}$ converges to an element $y \in \mathfrak{A}$ in the norm $| \cdot |$. But $yx \in A$ and so, by the k-property, $\{xx_{n_k}x\}$ converges to yx in the norm $\| \cdot \|$. Hence A is s.c.c.

6. Modular annihilator Banach algebras

LEMMA 6.1. *Let A be a semi-simple Banach algebra. Then every idempotent which is an s.c.c. element of A belongs to the socle of A.*

Proof. Let e be an idempotent which is s.c.c. Then $B = eAe$ is a finite-dimensional Banach algebra with identity e. Since A is semi-simple, B is also semi-simple (see [1; p. 6, Lemma 4.5]). Hence $e = e_1 + \cdots + e_n$, where e_i are minimal idempotents of B $(i = 1, 2, \cdots, n)$. Since $e_i Ae_i = e_i eAee_i = e_i Be_i = Ce_i$, each e_i also a minimal idempotent of A. Hence e belongs to the socle of A.

THEOREM 6.2. *Let A be a semi-simple Banach algebra. Then A is a modular annihilator algebra if and only if, for every maximal modular left* (*right*) *ideal*

M of A, there is a right (left) identity for A modulo M which is an s.c.c. *element of A.*

Proof. If A is a modular annihilator algebra, then every maximal modular left ideal M of A is of the form $M = \{x - xe : x \in A\}$, where e is a minimal idempotent. By [1; p. 14, Theorem 7.4], e is s.c.c. A similar statement holds for maximal modular right ideals of A. To prove the converse, let M be a maximal modular left ideal and let u be a right identity modulo M which is s.c.c. Let B be the closed subalgebra of A generated by u. That is, B is the closure of the algebra of all polynomials of the form $\alpha_1 u + \alpha_2 u^2 + \cdots \alpha_n u^n$, where n is an arbitrary positive integer. B is a non-radical s.c.c. Banach algebra. Now $M \cap B$ is a modular ideal of B and so can be extended to a maximal modular ideal M' of B with u being an identity for B modulo M'. Since the carrier space of B is discrete [1; p. 10, Theorem 6.6], there exists an idempotent $e \in B$ such that $e \notin M'$ [9; p. 168, Theorem (3.6.3)], and hence $e \notin M$. But e is an s.c.c. element of A. Therefore, by Lemma 6.1, there exists a minimal idempotent e_1 in A such that $e_1 \notin M$. The argument in the proof of Lemma 5.1 now shows that $r_A(M) \neq (0)$. A similar proof holds for a maximal modular right ideal of A. Thus A is modular annihilator.

THEOREM 6.3. *Let A be an A^*-algebra. Then A is a modular annihilator algebra if and only if every maximal communtative $*$-subalgebra of A is a modular annihilator algebra.*

Proof. If A is modular annihilator then, by [3; p. 517, Corollary], every maximal commutative $*$-subalgebra B of A is modular annihilator. Another proof of this fact can be given as follows: Let $\mathfrak{A}$ be the completion of A in an auxiliary norm. (In [11] it is shown that $\mathfrak{A}$ is unique up to $*$-isomorphism.) Since $\mathfrak{A}$ has dense socle [2; p. 287, Lemma 2.6], $\mathfrak{A}$ is dual. Let $\mathfrak{B}$ be the closure of B in $\mathfrak{A}$; $\mathfrak{B}$ is a dual $*$-subalgebra of $\mathfrak{A}$. For any $x \in B$, let $Sp_A(x)$, $Sp_B(x)$ and $Sp_{\mathfrak{A}}(x)$ denote the spectrum of x in A, B and $\mathfrak{A}$ respectively. Since B is maximal in A, $Sp_B(x) = Sp_A(x)$ and hence, since $|\cdot|$ is a Q-norm on A, it is also a Q-norm on B. (See [2; p. 285, Lemma 1.2].) Thus if M is a maximal modular ideal of B then M is closed with respect to $|\cdot|$ in B. Let u be an identity for B modulo M. Than u can be written in the form $u = \sum_\alpha \lambda_\alpha e_\alpha$, where $\{e_\alpha\}$ is the maximal orthogonal family of self-adjoint minimal idempotents in $\mathfrak{B}$; $\sum_\alpha \lambda_\alpha e_\alpha$ converges to u in the norm $|\cdot|$. (See the proof of [8; p. 21, Theorem 6].) But A is a modular annihilator $*$-subalgebra of $\mathfrak{A}$ and $Sp_{\mathfrak{A}}(u) \subseteq Sp_A(u)$. Hence, by [2; p. 287, Lemma 2.5], every $e_\alpha \in A$ and so every $e_\alpha \in A \cap \mathfrak{B} = B$; moreover every e_α is a minimal idempotent of A. Since $u \notin M$ and M is closed in $|\cdot|$, it follows that there exists at least one $e_\alpha \notin M$. Hence $l_B(M) = r_B(M) \neq (0)$ and consequently B is modular annihilator.

To prove the converse, let M be a maximal modular left ideal of A and u a right identity modulo M. We may assume that $u^* = u$; otherwise we take $u + u^*(1 - u)$. Let B be a maximal commutative $*$-subalgebra of A containing u. Since $M \cap B$ is a modular ideal of B and B is a modular annihilator algebra, there exists a minimal idempotent e in B such that $e \notin M \cap B$. But this means that e is a minimal idempotent of A and $e \notin M$. Hence $r_A(M) \neq (0)$. The continuity of the involution now completes the proof.

COROLLARY 6.4. *Let A be a modular annihilator A^*-algebra. Then every normal element $x \in A$ can be written in the form $x = \sum_\alpha \lambda_\alpha e_\alpha$, where $\{e_\alpha\}$ is an orthogonal family of self-adjoint minimal idempotents in A and $\{\lambda_\alpha\}$ is a family of scalars. The sum $\sum_\alpha \lambda_\alpha e_\alpha$ converges to x in the auxiliary norm of A.*

Proof. Let B be a maximal commutative $*$-subalgebra of A containing x. By the first paragraph of the proof of Theorem 6.3, B contains such a family $\{e_\alpha\}$.

COROLLARY 6.5. *Let A be an A^*-algebra. Then A is modular annihilator if and only if A has the spectral expansion property.*

Proof. For the definition of the spectral expansion property see [2; p. 288]. If A is modular annihilator then, by Corollary 6.4, A has the spectral expansion property. Conversely suppose A has the spectral expansion property. Then $|\cdot|$ is a Q-norm on A [2; p. 284] so that if M is a maximal modular left ideal of A then M is closed with respect to $|\cdot|$. Let u be a right identity modulo M; we may clearly assume that $u^* = u$. Then $u = \sum_\alpha \lambda_\alpha e_\alpha$, where $\{e_\alpha\}$ is an orthogonal family of self-adjoint idempotents in the socle S_A of A. Since $u \notin M$, there is at least one $e_\alpha \notin M$. As every $e_\alpha \in S_A$, this means that there exists a self-adjoint minimal idempotent $e \notin M$. It follows now that $r_A(M) \neq (0)$ and that A is modular annihilator.

THEOREM 6.6. *Let A be an A^*-algebra with the k-property and $\mathfrak{A}$ the completion of A. Then A is modular annihilator if and only if $\mathfrak{A}$ is modular annihilator.*

Proof. If A is modular annihilator then $\mathfrak{A}$ is modular annihilator since $\mathfrak{A}$ is dual. The converse follows from [12; p. 40, Theorem 3.7]. However we can give a direct proof of this. In fact, since A has the k-property, it is easy to see that if M is a maximal modular left ideal of A then $\mathfrak{M} = \mathrm{cl}_{\mathfrak{A}}(M)$ is a modular left ideal of $\mathfrak{A}$. Hence $r_{\mathfrak{A}}(\mathfrak{M}) \neq (0)$. Since A is a dense two-sided ideal of $\mathfrak{A}$ and $\mathfrak{A}$ is semi-simple $A \cap r_{\mathfrak{A}}(\mathfrak{M}) \neq (0)$. This shows that $r_A(M) \neq (0)$ and applying the continuity of the involution completes the proof.

Combining Theorems 5.2, 5.4, and 6.6 we obtain the following:

THEOREM 6.7. *Let A be an A^*-algebra with the k-property and let $\mathfrak{A}$ be the completion of A. Then the following statements are equivalent:*

(i) *A is a modular annihilator algebra.*
(ii) *A is a w.s.c.c. algebra.*
(iii) *$\mathfrak{A}$ is a modular annihilator algebra.*
(iv) *$\mathfrak{A}$ is a w.s.c.c. algebra.*

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