

# THE VOLUME OF A TUBE IN COMPLEX PROJECTIVE SPACE

BY

F. J. FLAHERTY

## 1. Introduction

The relationship between the volume of a tube around a submanifold and integral invariants of the submanifold has interested geometers for years. Steiner considered the problem as long ago as 1840 [10].

Many results on the subject appear in the 1940's but Hermann Weyl's work in 1939 [11] yields the definitive answer for tubes around submanifolds in the model spaces of Riemannian geometry (euclidean, spherical, or hyperbolic space). In fact, Weyl's results are so powerful that the first general Gauss-Bonnet theorem, proved by Allendoerfer [1] and Allendoerfer-Weil [2], used Weyl's formula in a fundamental manner. Of course, all of this took place before Chern provided us with his intrinsic proof of the general Gauss-Bonnet theorem [3].

In this paper we compute the volume of a tube around a compact subdomain, with smooth boundary, of a holomorphic submanifold of complex projective space. Essentially, we identify certain extrinsically defined functions as intrinsic scalar densities. The computation appears in Section 4.

A very crude estimate of the sum of the Betti numbers of the path space of a submanifold of a pinched manifold appears in work of Flaherty and Grossman [6]. In fact, in the present paper, we prove that the sum of the first  $\lambda$  Betti numbers of the path space of a compact holomorphic submanifold of complex projective space is dominated by a linear polynomial in  $\lambda$ .

Sections 2 and 3 serve as background for the main theorems, found in Sections 4 and 5. Section 2 recalls basic ideas and fixes notation for complex projective space necessary to the calculations in Section 4 while Section 3 reviews the local geometry of holomorphic submanifolds of Kaehler manifolds, on which the remainder of the paper rests.

We plan in the future to investigate the relation of this formula to equidistribution theory of holomorphic curves [13].

## 2. Complex projective space

We devote this section to the geometry of complex projective space, the ambient space for our submanifolds.

Let  $\mathbf{C}^{n+1}$  be the space of  $(n + 1)$ -tuples of complex numbers and  $e_0 \cdots e_n$  a frame field on  $\mathbf{C}^{n+1}$ ; then

$$de_A = \sum_B e_B \omega_A^B \quad (0 \leq B \leq n) \quad \text{and} \quad d\omega_B^A = \sum_C \omega_C^A \wedge \omega_B^C \quad (0 \leq C \leq n).$$

Received June 4, 1970.

Let  $P_n(\mathbf{C})$  be complex projective space and

$$\mathbf{C}^{n+1} - (0) \xrightarrow{\pi} P_n(\mathbf{C})$$

the canonical fibering. A frame  $e_0 \cdots e_n$  in  $\mathbf{C}^{n+1}$  is said to be adapted to  $P_n(\mathbf{C})$  if  $e_0$  is tangent to the fiber. Thus

$$d\pi = \sum e_i \omega_0^i + \sum \bar{e}_i \bar{\omega}_0^i \quad (1 \leq i \leq n).$$

If we suppose  $e_0 \cdots e_n$  unitary then the hermitian structure on  $P_n(\mathbf{C})$  is given by

$$ds^2 = 2 \sum \omega_0^j \bar{\omega}_0^j \quad (1 \leq j \leq n).$$

In homogeneous coordinates

$$ds^2 = (2/(z, z)^2)((z, z)(dz, dz) - (z, dz)(dz, z)).$$

The Kaehlerian connection on  $P_n(\mathbf{C})$  is given by

$$\pi_j^i = \omega_j^i - \delta_j^i \omega_0^0 \quad (1 \leq i, j \leq n).$$

and the curvature form  $(\Omega_j^i)$  is given by

$$\Omega_j^i = \theta^i \wedge \bar{\theta}^j + \delta_j^i \sum_k \theta^k \wedge \bar{\theta}^k$$

All of the details of these calculations may be found in [8].

### 3. Holomorphic submanifolds of Kaehler manifolds

Let  $Y_n$  be a Kaehler manifold, for a unitary coframe field  $\theta^1 \cdots \theta^n$  locally defined on  $Y$  the Kaehler form  $\varphi$  is given by

$$(3.1) \quad \varphi = \sqrt{-1} \sum_j \theta^j \wedge \bar{\theta}^j \quad (1 \leq j \leq n)$$

and the hermitian volume is given by  $\varphi^n/n!$

Let  $\theta^1 \cdots \theta^n$  be a local unitary coframe field and  $(\pi_j^i)$  the connection form matrix of the Kaehlerian connection then

$$(3.2) \quad d\theta^i + \sum_j \pi_j^i \wedge \theta^j = 0 \quad (1 \leq j \leq n).$$

Let  $X_m$  be a holomorphic submanifold of  $Y$ , then we may choose the  $\theta^r$ 's so that  $\theta^1 \cdots \theta^m$  is a unitary coframe field on  $X$ , moreover  $(\pi_\beta^\alpha)$ ,  $1 \leq \alpha, \beta \leq m$ , is the connection form matrix of the Kaehlerian connection on  $X$ .

Since  $\theta^r = 0$  on  $X$  for  $r = m + 1, \dots, n$ , it follows from (3.2) that

$$\sum_\alpha \pi_r^\alpha \wedge \theta^\alpha = 0 \quad (1 \leq \alpha \leq m).$$

As a result

$$(3.3) \quad \pi_\alpha^r = \sum_\beta \bar{S}_{\alpha\beta} \theta^\beta \quad (1 \leq \beta \leq m) \quad \text{with} \quad S_{\alpha\beta}^r = \bar{S}_{\beta\alpha}^r.$$

To complete the discussion of the local geometry of holomorphic submanifolds, let us recall that if  $(\bar{\Omega}_\beta^\alpha)$  is the curvature form matrix of  $(\pi_\beta^\alpha)$  and  $(\Omega_j^i)$  the curvature form of  $(\pi_j^i)$  then

$$(3.4) \quad \bar{\Omega}_\beta^\alpha = -\sum_{\gamma, \delta} \sum_r S_{\alpha\delta}^r \bar{S}_{\beta\gamma}^r \theta^\gamma \wedge \bar{\theta}^\delta + \Omega_\beta^\alpha$$

### 4. Volume of a tube in $P_n(\mathbf{C})$

First we will compute the volume element of a tube around a holomorphic submanifold of  $\mathbf{C}^n$ . By a tube of radius  $\sigma$  around a submanifold we mean the image of the normal  $\sigma$ -disc bundle under the exponential map. The calculation is simpler than the case of  $P_n(\mathbf{C})$  but necessary for the final formula. The added assumption of holomorphicity here is artificial as may be seen in [11].

Let  $X_m \rightarrow \mathbf{C}^n$  be a holomorphic submanifold, represented over a small neighborhood  $V$  by a holomorphic function  $z$ . Further let  $e_1 \cdots e_n$  be an adapted  $(1, 0)$ -unitary frame field; thus  $e_1 \cdots e_m$  is tangent to  $X$  and  $e_{m+1} \cdots e_n$  is normal to  $X$ . If  $\theta^1 \cdots \theta^n$  is the dual coframe field to  $e_1 \cdots e_n$  then since  $z$  is holomorphic and hence type-preserving

$$dz = \sum_\alpha e_\alpha \theta^\alpha + \sum_\alpha \bar{e}_\alpha \bar{\theta}^\alpha \quad (1 \leq \alpha \leq m).$$

A typical normal vector is of the form

$$(4.1) \quad w = z + \sum_r e_r t_r / \sqrt{2} + \sum_r \bar{e}_r \bar{t}_r / \sqrt{2} \quad (m + 1 \leq r \leq n).$$

We observe the following convention on indices from here on  $1 \leq \alpha, \beta, \dots \leq m, m + 1 \leq r, s, \dots \leq n$ . Differentiating (4.1),

$$\begin{aligned} dw &= dz + (1/\sqrt{2})(\sum_r de_r t_r + \sum_r e_r dt_r + \sum_r d\bar{e}_r \bar{t}_r + \sum_r \bar{e}_r d\bar{t}_r) \\ &= \sum_\alpha e_\alpha \theta^\alpha + \sum_\alpha \bar{e}_\alpha \bar{\theta}^\alpha + (1/\sqrt{2})(\sum_r (\sum_\alpha e_\alpha \omega_r^\alpha + \sum_s e_s \omega_r^s) t_r \\ &\quad + \sum_r e_r dt_r + \dots). \end{aligned}$$

Since the normal bundle is trivial over  $U$  we may choose a field of frames so that  $\omega_r^s$  and  $\bar{\omega}_r^s$  vanish on  $U$ . As a result

$$\begin{aligned} dw &= \sum_\alpha e_\alpha (\theta^\alpha + \sum_r \pi_r^\alpha t_r / \sqrt{2}) + \sum_\alpha \bar{e}_\alpha (\bar{\theta}^\alpha + \sum_r \bar{\pi}_r^\alpha \bar{t}_r / \sqrt{2}) \\ &\quad + \sum_r e_r dt_r / \sqrt{2} + \sum_r \bar{e}_r d\bar{t}_r / \sqrt{2}. \end{aligned}$$

If we restrict the Kaehler form  $\varphi$  to the tube

$$\varphi = \sqrt{-1} \sum_\alpha (\theta^\alpha + \sum_r \pi_r^\alpha t_r / \sqrt{2}) \wedge (\bar{\theta}^\alpha + \sum_r \bar{\pi}_r^\alpha \bar{t}_r / \sqrt{2}) + \frac{1}{2} \sum_r dt_r \wedge d\bar{t}_r$$

which by (3.3) yields

$$\begin{aligned} \varphi &= \sqrt{-1} \sum_\alpha \theta^\alpha \wedge \bar{\theta}^\alpha + \frac{1}{2} \sum_\alpha (\sum_r t_r S_{\alpha\gamma}^r) (\sum_r \bar{t}_r \bar{S}_{\alpha\beta}^r) \bar{\theta}^\gamma \wedge \theta^\beta \\ &\quad + \frac{1}{2} \sum_r dt_r \wedge d\bar{t}_r \\ &= \sqrt{-1} (\sum_{\alpha, \beta} (\delta_{\alpha\beta} - \frac{1}{2} \sum_\gamma \sum_r \bar{t}_r \bar{S}_{\alpha\gamma}^r \sum_r t_r S_{\gamma\beta}^r) \theta^\alpha \wedge \bar{\theta}^\beta \\ &\quad + \frac{1}{2} \sum_r dt_r \wedge d\bar{t}_r). \end{aligned}$$

Thus, the volume element for the tube is

$$(\sqrt{-1})^n \det (\delta_{\alpha\beta} - \frac{1}{2} \sum_{\gamma} (\sum_r t_r S_{\alpha\gamma}^r) (\sum_r \bar{t}_r \bar{S}_{\gamma\beta}^r) \cdot \wedge_{\alpha} (\theta^{\alpha} \wedge \bar{\theta}^{\alpha}) \wedge_r \frac{1}{2} (dt_r \wedge d\bar{t}_r)).$$

THEOREM. If  $X$  is a holomorphic submanifold of  $\mathbf{C}^n$ ,  $D$  a compact subdomain of  $X$  with smooth boundary and  $\nu_{\sigma}(D)$ , the tube of radius  $\sigma$  around  $D$  then

$$\text{vol } \nu_{\sigma}(D) = \int_D \wedge_{\alpha} \sqrt{-1} \theta^{\alpha} \wedge \bar{\theta}^{\alpha} \int_{|t_i| \leq \sigma^2} \det \left( \delta_{\alpha\beta} - \frac{1}{2} \sum_{\gamma} \left( \sum_r t_r S_{\alpha\gamma}^r \right) \left( \sum_r \bar{t}_r \bar{S}_{\gamma\beta}^r \right) \right) \wedge_r \frac{1}{2} \sqrt{-1} dt_r \wedge d\bar{t}_r$$

where  $\text{vol}$  is the hermitian measure on  $\mathbf{C}^n$ .

To compute the volume element for a tube around a holomorphic submanifold  $X$  of  $P_n(\mathbf{C})$ , observe that a typical point normal to  $X$  will be of the form  $\exp_z(t, \bar{t})$  where  $(t, \bar{t})$  is a complex normal vector. Since the canonical fibering  $\mathbf{C}^{n+1} - (0) \rightarrow P_n(\mathbf{C})$  is an hermitian submersion, the following diagram commutes

$$\begin{array}{ccc} T_p \mathbf{C}^{n+1} & \xrightarrow{\pi_*} & T_z P_n(\mathbf{C}) \\ \exp \downarrow & & \downarrow \exp_z \\ \mathbf{C}^{n+1} - (0) & \longrightarrow & P_n(\mathbf{C}) \end{array}$$

where  $\pi(p) = z$ .

Let  $e_0 \cdots e_n$  be a unitary frame field on  $\mathbf{C}^{n+1}$  adapted to  $P_n \mathbf{C}$  so that  $\pi_* e_1, \dots, \pi_* e_n$  are tangent to  $X$  and  $\pi e_0 = z$  where  $z$  is a local submanifold map of  $X$ . It follows from the diagram above that  $\pi(w) = \exp_z(t, \bar{t})$  where

$$w = e_0 t_0 + \sum_r e_r t_r / \sqrt{2} + \sum_r \bar{e}_r \bar{t}_r / \sqrt{2}$$

Since the frame is adapted, the derivative of  $w$  is

$$\begin{aligned} dw &= e_0 dt_0 + t_0 de_0 + d(\sum_r e_r t_r / \sqrt{2} + \sum_r \bar{e}_r \bar{t}_r / \sqrt{2}) \\ &= e_0(dt_0 + t_0 \omega_0^0) + t_0(\bar{e}_0 \bar{\omega}_0^0 + \sum_{\alpha} e_{\alpha} \theta^{\alpha} + \sum_{\alpha} \bar{e}_{\alpha} \bar{\theta}^{\alpha}) \\ &\quad + d(\sum_r e_r t_r / \sqrt{2} + \sum_r \bar{e}_r \bar{t}_r / \sqrt{2}). \end{aligned}$$

As in the case of a tube in  $\mathbf{C}^n$  we may assume that  $\pi'_s$  and  $\bar{\pi}'_s$  vanish locally. It follows then that

$$\begin{aligned} dw &= e_0(dt_0 + t_0 \omega_0^0) + t_0(\bar{e}_0 \bar{\omega}_0^0 + \sum_{\alpha} e_{\alpha} \theta^{\alpha} + \sum_{\alpha} \bar{e}_{\alpha} \bar{\theta}^{\alpha}) \\ &\quad + \sum_{\alpha,r} e_{\alpha} \pi_r^{\alpha} t_r / \sqrt{2} + \sum_{\alpha,r} \bar{e}_{\alpha} \bar{\pi}_r^{\alpha} \bar{t}_r / \sqrt{2} \\ &\quad + \sum_r e_r dt_r / \sqrt{2} + \sum_r \bar{e}_r d\bar{t}_r / \sqrt{2}. \end{aligned}$$

The Kaehler form  $\varphi$  for  $P_n(\mathbf{C})$  in the Fubini-Study metric of Section 2 is

$$(\sqrt{-1}/(w, w)^2)((w, w)(dw, dw) - (dw, w) \wedge (w, dw))$$

where  $(u \otimes \alpha, v \otimes \beta) = (u, v)\alpha \wedge \beta$  is the pairing used for vector-valued forms with  $u, v$  vectors and  $\alpha, \beta$  forms.

To compute  $\varphi$  on the tube, we normalize the  $t$ 's (homogeneous coordinates) and substitute:

$$(\sqrt{-1})t_0^2 (\sum_{\alpha} (\theta^{\alpha} + \sum_r \pi_r^{\alpha} t_r/t_0 \sqrt{2}) \wedge (\bar{\theta}^{\alpha} + \sum_r \bar{\pi}_r^{\alpha} \bar{t}_r/t_0 \sqrt{2}) + (1/2t_0^2) (\sum_r dt_r \wedge d\bar{t}_r - (\sum_r \bar{t}_r dt_r) (\sum_r t_r d\bar{t}_r))).$$

Hence the volume element for the tube is

$$\varphi^n/n! = (\sqrt{-1})^n t_0^{2n} \det (\delta_{\alpha\beta} - \frac{1}{2} \sum_{\gamma} (\sum_r t_r S_{\alpha\gamma}^r) (\sum_r \bar{t}_r \bar{S}_{\gamma\beta}^r)) \wedge_{\alpha} \theta^{\alpha} \wedge \bar{\theta}^{\alpha} \wedge_{\gamma} \frac{1}{2} dt_r \wedge d\bar{t}_r.$$

**THEOREM.** *If  $X$  is a holomorphic submanifold of  $P^n(\mathbf{C})$ ,  $D$  is a compact subdomain of  $X$  with smooth boundary then the volume of the tube  $v_{\sigma}(D)$  is*

$$\int_D \left( \frac{1}{(1 + t\bar{t})^{n+1}} \right) \wedge_{\alpha} (\sqrt{-1})\theta^{\alpha} \wedge \bar{\theta}^{\alpha} \cdot \int_{|t_i| \leq \sigma^2} \det \left( \delta_{\alpha\beta} - \frac{1}{2} \left( \sum_{\gamma} \left( \sum_r t_r S_{\alpha\gamma}^r \right) \left( \sum_r \bar{t}_r \bar{S}_{\gamma\beta}^r \right) \right) \right) \wedge_{\gamma} \frac{1}{2} (\sqrt{-1}) dt_r \wedge d\bar{t}_r.$$

*Proof.* Consider the substitution  $\tau_r = t_r/t_0$  in the fiber coordinates and use the fact that

$$t_0^2(1 + t\bar{t}) = 1$$

where  $t\bar{t} = \sum_r t_r \bar{t}_r$ . Note that  $t_0$  is the distance from  $z$  to  $w$ .

*Remarks.* 1. We have assumed here that the (holomorphic) curvature of  $P_n(\mathbf{C})$  is 1.

2. An analogous theorem holds in  $H_n(\mathbf{C})$ , the hyperbolic model space.

The remainder of this section will deal with the evaluation of the volume of the tubular neighborhood.

**THEOREM.** *If  $X$  is a holomorphic submanifold of  $\mathbf{C}^n$  and  $D$  is a compact subdomain of  $X$  with smooth boundary then the volume of the tube  $v_{\sigma}(D)$  around  $D$  is given by*

$$\text{vol} [v_{\sigma}(D)] = \frac{c_k}{2} \sum_{0 \leq \gamma \leq m} K_{\gamma} \frac{\sigma^{2(k+\gamma)}}{k(k+1) \cdots (k+\gamma)}$$

where  $k = n - m$  is the codimension of  $X$ ,  $c_k$  is the volume of the unit sphere in  $\mathbf{C}^k$  and  $K_{\gamma}$  are constants depending only on the curvature of  $X$ .

*Proof.* For the sake of simplicity the indices  $r, s, \dots$  will vary in the codimension range that is,  $1 \leq r, s, \dots \leq k$ . Let

$$\varphi(t, \bar{t}) = \varphi(t_1, \dots, t_k, \bar{t}_1, \dots, \bar{t}_k)$$

and denote the average of  $\varphi(t, \bar{t})$  on the unit sphere  $\sum_r \xi_r \bar{\xi}_r = 1$  by  $\langle \varphi(t, \bar{t}) \rangle$ .

Since the average value of a monomial

$$t_1^{\xi_1} \dots t_k^{\xi_k} \bar{t}_1^{\xi_1'} \dots \bar{t}_k^{\xi_k'}$$

plus its conjugate vanishes on  $S^{2k-1}$  unless  $\xi_r = \xi_r'$  for all  $r$ , one need only compute the average of monomials of the form

$$t_1^{\xi_1} \dots t_k^{\xi_k} \bar{t}_1^{\xi_1} \dots \bar{t}_k^{\xi_k}$$

Let  $t_r = \rho \zeta_r$  where  $\sum \zeta_r \bar{\zeta}_r = 1$ , further let  $t^{\xi} = t_1^{\xi_1} \dots t_k^{\xi_k}$ ,

$$|\xi| = \xi_1 + \dots + \xi_k \quad \text{and} \quad \xi! = \xi_1! \dots \xi_k!$$

the usual multiindex notation for  $\xi = (\xi_1, \dots, \xi_k)$ , and consider

$$(4.3) \quad \int_{\mathbf{C}^k} e^{-\rho^2} |t_1|^{2\xi_1} \dots |t_k|^{2\xi_k} \wedge \frac{1}{2}(\sqrt{-1}) dt_r \wedge d\bar{t}_r$$

which using the polar coordinates introduced above and Fubini's theorem may be written

$$\int_0^\infty e^{-\rho^2} \rho^{2(|\xi|+2k-1)} d\rho \int_{S^{2k-1}} \zeta^{\xi} \bar{\zeta}^{\xi} dv_k$$

where  $dv_k$  is the volume element of  $S^{2k-1}$  in  $\mathbf{C}^k$ . One may also apply Fubini's theorem directly to (4.3) yielding

$$\prod_{1 \leq r \leq k} \int e^{-t_r \bar{t}_r} (t_r \bar{t}_r)^{\xi_r} \frac{1}{2}(\sqrt{-1}) dt_r \wedge d\bar{t}_r.$$

Using polar coordinates in the  $(t_r, \bar{t}_r)$  plane this integral is equal to

$$\prod_r \int e^{-\rho^2} \rho^{2\xi_r+1} d\rho \wedge dv_1$$

which in turn, from the definition of the gamma function, may be written as

$$\prod_r (1/2)(2\pi)\Gamma(\xi_r + 1).$$

In summary we have

$$\int_0^\infty e^{-\rho^2} \rho^{2(|\xi|+2k-1)} d\rho \int_{S^{2k-1}} \zeta^{\xi} \bar{\zeta}^{\xi} dv_k = \pi^k \prod_{1 \leq r \leq k} \xi_r! = \pi^k \xi!.$$

Solving for the integral over  $S^{2k-1}$  one finds that

$$\frac{1}{2} \int_{S^{2k-1}} \zeta^{\xi} \bar{\zeta}^{\xi} dv_k = \frac{\pi^k \xi!}{2 \int_0^\infty e^{-\rho^2} \rho^{2(|\xi|+2k-1)} d\rho} = \frac{\pi^k \xi!}{\Gamma(|\xi| + k)}.$$

If  $\xi_1 = 0 = \dots = \xi_k$  then  $c_k = \pi^k / \frac{1}{2} \Gamma(k)$ . For the average value of  $t^{\xi} \bar{t}^{\xi}$  on  $S^{2k-1}$  we obtain

$$\langle t^{\xi} \bar{t}^{\xi} \rangle = \frac{\int \xi^{\xi} \bar{\zeta}^{\xi} dv_k}{\int_{S^{2k-1}} dv_k} = \frac{\pi^k \xi!}{\Gamma(|\xi| + k)} \cdot \frac{\Gamma(k)}{\pi^k} = \frac{\xi!}{k(k+1) \dots (k + |\xi| - 1)}$$

Now

$$\det (\delta_{\alpha\beta} + \frac{1}{2} \sum_{\gamma} (- \sum_r t_r S'_{\alpha\gamma})(\sum_r \bar{t}_r \bar{S}'_{\gamma\beta}))$$

may be expanded in the form  $1 + \eta_2 + \dots + \eta_{2m}$  where

$$\eta_{2\gamma}(t, \bar{t}) = \sum_{\xi+\xi'=2\gamma} A_{\xi,\xi'} t^{\xi} \bar{t}^{\xi'}$$

Define the scalar invariants  $S_{\gamma}$  on  $X$  by

$$S_{\gamma} = k(k + 1) \dots (k + \gamma - 1) \langle \eta_{2\gamma}(t, \bar{t}) \rangle.$$

It is easy to see then that

$$S_{\gamma} = \sum_{\xi+\xi'=2\gamma} \xi! A_{\xi,\xi'}$$

And as a result

$$\int_{t\bar{t} \leq \sigma^2} \eta_{2\gamma}(t, \bar{t}) \wedge_r \frac{\sqrt{-1}}{2} dt_r \wedge d\bar{t}_r = \frac{c_k S_{\gamma}}{k(k + 1) \dots (k + \gamma - 1)} \cdot$$

$$\int_0^{\rho} \rho^{2(\gamma+k)-1} d\rho = \frac{1}{2} \frac{c_k S_{\gamma}}{k(k + 1) \dots (k + \gamma)} \sigma^{2(\gamma+k)}.$$

If we let  $K_{\gamma} = \int_D S_{\gamma} \wedge_{\alpha} (\sqrt{-1})\theta^{\alpha} \wedge \bar{\theta}^{\alpha}$  then

$$\text{vol} (\nu_{\sigma}(D)) = \frac{c_k}{2} \sum_{0 \leq \gamma \leq m} K_{\gamma} \frac{\sigma^{2(\gamma+k)}}{k(k + 1) \dots (k + \gamma)}$$

where  $K_0 = 1$ .

**THEOREM.** *If  $X$  is a holomorphic submanifold of  $P_n(\mathbf{C})$  and  $D$  is a compact subdomain of  $X$  with smooth boundary then the volume of the tube  $\nu_{\sigma}(D)$ ,  $\sigma$  the radius of the tube around  $D$ , is given by the formula*

$$\text{vol} (\nu_{\sigma}(D)) = c_k \sum_{0 \leq \gamma \leq m} K_{\gamma} J_{\gamma}(a)$$

where the  $K_{\gamma}$  are as in the previous theorem and

$$J_{\gamma}(a) = \int_0^a (\sin b)^{2\gamma+2k-1} (\cos b)^{2m-2\gamma+1} db$$

where  $\tan a = \sigma$ .

*Proof.* We proceed as in the previous theorem, using (4.2), the formula for the volume element of the tube

$$\int_{t\bar{t} \leq \rho^2} \frac{\eta_{2\gamma}(t, \bar{t})}{(1 + t\bar{t})^{n+1}} \wedge_r \frac{\sqrt{-1}}{2} dt_r \wedge d\bar{t}_r = \frac{c_k S_{\gamma}}{k(k + 1) \dots (k + \gamma - 1)}$$

$$\cdot \int_0^{\sigma} \frac{\rho^{2\gamma+2k-1}}{(1 + \rho^2)^{n+1}} d\rho.$$

The notation is identical to that used in the last theorem. If we substitute

$\tan b = \rho$  the integral becomes

$$\frac{c_k S_\gamma}{k(k+1) \cdots (k+\gamma-1)} \int_0^a (\sin b)^{2\gamma+2k-1} (\cos b)^{2m+2\gamma+1} db$$

where  $\tan a = \sigma$ . Let

$$J^\gamma(a) = \frac{\int_0^a (\sin b)^{2\gamma+2k-1} (\cos b)^{2m-2\gamma+1} db}{k(k+1) \cdots (k+\gamma-1)}.$$

Then the volume of the tube is

$$c_k \sum K_\gamma J_\gamma(a).$$

The only remaining detail at this stage is to determine the nature of the scalar functions  $S_\gamma$ . More precisely, we wish to show that the  $S_\gamma$  are local representations of globally defined scalar invariants on  $X$ .

Now  $S_\gamma$  is a polynomial in  $S_{\alpha\beta}^r, \bar{S}_{\alpha\beta}^r$  with the properties that  $S_\gamma$  is invariant under the unitary groups  $U(m), U(k)$  in the sense that if  $S_{\alpha\beta}^r$  is transformed into

$$\sum_p U_{rp} S_{\alpha\beta}^p$$

where  $(U_{rp}) \in U(k)$  or if  $S_{\gamma\beta}^r$  is transformed into

$$\sum_\gamma U_{\alpha\gamma} S_{\gamma\beta}^r$$

where  $(U_{\alpha\gamma}) \in U(m)$ ,  $S_\gamma$  does not change. In fact,  $\eta_{2\gamma}$  is a sum of principal minors of order  $\gamma$  from the matrix

$$-\sum_\epsilon (\sum_r t_r S_{\alpha\epsilon}^r) (\sum_r \bar{t}_r \bar{S}_{\epsilon\beta}^r)$$

and as a result  $S_\gamma$  is a sum of averages of principal minors from the same matrix. Thus by Weyl's theory of vector invariants [12] the  $S_\gamma$  are polynomial in the hermitian forms

$$\sum_r S_{\alpha\epsilon}^r \bar{S}_{\beta\delta}^r.$$

Thus,

$$\langle \det (-\frac{1}{2} \sum_\epsilon (\sum_r t_r S_{\alpha\epsilon}^r) (\sum_r \bar{t}_r \bar{S}_{\epsilon\beta}^r)) \rangle$$

is a constant multiple of

$$\sum \delta(\alpha_1 \cdots \alpha_\gamma)^{\beta_1 \cdots \beta_\gamma} \delta(\lambda_1 \cdots \lambda_\gamma)^{\mu_1 \cdots \mu_\gamma} S_{\alpha_1 \lambda_1 \beta_1 \mu_1} \cdots S_{\alpha_\gamma \lambda_\gamma \beta_\gamma \mu_\gamma}$$

where

$$S_{\alpha\lambda\beta\mu} = \sum_r S_{\alpha\mu}^r \bar{S}_{\lambda\beta}^r$$

and

$$\delta(\alpha_1 \cdots \alpha_\gamma)^{\beta_1 \cdots \beta_\gamma}$$

is the generalized Kronecker symbol: equal to the sign of the permutation if  $\beta_1, \dots, \beta_\gamma$  is a rearrangement of  $\alpha_1, \dots, \alpha_\gamma$  or equal to zero otherwise. The summation extends over all  $\gamma$ -tuples selected from  $1, 2, \dots, m$ .



For a submanifold of  $\mathbf{C}^n$ ,  $S_{\alpha\lambda\beta\mu}$  is easily seen to be the hermitian curvature tensor. It follows from (3.4) that for a submanifold of  $P_n(\mathbf{C})$

$$S_{\alpha\beta\lambda\mu} = \frac{1}{2} (\delta_{\alpha\beta} \delta_{\lambda\mu} + \delta_{\beta\mu} \delta_{\alpha\lambda}) - \tilde{R}_{\alpha\beta\lambda\mu}$$

where  $\tilde{R}_{\alpha\beta\lambda\mu}$  is the hermitian curvature of the induced structure: namely if  $\tilde{R}$  is the curvature operator then by (3.4),

$$\tilde{R}_{\alpha\beta\lambda\mu} = (\tilde{R}(e_\lambda, \bar{e}_\mu)e_\beta, \bar{e}_\alpha) = \tilde{\Omega}_\beta^\alpha(e_\lambda, \bar{e}_\mu).$$

The constant in question may be determined by specializing the formula to a simple geometric figure of codimension 1 and with  $S_{\alpha\beta} = \delta_{\alpha\beta}$ . As a result:

$$\left\langle \det \left( -\frac{1}{2} \sum_{\epsilon} (\sum t_r S_{\alpha\epsilon}^r) (\sum \bar{t}_r \bar{S}_{\epsilon\beta}^r) \right) \right\rangle = \frac{1}{2^{\gamma} \gamma! k(k+1) \cdots (k+\gamma-1)} \cdot \sum \delta \left( \begin{matrix} \beta_1 \cdots \beta_\gamma \\ \alpha_1 \cdots \alpha_\gamma \end{matrix} \right) \delta \left( \begin{matrix} \mu_1 \cdots \mu_\gamma \\ \lambda_1 \cdots \lambda_\gamma \end{matrix} \right) S_{\alpha_1 \lambda_1 \beta_1 \mu_1} \cdots S_{\alpha_\gamma \lambda_\gamma \beta_\gamma \mu_\gamma}$$

Rewriting the formula for the volume of the tube and calling the integral of the expression inside the summation  $W_\gamma$  we obtain

$$(C^n) \quad \text{vol}(\nu_\sigma(D)) = c_k \sum_{0 \leq \gamma \leq m} \frac{W_\gamma}{2^{\gamma+1} \gamma! k(k+1) \cdots (k+\gamma)} \sigma^{2(\gamma+k)},$$

$$(P_n(C)) \quad \text{vol}(\nu_\sigma(D)) = c_k \sum_{0 \leq \gamma \leq m} \frac{1}{\lambda!} W_\gamma J_\gamma(a).$$

### 5. Topology of compact holomorphic submanifolds of $P_n(\mathbf{C})$

Let  $X$  be a compact holomorphic submanifold of  $P_n(\mathbf{C})$  and let  $p$  be a point not in  $X$ . Denote the space of paths from  $p$  to  $X$  with the compact-open topology by  $\Omega(p, X)$ ; denote the  $j$ th Betti number of  $\Omega(p, X)$  by  $b_j[\Omega(p, X)]$ . It follows from the Morse inequalities that the number of geodesics normal to  $X$  initially and terminating at  $p$  of index at most  $\lambda$  is greater than or equal to  $\sum_j b_j[\Omega(p, X)]$ ,  $0 \leq j \leq \lambda$ . A brief summary of the relevant Morse Theory may be found in [5].

Following the same plan as in Proposition of Section 3 of [5] we now estimate the location of focal points in a Kaehler manifold. The Kaehlerian sectional curvature of a Kaehler manifold is defined at any tangent plane to be the riemannian sectional curvature of the tangent plane divided by  $\frac{1}{4}(1 + 3 \cos^2 \alpha)$  where  $\alpha$  is the angle between the tangent plane and its corresponding holomorphic tangent plane. Denote the Kaehler curvature by  $K$ . Note that the Kaehler curvature of  $P_n(\mathbf{C})$  is the holomorphic curvature and hence constant.

**PROPOSITION.** *Let  $Y$  be a Kaehler manifold with Kaehler curvature restricted to  $[h, 1]$ ,  $h > 0$ . Let  $X$  be a Kaehler submanifold of  $Y$  and suppose that the proper values of all of the second fundamental forms of  $X$  lie in the*

interval  $[-b, b]$ ,  $b > 0$ . Then a normal geodesic to  $X$  with length at least

$$(2/\sqrt{h}) \operatorname{arc\,cot} - 2b/\sqrt{h}$$

has a focal point. Moreover, if the length of the geodesic is at least

$$\lambda\pi + (2/\sqrt{h}) \operatorname{arc\,cot} - 2b/\sqrt{h}$$

then there are at least  $2\lambda$  focal points.

*Proof.* Consider Jacobi fields on  $P_n(\mathbf{C})$  with holomorphic curvature  $h$  given by

$$\left( \frac{2y'}{\sqrt{h}} \sin \frac{\sqrt{h}}{2} t + y \cos \frac{\sqrt{h}}{2} t \right) U(t)$$

where  $U$  is a parallel vector field along the geodesic with initial vector a unit proper vector of the proper value  $a$  and where  $y \neq 0$  and  $ay = -y'$ . This Jacobi field vanishes for

$$t = (2/\sqrt{h}) \operatorname{arc\,cot} (+2a/\sqrt{h}).$$

Using the canonical complex structure there is another Jacobi field vanishing for

$$t = (2/\sqrt{h}) \operatorname{arc\,cot} (-2a/\sqrt{h}).$$

Thus it follows from the Morse index theorem that for

$$t > (2/\sqrt{h}) \operatorname{arc\,cot} (2b/\sqrt{h})$$

there are at least two focal points, that is, the focal points occur in pairs. The latter part of the proposition is clear.

**PROPOSITION.** Let  $X$  be a compact holomorphic submanifold of  $P_n(\mathbf{C})$  with proper values of all second fundamental forms in the interval  $[-b, b]$ . Let

$$c_\lambda = \sum b_j[\Omega(p, X)], \quad 0 \leq j \leq 2\lambda, \quad \text{and} \quad \sigma = \lambda\pi + 2 \operatorname{arc\,cot} - 2b;$$

then

$$c_\lambda \leq \operatorname{vol} \nu_\sigma(X) / \operatorname{vol} P_n(\mathbf{C}).$$

*Proof.* By the above proposition each normal geodesic to  $X$  of length at least  $2 \operatorname{arc\,cot} - 2b$  has index at least  $2\lambda + 2$ . Thus any geodesic index of most  $2\lambda$  has length at most  $\sigma$ . By a similar argument as in Proposition 3.4 of [7] our proposition follows.

**THEOREM.** If  $X$  is a compact holomorphic submanifold of  $P_n(\mathbf{C})$  with proper values of the second fundamental forms restricted to  $[-b, b]$ ,  $b > 0$  then the sum of the first  $2\lambda$  Betti numbers of  $\Omega(p, X)$  is no larger than

$$\frac{(\lambda\pi + 2 \operatorname{arc\,cot} (-2b))\pi^n c_k \operatorname{vol}(X)}{n!} \sum_{0 \leq \lambda \leq m} A_\lambda b^{2\gamma}$$

where the  $A_\gamma$  are constants.

Since we are interested in estimating

$$(5.1) \quad \int_x \frac{C_k S_\gamma}{k(k+1) \cdots (k+\gamma-1)} \int_0^\sigma \frac{\rho^{2\gamma+2k-1}}{(1+\rho^2)^{n+1}} d\rho \wedge (\sqrt{-1})\theta^\alpha \wedge \bar{\theta}^\alpha$$

let us first estimate the integral from 0 to  $\sigma$ . Now

$$\int_0^\sigma \frac{\rho^{2\gamma+2k-1}}{(1+\rho^2)^{n+1}} d\rho \leq \sigma(\rho_\gamma(n, k))$$

where  $\rho_\gamma(n, k)$  denotes the max of the integrand.

Recall that the  $(\bar{S}_{\alpha\beta}^r)$  are the coefficients of the second fundamental forms in the normal directions, that is,

$$\bar{S}_{\alpha\beta}^r = (D_{e_\beta} e_\alpha, e_r)$$

where  $D$  is the associated covariant derivative and  $e_1 \cdots e_n$  is an adapted frame field. If  $L(w, w)$  is the complex second fundamental form and  $S(u, u)$  is the real second fundamental form then an elementary calculation reveals that

$$L_{Pv}(Pu, Pu) = (1/\sqrt{2})(S_v(u, u) + (\sqrt{-1})S_{rv}(u, u))$$

where  $P$  is the type  $(1, 0)$  projection.

Hence

$$|L_{Pv}(Pu, Pu)|^2 \leq |S_v(u, u)|^2.$$

As a result

$$|S_\gamma| \leq C_\gamma b^{2\gamma}$$

where  $C_\gamma$  is a constant.

Since the volume of  $P_n(\mathbf{C})$ , in the Fubini-Study metric is  $\pi^n/n!$ , the integral (5.1) is bounded above by

$$\frac{(\lambda\pi + 2 \arccot(-2b))\pi^n c_k \text{vol}(X)}{n!} \sum_{0 \leq \gamma \leq m} A_\gamma b^{2\gamma}$$

where  $A_\gamma$  incorporates all of the constants.

In other words, the sum of the first  $2\lambda$  Betti numbers of  $\Omega(p, X)$  grows like a first degree polynomial in  $\lambda$ .

*Added in proof.* A similar formula for the volume of a tube in  $P_n(\mathbf{C})$  has been found, using different techniques, by R. A. Wolf in his Berkeley thesis, 1968.

BIBLIOGRAPHY

1. C. B. ALLENDOERFER, *The Euler number of a Riemannian manifold*, Amer. J. Math., vol. 62 (1940), pp. 243-248.
2. C. B. ALLENDOERFER AND A. WEIL, *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc., vol. 53 (1943), pp. 101-129.

3. S. S. CHERN, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math., vol. 45 (1944), pp. 747-752.
4. ———, *On the kinematic formula in integral geometry*, J. Math. Mech., vol. 16 (1966), pp. 101-118.
5. F. J. FLAHERTY, *Spherical submanifolds of pinched manifolds*, Amer. J. Math., vol. 89 (1967), pp. 1109-1114.
6. F. J. FLAHERTY AND N. GROSSMAN, *Submanifolds of pinched manifolds*, Topology, vol. 10 (1971), pp. 37-42.
7. N. GROSSMAN, *On the topology of taut Riemannian manifolds with positive ricci pinching*, J. Differential Geometry, vol. 3 (1969), pp. 393-410.
8. S. KOBAYASHI AND K. NOMIZU, *Foundations of differential geometry*, vol. II, Interscience, New York, 1969.
9. L. SANTALÓ, *Integral geometry in hermitian spaces*. Amer. J. Math., vol. 74 (1952), pp. 423-434.
10. J. STEINER, *Über parallele Flächen*, Monatsb. Deutsch. Akad. Wiss. Berlin, 1840, pp. 114-118, also Werke, vol. 2 (1886), pp. 171-176.
11. H. WEYL, *On the volume of tubes*, Amer. J. Math., vol. 61 (1939), pp. 461-472.
12. ———, *Classical groups*, Princeton Univ. Press, Princeton, 1939.
13. H. H. WU, *The equidistribution theory of holomorphic curves*, Annals of Math. Studies, no. 64, Princeton, 1970.

OREGON STATE UNIVERSITY  
CORVALLIS, OREGON