RADICAL GROUPS OF FINITE ABELIAN SUBGROUP RANK

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Following Plotkin we term a group *radical,* if each of its non-trivial epimorphic images possesses a non-trivial locally nilpotent normal subgroup. A group has been termed *of finite abelian subgroup rank*, if its elementary abelian primary subgroups are finite and if its torsionfree abelian subgroups are of finite rank. For the sake of brevity and for the moment we term $\mathfrak{R}$-group every radical group of finite abelian subgroup rank. Subgroups and epimorphic images of $\mathfrak{R}$-groups are $\mathfrak{R}$-groups; and extensions of $\mathfrak{R}$-groups by $\mathfrak{R}$-groups are $\mathfrak{R}$-groups [Discussion of Theorem 6.1, (A)]. $\mathfrak{R}$-groups are countable; and every epimorphic image, not 1, of an $\mathfrak{R}$-group possesses an abelian characteristic subgroup, not 1, which is either finite, elementary and primary or torsion-free of finite rank [Theorem 6.1, (a)]. The intersection of all subgroups of finite index in an $\mathfrak{R}$-group is a nilpotent characteristic subgroup without proper subgroups of finite index [Theorem 6.1, (b)].

If an $\mathfrak{R}$-group is a torsion group, then it is locally finite-soluble and its Sylow subgroups of equal characteristic are conjugate; it is an extension of a radicable abelian group by a residually finite group [Theorem 6.1, (c) + Proposition 4.5].

If 1 is the only normal torsion subgroup of an $\mathfrak{R}$-group, then it is soluble of finite rank and its torsion subgroups are finite of bounded order; it is an extension of a torsion free nilpotent group by a noetherian and almost abelian group [Theorem 6.1, (d) + Proposition 5.5].

It is easy to construct abelian groups of finite abelian subgroup rank which are not of finite rank [in the sense of Prüfer]; and $\mathfrak{R}$-groups need not be soluble [Discussion of Theorem 6.3, (A)].

The $\mathfrak{R}$-groups of finite rank are then characterized by the following equivalent properties: A group is a radical group whose abelian subgroups are of finite rank if, and only if, it is of finite rank and an extension of a hypercentral torsion group whose primary components are artinian and almost abelian by a soluble group [Theorem 6.3].

More especially we show that a group G is soluble of finite rank and the number of primes which are orders of elements in G is finite if, and only if, G is radical and its elementary abelian and torsion free abelian subgroups are of finite rank [Theorem 7.1].

It is noteworthy that a group G possesses an $\mathfrak{R}$-subgroup of finite index if, and only if, G is of finite abelian subgroup rank and its infinite epimorphic images possess normal subgroups, not 1, which are finite or radical [Theorem 8.1].

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Two remarks on method seem to be in order: we get very similar results for primary groups and for torsionfree groups; in other words: it does not make too much difference whether the characteristic is a prime or 0. Actually it does not matter too much whether a group is torsionfree; what is really important is the absence of non-trivial normal torsion subgroups; see in particular Proposition 5.5. —As always it is important to have information on the automorphism groups which are induced in normal subgroups of our groups. But as our groups need not be soluble and as we only assume our groups to be radical, such information as is available about the automorphism groups of soluble groups—see e.g. Wehrfritz—can only be used with circumspection. This situation becomes aggravated by the fact that the automorphism groups even of abelian groups of finite rank may be quite wild: remember that the automorphism group of the abelian group of type p^∞ is the group of integral p-adic numbers prime to p and this group is uncountably infinite; likewise the group of automorphisms of the additive group of rational numbers is the direct product of a cyclic group of order 2 and a free abelian group of countably infinite rank.

Notations. $x \circ y = x^{-1}y^{-1}xy = x^{-1}x^y$.

$e \circ A$ = set of all commutators $e \circ a$ with $a \in A$.

$\{x, y, z, \cdots\}$ = subgroup generated by $x, y, z, \cdots$.

$A \circ B = \{a \circ b \mid a \in A, b \in B\}$.

$G' = G \circ G$ = commutator subgroup of G.

$G^{(0)} = G, G^{(i+1)} = (G^{(i)})'$.

$\mathfrak{z}X = \{a \mid a \in X, a \circ x = 1 \text{ for all } x \in X\}$ = center of X.

$\mathfrak{z}_0 X = 1, \mathfrak{z}_{i+1} X/\mathfrak{z}_i X = \mathfrak{z}(X/\mathfrak{z}_i X)$.

hypercenter of X = intersection of all normal subgroups Y of X with $\mathfrak{z}(X/Y) = 1$.

$\mathfrak{hp}X$ = Hirsch-Plotkin-radical of X = product of all locally nilpotent normal subgroups of X; see Schenkman [p. 205].

$\mathfrak{a}X$ = hyporesiduum of X = subgroup of X, generated by all subgroups of X without proper subgroups of finite index in X.

$\mathfrak{res}\, X$ = residuum of X = intersection of all subgroups of finite index in X.

socle of X = product of all minimal normal subgroups of X.

$\mathfrak{t}X$ = product of all normal torsion subgroups of X.

$\mathfrak{n}_G S$ = normalizer of the subgroup S of $G = \{a \mid a^{-1}Sa = S \text{ and } a \in G\}$.

$\mathfrak{c}_X Y$ = centralizer of the subset Y in the subgroup X
$= \{a | a \in X \text{ and } a \circ y = 1 \text{ for all } y \in Y\}$.

$o(X)$ = order of group X.

torsion element = element of finite (positive) order.

p-element = element of order a power of p.

p-Sylow subgroup = maximal p-subgroup.

p'-element = torsion element of order prime to p.

The subgroup C of G is a complement of the normal subgroup N of G if

$G = NC$ and $1 = N \cap C$; and G splits over its normal subgroup N if such a complement C exists.

Factor of the group G = epimorphic image of a subgroup of G.

The automorphism σ of G stabilizes the chain of subgroups S_i of G with $S_0 = 1$, S_i a normal subgroup of S_{i+1} and $S_n = G$, if σ fixes every element in every S_{i+1}/S_i.

$\mathrm{Hom}(U, V)$ = group of homomorphisms of the abelian group U into the abelian group V.

The group G is

nilpotent (of finite class), if $G = \mathfrak{z}_i G$ for almost all i;

hypercentral, if non-trivial epimorphic images of G possess non-trivial centers;

soluble, if $G^{(i)} = 1$ for almost all i;

hyperabelian, if non-trivial epimorphic images of G possess non-trivial abelian normal subgroups;

radical, if non-trivial epimorphic images of G possess non-trivial locally nilpotent normal subgroups;

almost abelian, if there exists an abelian subgroup of finite index in G;

artinian, if the minimum condition (= descending chain condition) is satisfied by the subgroups of G;

noetherian, if every subgroup is finitely generated

= ascending chain conditions for subgroups

= maximum condition for subgroups;

of rank r (in the sense of Prüfer), if every finitely generated subgroup may be generated by r or fewer elements;

of finite rank, if G is of rank r for almost all positive integers r;

radicable, if every element in G is for every positive integer n the nth power of an element in G;

elementary, if all its elements have squarefree order;

a p-group, if all its elements are p-elements;

primary, if G is a p-group for some prime p;

an ω-group, if G is a torsion group the order of whose elements are divisible by primes in the set ω only;

an n-group, if G is an ω-group for ω the set of primes dividing n;

a p'-group, if G is a torsion group all of whose elements are of order prime to p;

an ω'-group, if G is a torsion group all of whose elements are of order prime to every prime in ω.

1. It is our aim in this section to investigate elementary abelian p-groups Γ of automorphisms of abelian p-groups A. As usual in such a situation it will prove convenient to denote the composition of the elements a, b of A by addition $a + b$, the effect of the automorphism σ on the element a by the product $a\sigma$ and the composition of automorphisms as multiplication and to make use of the fact that Γ is part of the endomorphism ring of A.

LEMMA 1.1. *Assume that the automorphism σ of the abelian p-group A meets the following requirements*:

(a) $\sigma^p = 1$.
(b) $px = 0$ *implies* $x\sigma = x$.
(c) $p = 2$ *and* $4x = 0$ *imply* $x\sigma = x$.
Then $p(\sigma - 1) = 0$.

For related results see Robinson [1; p. 55, Lemma 2.36] and also Baer [4; p. 525, Lemma].

Proof. Denote by $A(i)$ the totality of elements x in A with $p^i x = 0$. Then every $A(i)$ is a fully invariant subgroup of A and

$$0 = A(0) \subseteq A(1) \subseteq \cdots \subseteq A(i) \subseteq A(i+1) \subseteq \cdots \subseteq \bigcup_{i=0}^{\infty} A(i) = A;$$
$$pA(i+1) \subseteq A(i).$$

Condition (b) may be restated as

(b*) $A(1)(\sigma - 1) = 0$

and condition (c) as

(c*) $p = 2$ implies $A(2)(\sigma - 1) = 0$.

Because of (b*) we may make the inductive hypothesis

$$A(i)(\sigma - 1) \subseteq A(i-1) \quad \text{for some positive } i.$$

Then

$$pA(i+1)(\sigma - 1) \subseteq A(i)(\sigma - 1) \subseteq A(i-1)$$

so that

$$p^i A(i+1)(\sigma - 1) \subseteq p^{i-1} A(i-1) = 0$$

and hence $A(i+1)(\sigma - 1) \subseteq A(i)$. Thus we have shown by complete induction:

(1) $A(i)(\sigma - 1) \subseteq A(i-1)$ for all positive i.

If $p = 2$, then (c*) permits us to make the inductive hypothesis

$$A(i)(\sigma - 1) \subseteq A(i-2)$$

for some $i > 1$. It follows then as before

$$2A(i+1)(\sigma - 1) \subseteq A(i)(\sigma - 1) \subseteq A(i-2),$$
$$2^{i-1} A(i+1)(\sigma - 1) \subseteq 2^{i-2} A(i-2) = 0,$$
$$A(i+1)(\sigma - 1) \subseteq A(i-1);$$

and we have shown by complete induction:

(2) If $p = 2$, then $A(i)(\sigma - 1) \subseteq A(i - 2)$ for $i > 1$.

If our lemma were false, then $p(\sigma - 1) \neq 0$. This is equivalent to

$$A(\sigma - 1) \nsubseteq A(1).$$

Since every element in A is in some $A(i)$, it follows that $A(\sigma - 1) \nsubseteq A(1)$ is equivalent with the existence of some i such that $A(i)(\sigma - 1) \nsubseteq A(1)$; and among these there is a minimal one m. Consequently we have

(b**) $A(i)(\sigma - 1) \subseteq A(1)$ for $i < m$, $A(m)(\sigma - 1) \nsubseteq A(1)$

and especially

$$pA(m)(\sigma - 1) \subseteq A(m - 1)(\sigma - 1) \subseteq A(1),$$
$$p^2A(m)(\sigma - 1) \subseteq pA(1) = 0,$$

so that

(b***) $A(m)(\sigma - 1) \subseteq A(2)$;

and from (1), (2) and (c*) we deduce

(b$^+$) $A(m)(\sigma - 1)^3 = 0$
(c$^+$) $A(m)(\sigma - 1)^2 = 0$ for $p = 2$.

Application of (a) and the Binomial Theorem gives

(+) $0 = \sum_{i=1}^{p} C(p, i)(\sigma - 1)^i$.

Therefore

$$0 = A(m)\left(\sum_{i=1}^{p} C(p, i)(\sigma - 1)^i\right) = A(m)(p(\sigma - 1) + C(p, 2)(\sigma - 1)^2)$$

by (b$^+$). As $A(m)(\sigma - 1) \subseteq A(2)$ by (b***), we have

$$A(m)(\sigma - 1)^2 \subseteq A(1)$$

by (1). Since $C(p, 2) \equiv 0 \bmod p$ for $p \neq 2$ and since we may use (c$^+$) for $p = 2$, we obtain

$$\begin{aligned} 0 &= A(m)(p(\sigma - 1) + C(p, 2)(\sigma - 1)^2) \\ &= A(m)(p(\sigma - 1)) \\ &= pA(m)(\sigma - 1), \end{aligned}$$
$$A(m)(\sigma - 1) \subseteq A(1).$$

This contradicts (b**) and proves our Lemma.

COROLLARY 1.2. *If Γ is a group of automorphisms of the abelian p-group A such that*

(a) $\Gamma^p = 1$

(b) $px = 0$ *implies* $x\Gamma = x$

(c) $p = 2$ *and* $4x = 0$ *imply* $x\Gamma = x$,

then

(A) *the mapping* $\sigma \rightarrow \sigma - 1$ *induces an isomorphism of* Γ *into* Hom $(A/pA, \bar{A})$ *with* $\bar{A}$ *the socle of* A;

(B) Γ *is an elementary abelian p-group*;

(C) *if, in addition,* A *is of finite rank* n, *then* Γ *is of rank* n^2.

Terminological reminder. The *socle* $\bar{A}$ of the abelian p-group A is just the totality of elements x in A with $px = 0$; it is therefore identical with $A(1)$ as defined in the proof of Lemma 1.1.

Proof. Every element σ in Γ meets the requirements (a), (b), (c) of Lemma 1.1 so that $p(\sigma - 1) = 0$. Hence $p(A(\sigma - 1)) = 0$ so that $\sigma - 1$ is for every σ in Γ *a* homomorphism into the socle $\bar{A}$ of A.

It is clear that the mapping $\sigma \rightarrow \sigma - 1$ is one to one. If α, β are elements in Γ, then every element in $\bar{A}$ is, by (b), a fixed element of (α and) β. If a is an element in A, then

$$a(\alpha\beta - 1) = a(\alpha - 1)\beta + a(\beta - 1) = a(\alpha - 1) + a(\beta - 1)$$

so that

$$\alpha\beta - 1 = (\alpha - 1) + (\beta - 1);$$

and this completes the proof of (A); and (B) is an immediate consequence of (A).

If finally A is of finite rank n, then $\bar{A}$ has order p^h with $h \leq n$. It follows that A is the direct sum of h groups of rank 1 (cyclic or of type p^∞); see Fuchs [p. 65]. The order of A/pA is therefore a divisor p^m of p^n. Since every homomorphism of A into $\bar{A}$ maps pA onto 0, the group Γ is essentially the same as a group of homomorphisms of A/pA into $\bar{A}$. The group of all homomorphisms of the elementary abelian group A/pA of order p^m into the elementary abelian group $\bar{A}$ of order p^h is an elementary abelian group of order p^{mh}, and this proves (C).

Remark 1.3. If $A = pA$, the homomorphisms of A/pA into $\bar{A}$ must be trivial, so $\Gamma = 1$; cp. Baer [4; p. 525, Lemma].—The indispensability of condition (c) in Lemma 1.1 and Corollary 1.2 may be seen from the example of an abelian 2-group A and its automorphism $a \rightarrow -a$. This automorphism is an involution fixing all elements of order 2. But selecting A properly, none of the conclusions holds.

PROPOSITION 1.4. *If* A *is an abelian p-group of finite rank* n, *then every p-group of automorphisms of* A *is finite of rank* $\frac{1}{2}n(5n - 1)$.

Proof. Abelian p-groups are of finite rank if, and only if, they are artinian; see Fuchs [p. 65, Theorem 19.2 or p. 68, Exercise 19]. Application of Baer [4; p. 526, Corollary] shows the existence of a positive integer k with the following property:

(+) If the order of the automorphism φ of A is finite and fixes every element $a \in A$ with $p^k a = 0$, then $\varphi = 1$.

Accordingly we denote by K the totality of elements $a \in A$ with $p^k a = 0$. Property (+) shows then that every torsion group Γ of automorphisms of A induces in the characteristic subgroup K of A a group of automorphisms $\Gamma^* \cong \Gamma$.

Since the rank of A is n, the order of K is a divisor of p^{kn}. Thus K is finite so that Γ^* and hence Γ is likewise finite.

Now we make use of the hypothesis that Γ is a p-group. Then we may apply a result of Roseblade [p. 408, Lemma 5] on the p-group Γ^* of the finite abelian p-group K of rank n. It follows that the rank of Γ^* and hence the rank of Γ is $\frac{1}{2}n(5n - 1)$.

2. In the present section estimates are derived for the rank of G/A whenever A is an abelian normal subgroup of G and G/A is an elementary abelian p-group. The general case will be discussed after we have treated several special cases. Our first result will be stated and proved in a somewhat more general setting.

(2.1) *Assume that the group G and its normal subgroup N meet the following requirements*:

(a) *G/N is an abelian torsion group.*

(b) *If the prime p is the order of an element in G/N, then p is not the order of an element in N.*

(c) *If S is a subgroup of G and $N \cap S = 1$, then S is finite.*

(d) *N is locally finite.*

Then

(A) *G splits over N;*

(B) *any two complements of N in G are conjugate in G;*

(C) *a subgroup S of G is a complement of N if, and only if, S is maximal with respect to the property $N \cap S = 1$;*

(D) *G/N is finite and $[G{:}N]$ is the order of a maximal abelian $[G{:}N]$-subgroup of G;*

(E) *G is locally finite.*

Proof. Abelian torsion groups are locally finite. Therefore the extension G of the locally finite group N by the locally finite quotient group G/N is locally finite (see for instance Kurosh [p. 153]); and we have shown (E).

Consider a finitely generated subgroup S of G. We deduce from (E) that S is finite. Because of $NS/N \cong S/(N \cap S)$ and (b) we may deduce from Cauchy's Theorem that the orders of $N \cap S$ and the abelian group $S/(N \cap S)$

are relatively prime. Application of the Theorems of Schur-Zassenhaus shows: there exist complements of $N \cap S$ in S and any two complements are conjugate in S; see Zassenhaus [p. 125, Satz 25 and p. 126, Satz 27]. Thus we have shown:

(+) If S is a finitely generated subgroup of G, then S is finite and splits over (its normal Hall subgroup) $N \cap S$; all the complements of $N \cap S$ in S are conjugate in S.

There exist subgroups X of G with $N \cap X = 1$ and among these there exists a maximal one, say M (Maximum Principle of Set Theory). By (c), M is finite. If x is any element in G, then $S = \{M, x\}$ is finitely generated. Application of (+) shows the finiteness of S and the existence of a complement T of $N \cap S$ in S. Then

$$[T \cap M(N \cap S)] \cap (N \cap S) = T \cap (N \cap S) = 1$$

and

$$\begin{aligned}[T \cap M(N \cap S)](N \cap S) &= T(N \cap S) \cap M(N \cap S)\\ &= S \cap M(N \cap S)\\ &= M(N \cap S)\end{aligned}$$

by Dedekind's Modular Law so that $T \cap M(N \cap S)$ is a complement of $N \cap S$ in $M(N \cap S)$. Since M is likewise a complement of $N \cap S$ in $M(N \cap S)$, and since $M(N \cap S)$ is finite, application of (+) shows that M and $T \cap M(N \cap S)$ are conjugate in S. Hence there exists an element a in S with

$$M = [T \cap M(N \cap S)]^a \subseteq T^a.$$

Because of $T \subseteq S$ and $1 = T \cap (N \cap S) = T \cap N$ we have $1 = T^a \cap N$; and from the maximality of M we deduce $M = T^a$. It follows that M is a complement of $N \cap S$ in S, as is T. So x belongs to $S = (N \cap S)M \subseteq NM$; and we have shown $G = MN$. Hence M is a complement of N in G, proving (A) and (C) and the finiteness of G/N (since $M \cong G/N$ is finite). Noting that by (a), (b) a subgroup S of G satisfies $N \cap S = 1$ if, and only if, S is abelian with $S^{[G:N]} = 1$, we see that (D) too is true.

If U and V are complements of N in G, then they are both finite. Hence $S = \{U, V\}$ is finitely generated and U, V are both complements of $N \cap S$ in S. Application of (+) shows that U and V are conjugate in S, proving the validity of (B).

Remark 2.2. Denote by $G = \{a\} \otimes \{b\}$ the direct product of the infinite cyclic group $\{a\}$ and the cyclic group $\{b\}$ of order a prime p. Let $N = G^p = \{a^p\}$. Then G/N is a finite elementary abelian p-group, N is free of elements of order p, and our condition (c) likewise holds. But G does not split over N, showing the indispensability of (d).

(2.3) *Assume that the hyperabelian group G and its normal abelian subgroup A meet the following requirements*:

(a) *G/A is a torsion group.*

(b) *If the abelian normal subgroup S of G is a torsion group, then $S = 1$.*

(c) *The rank $\mathfrak{r}A$ of A is finite.*

Then G is soluble and of finite rank, and every p-subgroup of G/A is of rank

$$(p-1)(p-2)^{-1}\mathfrak{r}A \text{ for } p \neq 2 \quad \text{and} \quad 3\,\mathfrak{r}A \text{ for } p = 2.$$

Proof. There exists a maximal abelian normal subgroup B of G which contains A (Maximum Principle of Set Theory). The torsion subgroup tB of the abelian group B is a characteristic subgroup of the normal subgroup B and hence it is an abelian normal torsion subgroup of G. Application of (b) shows $tB = 1$; and we have shown that B is torsionfree. Assume now by way of contradiction that $B \neq \mathfrak{c}_G B$. Since G is hyperabelian, there exists a normal subgroup C of G with $C' \subseteq B \subset C \subseteq \mathfrak{c}_G B$; and we note that $B \subseteq \mathfrak{z}C$. Since G/A is, by (a), a torsion group, its factor C/B is a torsion group as well. If x and y are elements in C, then there exists a positive integer k such that x^k belongs to B; and $x \circ y$ belongs to B too. From $B \subseteq \mathfrak{z}C$ we deduce now that

$$1 = x^k \circ y = (x \circ y)^k.$$

Hence $x \circ y$ belongs to $tB = 1$, proving the commutativity of C. This contradicts the maximality of B; and thus we have shown that

(1) $B = \mathfrak{c}_G B$ is torsionfree.

Since $B/A \subseteq G/A$ is, by (a), a torsion group, it follows from (1) that

(2) $\mathfrak{r}A = \mathfrak{r}B$.

It is a consequence of (1) that G/B is essentially the same as the group of automorphisms of B, induced by G in B. It is a consequence of $A \subseteq B$ and (a) that G/B is a torsion group; and G/B is hyperabelian, since G is hyperabelian. Since B is torsionfree of finite rank [by (1), (2)], we may apply Baer [3; p. 167, Hauptsatz 2]. Hence

(3) G/B is finite.

Thus G is an extension of the abelian group B by the finite hyperabelian (and hence soluble) group G/B, and we conclude:

(4) G is soluble.

Finally (2) and (3) yield

(5) G is of finite rank.

We will now determine an upper bound for the order of a p-Sylow subgroup

S/B of G/B. Obviously S/B is essentially the same as some p-subgroup of the automorphism group of B, and the order of the p-Sylow subgroups of this automorphism group is bounded by p^{M_p}, where

$$M_p \leq \sum_{i=0}^{\infty} p^{-i}(p-1)^{-1}\mathfrak{r}A = (p/(p-1)^2)\mathfrak{r}A;$$

see Burnside [Note G, p. 484].

Hence $M_2 \leq 2\mathfrak{r}A$ and $M_p \leq (1/(p-2))\mathfrak{r}A$ for $p \neq 2$. The rank of any p-subgroup of the automorphism group of B is M_p. Since the rank of any p-subgroup of G/A cannot exceed $M_p + \mathfrak{r}A$, we obtain the last statement of (2.3).

(2.4) *Assume that the group G and its normal subgroup A meet the following requirements:*

(a) *G is a p-group.*

(b) *$A = \mathfrak{c}_G A$.*

(c) *A is of finite rank n.*

Then

(A) *G/A is finite;*

(B) *the rank of G is $\frac{1}{2}n(5n + 1)$.*

Proof. It is an immediate consequence of our hypotheses that G/A is essentially the same as a p-group of automorphisms of the abelian p-group A of rank n. Application of Proposition 1.4 shows that G/A is finite and its rank is $\frac{1}{2}n(5n - 1)$. Since the rank of G is the sum of the ranks of A and of G/A, it is $\frac{1}{2}n(5n + 1)$.

Proposition 2.5. *Assume that the prime number p and the abelian normal subgroup A of G meet the following requirements:*

(a) *G/A is a hypercentral p-group.*

(b) *The rank of every abelian p-subgroup of G is finite.*

(c) *The rank of every free abelian subgroup of A is finite.*

Then there exist integers a and b with the following properties:

(A) *$A/\mathfrak{t}A$ and the torsionfree abelian subgroups of G are of finite rank b.*

(B) *There exists an elementary abelian p-subgroup of G whose rank is a; and the p-subgroups of G are of finite rank $\frac{1}{2}a(5a + 1) + 2b + ab$.*

(C) *G/A is of finite rank $\frac{1}{2}a(5a + 1) + 3b + ab$.*

Proof. We recall a group is a that p'-group if it is a torsion group the orders of whose elements are prime to p. The totality of elements of (finite) order prime to p in the abelian group A is a characteristic p'-subgroup Q of A. The torsion elements of G of order prime to p belong, by (*a*), to A and hence to Q. Thus we have shown:

(1) The totality Q of torsion elements in G of order prime to p is a characteristic p'-subgroup of G with $Q \subseteq A$.

Consider next a subgroup S of G with $Q \subseteq S$ and S/Q an elementary abelian

p-group. If U is a subgroup of S with $Q \cap U = 1$, then $U \cong UQ/Q \subseteq S/Q$ so that U is an elementary abelian p-group. Application of (b) shows the finiteness of U. Hence we may apply (2.1) on the pair Q, S so that in particular S splits over Q and S/Q is a finite group, isomorphic to a subgroup of S. Thus we have shown:

(2) Every elementary abelian p-subgroup of G/Q is finite and isomorphic to a subgroup of G.

It is an immediate consequence of (2) that

(3) the rank of every abelian p-subgroup of G/Q is finite.

The rank of the torsionfree abelian group A/tA is equal to the rank of a free abelian subgroup of A/tA; and every free abelian subgroup of A/tA is isomorphic to a free abelian subgroup of A whose rank is finite by (c). It follows that the rank b of A/tA is finite. Every free abelian subgroup of A is isomorphic to a subgroup of A/tA so that its rank is likewise b. Thus it follows:

(4) The rank b of A/tA is finite and the rank of every torsionfree abelian subgroup of A is b.

If F is a subgroup of G with $Q \subseteq F$ and free abelian F/Q, then $Q \subseteq F \cap A$ by (1) and

$$[F/Q]/[(F \cap A)/Q] \cong F/(F \cap A) \cong FA/A \subseteq G/A$$

is by (a) a p-group. Since the subgroup $(F \cap A)/Q$ of the free abelian group F/Q is by Fuchs [p. 46, Theorem 12.2] a free abelian group, F/Q and $(F \cap A)/Q$ are free abelian groups of equal rank. By construction,

$$Q = F \cap tA = (F \cap A) \cap tA$$

so that

$$(F \cap A)/Q = (F \cap A)/((F \cap A) \cap tA) \cong (F \cap A)tA/tA \subseteq A/tA;$$

and this implies by (4) that

(5) torsionfree abelian subgroups of G/Q are of rank b.

If we let $G^* = G/Q$ and $A^* = A/Q$, then $G^*/A^* \cong G/A$ and it follows from (a), (2), (3) and (5) that

(6) G^*/A^* is a hypercentral p-group; abelian p-subgroups of G^* are of finite rank a [and a is the rank of some elementary abelian p-subgroup of G]; and every torsionfree abelian subgroup of G^* is of finite rank b.

Among the abelian normal subgroups of G^* which contain A^* there exists a maximal one, say B^* (Maximum Principle of Set Theory). G^*/B^* is an epimorphic image of G^*/A^* and as such G^*/B^* is a hypercentral p-group.

Hence every normal subgroup, not 1, of G^*/B^* contains a center element, not 1. Since B^* is a maximal abelian normal subgroup of G^*, it follows therefore that

(7) $B^* = \mathfrak{c}_{G^*} B^*$.

It is a consequence of (1) that

(8) 1 is the only p'-element in G^*.

The totality T^* of all torsion elements in the abelian group B^* is a characteristic subgroup of B^* which is a p-group by (8) and which is a normal subgroup of G^*, since B^* is a normal subgroup of G^*. Furthermore B^*/T^* is torsionfree. The rank of B^*/T^* is equal to the rank of a free abelian subgroup of B^*/T^* which is isomorphic to a subgroup of B^*. Thus application of (6) shows:

(9) The rank a of the abelian p-group T^* is finite and the rank of an elementary abelian p-subgroup of G; and the rank of the torsionfree abelian group B^*/T^* is b.

It is a consequence of (7) that G^*/B^* is essentially the same as the group Γ of automorphisms, induced by G^* in B^*. Since G^*/B^* is a hypercentral p-group, as we remarked before, Γ is a hypercentral p-group of automorphisms of B^*. Since T^* is a characteristic subgroup of B^*, it is transformed into itself by every element of Γ. It follows that the totality Λ of automorphisms in Γ, fixing every element in T^*, is a normal subgroup of Γ with Γ/Λ essentially identical with the group of automorphisms, induced in B^* by Γ. The rank a of the abelian p-group T^* is finite by (9). We may apply Proposition 1.4 to show that

(10.a) Γ/Λ is a finite p-group of rank $\frac{1}{2}a(5a - 1)$.

The totality Δ of the automorphisms in Γ fixing every element in B^*/T^* is a normal subgroup of Γ with Γ/Δ essentially the same as the group of automorphisms, induced by Γ in B^*/T^*. Since Γ is a hypercentral p-group, so is Γ/Δ; and we deduce from (9) that B^*/T^* is a torsionfree abelian group of rank b. Application of Burnside [p. 484, Note G] shows that

(10.b) Γ/Δ is a finite p-group of order p^M with

$$\begin{aligned} M \leq [bp(p-1)^{-2}] &\leq 2b \quad \text{for } p = 2 \\ &\leq b \quad \text{for } 2 < p. \end{aligned}$$

Combining (10.a) and (10.b) one obtains

(10) $\Gamma/(\Lambda \cap \Delta)$ is a finite p-group of rank $\frac{1}{2}a(5a - 1) + 2b$.

The automorphisms in $\Lambda \cap \Delta$ fix every element in T^* and every element in B^*/T^*. They belong therefore to the stabilizer of the subgroup T^* of the abelian group B^*. But the stabilizer is well known and easily seen to be isomorphic to $\mathrm{Hom}(B^*/T^*, T^*)$, the group of homomorphisms of B^*/T^* into T^*. Noting (9), it follows that

(11) $\Lambda \cap \Delta$ is an abelian p-group of rank ab.

Combination of (10), (11) and $\Gamma \cong G^*/B^*$ shows that

(12) G^*/B^* is a hypercentral p-group of rank $\frac{1}{2}a(5a - 1) + 2b + ab$.

Since the sum of the ranks of B^*/T^* and of T^* is the rank of B^*, we deduce from (9) that B^* is of rank $a + b$ and that G^* is by (12) of rank

$$\tfrac{1}{2}a(5a + 1) + 3b + ab;$$

and noting that G/A is an epimorphic image of $G^* = G/Q$ with $Q \subseteq A$, we derive (C) from this estimate for the rank of G^*. It is clear furthermore that (A) is a consequence of (4).

Consider finally a p-subgroup P of G. Then $P \cap Q = 1$, since Q is a P'-subgroup. Hence $P \cong QP/Q = P^* \subseteq G^*$; and P^* is an extension of

$$P^* \cap B^* = P^* \cap T^* \subseteq T^*$$

by $P^*/(P^* \cap B^*) \cong B^*P^*/B^* \subseteq G^*/B^*$. Since T^* has rank a by (9), and since G^*/B^* is by (12) a p-group of rank

$$\tfrac{1}{2}a(5a - 1) + 2b + ab,$$

it follows that $P \cong P^*$ is of rank $\frac{1}{2}a(5a + 1) + 2b + ab$, and this proves (B).

3. In this section we discuss normal subgroups A of groups G with free abelian G/A. We begin by exploring a general situation we will have to face later.

(3.1) *If σ is an automorphism of the group G, if N is a product of σ-admissible, normal, finite subgroups of G, if g is an element in G with*

$$g^\sigma \equiv g \mod N,$$

then the set of elements g^{σ^i} is finite.

Proof. The element $g^\sigma g^{-1}$ is, by hypothesis, an element in N and belongs consequently to a product $P \subseteq N$ of finitely many σ-admissible, normal, finite subgroups of G. It follows that σ induces an automorphism in G/P and Pg is fixed by σ. Hence

$$Pg = (Pg)^{\sigma^i} \quad \text{for every integer } i,$$

so that the finite set Pg contains all the elements g^{σ^i} and the set of elements g^{σ^i} is finite too.

(3.2) *If A is an abelian normal subgroup of G with abelian G/A, if e is* *an element in* G, *then the set* $e \circ A$ *of all commutators* $e \circ a$ *for* a *in* A *is a normal subgroup of* G *with* $e \circ A \subseteq A$. *If* $A/[e \circ A]$ *is a product of finite normal subgroups of* $G/[e \circ A]$, *then* $G/A\mathfrak{c}_G e$ *is a torsion group.*

Proof. Mapping a in the abelian normal subgroup A onto the commutator $e \circ a$ is an endomorphism of A, since

$$e \circ (ab) = (e \circ b)(e \circ a)^b = (e \circ b)(e \circ a).$$

Hence $e \circ A$ is a subgroup of A. Since G/A and A are abelian, the elements e and $e^g = e(e \circ g)$ for g in G induce the same automorphism in A. Hence $(e \circ a)^g = e^g \circ a^g = e \circ a^g$ so that $(e \circ A)^g = e \circ A$, proving that $e \circ A$ is a normal subgroup of G.

Assume now that $A/e \circ A = A^*$ is a product of finite normal subgroups of $G/e \circ A = G^*$. If g is an element in G, then $g^* = (e \circ A)g$ is an element in G^*. If s is an element in G, then s^* induces an automorphism in G^* with

$$e^{*s^*} \equiv e^* \mod A^*,$$

since $G^*/A^* \cong G/A$ is abelian. Application of (3.1) shows that the set of elements $e^{*s^{*i}}$ with integral i is finite. Consequently there exists a positive integer $n = n(s)$ such that

$$e^* = e^{*s^{*n}}.$$

The element $e^{-1}e^{s^n}$ belongs therefore to $e \circ A$. This implies the existence of an element t in A with

$$e^{-1}e^{s^n} = e \circ t = e^{-1}e^t.$$

Hence

$$e^{s^n} = e^t \quad \text{and} \quad e = e^{s^n t^{-1}},$$

so that $s^n t^{-1}$ belongs to $\mathfrak{c}_G e$ and s^n belongs to $A\mathfrak{c}_G e$. Thus $G/A\mathfrak{c}_G e$ is a torsion group.

(3.3) *Assume that the normal subgroup* A *of* G *meets the following requirements*:

(a) A *is torsionfree abelian.*

(b) G/A *is abelian.*

(c) *If* X *is a normal subgroup of* G *with* $1 \subset X \subseteq A$, *then* A/X *is a product of finite normal subgroups of* G/X.

(d) $A \not\subseteq \mathfrak{z}G$.

Then there exists an abelian subgroup S *of* G *with* $A \cap S = 1$ *and* G/AS *a torsion group.*

Proof. Because of (d) there exists an element e in G which does not centralize A. The set $e \circ A$ is, by (3.2), a normal subgroup of G with $1 \subset e \circ A \subseteq A$. It is a consequence of (c) that $A/[e \circ A]$ is a product of finite normal subgroups of $G/[e \circ A]$. Application of (3.2) proves that

(1) $G/A\mathfrak{c}_G e$ is a torsion group.

Let $V = A \cap \mathfrak{c}_G e = \mathfrak{c}_A e$. Every element e^g induces in A the same automorphism as e, since A and G/A are abelian. The automorphisms, induced by e and e^g in A, have consequently the same fixed elements so that $\mathfrak{c}_A e = \mathfrak{c}_A e^g = (\mathfrak{c}_A e)^g$. Thus V is a normal subgroup of G. If $1 \subset V$, then A/V would be, by (c), a product of finite normal subgroups of G/V; and A/V would in particular be a torsion group. The mapping which maps every element a in A onto the commutator $e \circ a$ is an endomorphism of A with kernel V. We obtain $A/V \cong e \circ A$, a contradiction because A/V is a torsion group while $1 \subset e \circ A \subseteq A$ is, by (a), torsionfree. Hence $1 = V = A \cap \mathfrak{c}_G e$; and from the commutativity of G/A we deduce the commutativity of $\mathfrak{c}_G e$. Now it follows from (1) that $\mathfrak{c}_G e$ is the desired subgroup S.

Remark 3.4. If $G = \{a, b, c\}$ with $a \circ b = c \in \mathfrak{z}G$ and $\{c\} = A$, then there does not exist an abelian subgroup S of G with $1 = A \cap S$ and torsion group G/AS. Thus condition (d) is indispensable.

Remark 3.5. M. F. Newman [p. 357, Theorem 3.3] has shown that (3.3) remains valid if conditions (a) and (c) are replaced by the condition: A is an abelian minimal normal subgroup of G.

LEMMA 3.6. *Assume that the group G and its normal subgroup N meet the following requirements*:

(a) *If K is a normal subgroup of G with $K \subset N$, then there exists a normal subgroup L of G with $K \subset L \subseteq N$ and finite L/K.*

(b) *Every free abelian subgroup of G is of finite rank.*

Then every free abelian subgroup of G/N is isomorphic to a subgroup of G and hence in particular of finite rank.

Proof. Consider a free abelian subgroup F/N of G/N. There exist torsionfree abelian subgroups of F and among these there exists a maximal one, say A (Maximum Principle of Set Theory). It is a consequence of (a) that N is a torsion group. Hence $A \cap N = 1$ so that

$$A = A/(A \cap N) \cong AN/N \subseteq F/N.$$

Since F/N is free abelian, so is $A \cong AN/N$; see Fuchs [p. 45, Theorem 12.1]. Furthermore A is of finite rank by (b).

Consider an element f in F and form the set $\mathfrak{S}$ of all the normal subgroups X of G with $X \subseteq N$ and $X\{A, f^i\}/X$ non-abelian for every positive integer i. Consider a non-vacuous subset $\mathfrak{X}$ of $\mathfrak{S}$ which is linearly ordered by inclusion. Form $T = \bigcup_{X \in \mathfrak{X}} X$. This is a normal subgroup of G with $T \subseteq N$. If T were not in $\mathfrak{S}$, then there would exist a positive integer k such that $T\{A, f^k\}/T$ is abelian. This is equivalent to $A \circ f^k \subseteq T$. Since A is free abelian of finite rank, A is finitely generated. The normal subgroup K of G which is generated by $A \circ f^k$ is consequently spanned by finitely many classes of elements, conjugate in G; and $K \subseteq T = \bigcup_{X \in \mathfrak{X}} X$. Since $\mathfrak{X}$ is linearly ordered by inclusion, there exists a normal subgroup H in $\mathfrak{T}$ with $K \subseteq H$. Then $H\{A, f^k\}/H \neq 1$ is both abelian and non-abelian, a contradiction proving that T belongs to $\mathfrak{S}$.

Assume now by way of contradiction that $\mathfrak{S}$ is not vacuous. Then we may apply the Maximum Principle of Set Theory on $\mathfrak{S}$ and there exists a maximal M in $\mathfrak{S}$. Since $N\{A, f\}/N \subseteq F/N$ is abelian, N does not belong to $\mathfrak{S}$. Hence $M \subset N$. Since M is a normal subgroup of G, there exists, by (a), a normal subgroup W of G with $M \subset W \subseteq N$ and finite W/M. From the maximality of M we deduce the existence of a positive integer h such that $W\{A, f^h\}/W$ is abelian. This is equivalent to $A \circ f^h \subseteq W$, and f^h normalizes WA. Since W/M is finite and M, W are normal subgroups of G, there exists a positive integer j such that f^j induces the 1-automorphism in W/M. The element f^{jh} induces consequently in AW/M an automorphism which fixes every element in W/M and every element in $(AW/M)/(W/M)$ so that $AW \circ f^{jh} \subseteq W$ and $W \circ f^{jh} \subseteq M$. It follows that

$$a^{f^{jhi}} \equiv a(a \circ f^{jh})^i \mod M$$

for every positive i and every $a \in A$. Since W/M is finite, there exists a positive integer e such that $(W/M)^e = 1$. It follows that $(a \circ f^{jh})^e \in M$ and hence

$$a^{f^{jhe}} \equiv a \mod M$$

for every $a \in A$. Hence AM/M is centralized by f^{jhe} so that $M\{A, f^{jhe}\}/M$ is abelian. But M belongs to $\mathfrak{S}$, a contradiction showing that $\mathfrak{S}$ is vacuous. Thus 1 does not belong to $\mathfrak{S}$ so that $\{A, f^i\}$ is abelian for some positive i. Remember that A is a maximal torsionfree abelian subgroup of F. Hence there exists a positive integer n with $f^{in} \in A$. We conclude that $(F/N)/(AN/N)$ is an abelian torsion group. Since F/N is a free abelian group, we conclude that F/N and $AN/N \cong A$ are free abelian groups of the same finite rank. Thus they are isomorphic, which completes the proof of Lemma 3.6.

Remark 3.7. Let F be the free group on the two free generators a, b; and let

$$G = F/F''(F')^3.$$

If we let $x = aF''(F')^3$ and $y = bF''(F')^3$, then $G = \{x, y\}$. Furthermore G' is a countably infinite, elementary abelian 3-group and G/G' is a free abelian group of exact rank 2 which is essentially identical with the group Γ of automorphisms, induced in G' by G. Denote by X and Y the automorphisms, induced in G' by x and y respectively. Then the ring of endomorphisms of the elementary abelian 3-group G', spanned by Γ, is a ring of polynomials in X, X^{-1}, Y, Y^{-1} with coefficients in the prime field of characteristic 3; and it is a subring of the field of rational functions in the independent variables X, Y with coefficients in the prime field of characteristic 3.

Assume by way of contradiction the existence of a non-cyclic torsionfree abelian subgroup U of G. Then $U \cap G' = 1$ so that U is isomorphic to a subgroup of G/G'. Hence U is free abelian of rank 2 so that $[G/G']/[UG'/G']$ is a finite group. Consequently there exists a positive integer k with $G^k \subseteq UG'$. Naturally $[G/G']^k = G^kG'/G'$ is likewise free abelian of rank 2. Application of Dedekind's Modular Law shows $G'G^k = G'(G^k \cap U)$ so that

$$G^k \cap U = (G^k \cap U)/(G^k \cap U \cap G') \cong G'G^k/G'$$

is free abelian of rank 2. It follows that

$$U \cap G^k = \{x^k c, y^k d\} \quad \text{with } c, d \text{ in } G'.$$

From the commutativity of U and G' we deduce

$$1 = (x^k c) \circ (y^k d) = (x^k \circ d)(x^k \circ y^k)(c \circ y^k).$$

Since $c, d \in G'$, there exist endomorphisms C and D of Γ with $c = (x \circ y)^C$ and $d = (x \circ y)^D$. If we denote by X and Y the automorphisms of G', induced by x and y respectively, then the preceding equation leads to the following equation, relating endomorphisms of G' in the endomorphism ring of G':

$$0 = (1 - X^k)D + \textstyle\sum_{i=0}^{k-1} X^i \sum_{i=0}^{k-1} Y^i + (Y^k - 1)C.$$

As noted before the ring of endomorphisms of G', spanned by Γ, is a subring of a field and hence free of zero divisors. Since two of the summands of the above equation are divisible by $\sum_{i=0}^{k-1} X^i$, and since X and Y are independent, it follows that

$$C = \textstyle\sum_{i=0}^{k-1} X^i C^*$$

with suitable C^* in our ring of endomorphisms; and similarly we have

$$D = \textstyle\sum_{i=0}^{k-1} Y^i D^*.$$

But then our equation reduces to

$$0 = (1 - X)D^* + 1 + (Y - 1)C^*$$

because of the absence of zero divisors; and this is a contradiction, since 1 is not contained in the ideal spanned by $1 - X$ and $1 - Y$.

Thus we have shown that every torsionfree abelian subgroup of G is cyclic although G/G' is a free abelian group of exact rank 2. Hence condition (a) in Lemma 3.6 is indispensable; and we cannot substitute for (a) the requirement that N be locally finite.

4. We have defined elsewhere that the group G is *of finite abelian subgroup rank*, if all its primary abelian subgroups and all its torsionfree abelian subgroups are of finite rank; for a detailed discussion of this concept see Baer [6; p. 94/95]. We mention that G is of finite abelian subgroup rank if, and only if, every elementary abelian p-subgroup of G is finite and every free abelian subgroup of G is of finite rank.

LEMMA 4.1. *If A is an abelian normal subgroup of the group G of finite abelian subgroup rank, then G/A too is of finite abelian subgroup rank.*

Proof. Assume first that A is a torsion group. Then A is the product of finite characteristic subgroups; see Fuchs [p. 65, Theorem 19.2 and p. 68, (19)]. Consider a subgroup S of G such that $S' \subseteq A \subset S$. If firstly S/A is an elementary abelian p-group, then the finiteness of S/A is a consequence of Proposition 2.5; and if S/A is a free abelian group, then the finiteness of the rank of S/A is contained in Lemma 3.6.

We turn next to the general case. The totality tA of the torsion elements of A is a characteristic torsion subgroup of A and hence a normal subgroup of G. It is a consequence of what has been shown in the first part of the proof that $G^* = G/tA$ is of finite abelian subgroup rank. By construction $A^* = A/tA$ is a torsionfree abelian normal subgroup of G^*. Consider a subgroup S of G^* such that $S' \subseteq A^* \subset S$. If firstly S/A^* is an elementary abelian p-group, then the finiteness of S/A^* is contained in Proposition 2.5; and if S/A^* is free abelian, then the finiteness of the rank of the torsionfree group S is a consequence of a theorem by Čarin [2; Theorem 6, p. 910]. Hence $G^*/A^* \cong G/A$ is of finite abelian subgroup rank.

We shall say that the group G is of *bounded abelian subgroup rank*, if there exist positive integers $r(0)$, $r(p)$ for p *a* prime such that

(I) every abelian p-subgroup has rank $r(p)$ and
(II) every torsionfree abelian subgroup of G has rank $r(0)$.

Clearly (cp. Fuchs [p. 68, Exercise 18, (a)]) condition (I) is equivalent to the following requirement:

(I′) every elementary abelian p-subgroup of G has order dividing $p^{r(p)}$.

Every group of bounded abelian subgroup rank is obviously of finite abelian subgroup rank. The converse is false as may be seen from the following example: Let $E(i)$ be an elementary abelian p-group of order p^i and G the free product of all the $E(i)$. Then every torsionfree abelian subgroup of G is cyclic, every abelian q-subgroup with $p \neq q$ is trivial, and every abelian p-subgroup is isomorphic to a subgroup of some $E(i)$. This is a consequence of the Kurosh Subgroup Theorem; see Specht [p. 189, Satz 8]. Hence G is of finite abelian subgroup rank, but not of bounded abelian subgroup rank.

The group G is of *bounded abelian factor rank*, if there exist positive integers $r(p)$ (for every prime p) such that

(I) every abelian p-factor of G has rank $r(p)$.

Clearly (I) is equivalent to the requirement:

(I′) every elementary abelian p-factor of G has order dividing $p^{r(p)}$.

Assume that U is a free abelian factor of G. Then U/U^p is a p-factor of G and the ranks of U and U/U^p are the same. Thus we obtain from (I):

(II) Every torsionfree abelian factor of G has rank $r(0) = \mathrm{Min}(r(p))$.

Note that the set of integers $r(p)$ need not be bounded.

It is evident that groups of finite rank are of bounded abelian factor rank and that every factor of a group of bounded abelian factor rank is likewise of bounded abelian factor rank.

(4.2) *If the group G possesses a normal subgroup N such that G/N and N are of bounded abelian factor rank, then so is G.*

Proof. Let T be a normal subgroup of the subgroup S of G. Then certainly S/T is an extension of $(TN \cap S)/T$ by $S/(TN \cap S)$. However, $TN \cap S = T(N \cap S)$ by Dedekind's Modular Law because $T \subseteq S$ and $TN = NT$; and by the Second Isomorphism Theorem

$$(TN \cap S)/T = T(N \cap S)/T \cong (N \cap S)/[T \cap (N \cap S)] = (N \cap S)/(N \cap T),$$

$$S/(TN \cap S) \cong STN/TN = SN/TN \cong (SN/N)/(TN/N).$$

If S/T is an elementary abelian p-group, its rank is exactly the sum of the ranks of

$$S/(TN \cap S) \cong (SN/N)/(TN/N) \quad \text{and} \quad (TN \cap S)/T \cong (S \cap N)/(T \cap N),$$

which are both bounded by hypothesis. If S/T is torsionfree and abelian, then its rank is the sum of the rank of $(S \cap N)/(T \cap N)$ and the rank of the quotient group of $(SN/N)/(TN/N)$ modulo its torsion subgroup. Both of these ranks are bounded by hypothesis, and so is their sum; and (4.2) is proved completely.

Abelian groups of finite abelian subgroup rank are of bounded abelian factor rank. We will generalize this statement and Lemma 4.1 in the following

LEMMA 4.3. *If N is a locally hypercentral normal subgroup of the group G of finite abelian subgroup rank, then G/N is of finite abelian subgroup rank and N is hypercentral and of bounded abelian factor rank.*

Terminological reminder. A group is *locally hypercentral*, if every finitely generated subgroup of it is hypercentral. But a group is hypercentral and finitely generated if, and only if, it is noetherian and nilpotent (of finite class); see Baer [2; p. 203, Theorem]. Thus local hypercentrality and local nilpotency are equivalent requirements.

Proof. It is an immediate consequence of our hypothesis and Baer [6; p. 98, Theorem and p. 96, Lemma] that

(1) N is hypercentral,

(2) primary subgroups of N are artinian,

(3) torsionfree epimorphic images of N are of finite rank and nilpotent of finite class.

It is a consequence of (1) that

(4) the set T of all torsion elements in N is a characteristic subgroup of N and T is the direct product of its primary components T_p (see Specht [p. 382, Satz 12 and p. 380, Satz 11]).

Application of (1), (2), and Baer [5; p. 21, Satz 4.1 and p. 7/8, Satz 2.1] shows the existence of an abelian characteristic subgroup A_p of T_p with finite T_p/A_p (and $A_p = (A_p)^p$). As a characteristic subgroup of the characteristic subgroup T_p of the characteristic subgroup T of the normal subgroup N of G each A_p is a normal subgroup of G.

It is a consequence of (2), (3), Baer [5; p. 21, Satz 4.1 and p. 7/8, Satz 2.1] and (4.2) that N is of bounded abelian factor rank.

Consider a subgroup S of G such that $S' \subseteq T \subset S$ and S/T is torsionfree. By (1) and (2) the requirements of Lemma 3.6 are satisfied for S and T; hence every free abelian subgroup of S/T is isomorphic to a subgroup of S itself. Thus all free abelian subgroups of the torsionfree abelian group S/T are of finite rank and S/T is of finite rank.

Consider next a subgroup S of G such that $S' \subseteq T \subset S$ and S/T is an elementary abelian p-group. Denote by $T_{p'}$ the direct product of all T_q with $q \neq p$. Clearly $S/T_{p'}$ is an extension of the p-group $T/T_{p'} \cong T_p$ by the p-group S/T so that $S/T_{p'}$ is a p-group. Assume that R is a subgroup of S such that $R' \subseteq T_{p'} \subset R$ and $R/T_{p'}$ is elementary abelian. It is easily checked that R and $T_{p'}$ satisfy all the conditions of (2.1); and we may conclude—see (2.1, D)—that $R/T_{p'}$ is finite. Hence $S/T_{p'}$ (being a p-group) is of finite abelian subgroup rank, and the same is true for

$$(S/T_{p'})/(A_pT_{p'}/T_{p'}) \cong S/A_pT_{p'}$$

by Lemma 4.1. The quotient group $S/A_pT_{p'}$ is an extension of the finite p-group $T/A_pT_{p'} \cong T_p/A_p$ by the elementary abelian p-group

$$S/T \cong (S/A_pT_{p'})/(T/A_pT_{p'}).$$

Thus $S/A_pT_{p'}$ is of finite exponent and nilpotent of finite class. As $S/A_pT_{p'}$ is of finite abelian subgroup rank, every abelian normal subgroup of $S/A_pT_{p'}$ is of finite rank and of finite exponent, hence finite. Among these abelian normal subgroups there exists a maximal one which (by the nilpotency of $S/A_pT_{p'}$) is self-centralizing. The quotient group of $S/A_pT_{p'}$ modulo this normal subgroup is essentially the same as the group of automorphisms induced by $S/A_pT_{p'}$ in its finite normal subgroup, so it is finite. Thus $S/A_pT_{p'}$ is finite and its quotient group $(S/A_pT_{p'})/(T/A_pT_{p'}) \cong S/T$ is finite too. Hence we have shown

(5) G/T is of finite abelian subgroup rank.

By (3), the quotient group N/T is nilpotent of finite class. Thus there exists a series of normal subgroups of G

$$T = K_1 \subseteq K_2 \subseteq \cdots \subseteq K_t = N,$$

such that $K_{i+1}/K_i = \mathfrak{z}(N/K_i)$ for all i. Therefore K_{i+1}/K_i is an abelian normal subgroup of G/K_i. We will show by induction that G/K_i is of finite abelian subgroup rank. The initial step ($i = 1$) is the content of (5). Assume now that $i > 1$ and G/K_{i-1} is of finite abelian subgroup rank. Then, by Lemma 4.1, $(G/K_{i-1})/(K_i/K_{i-1}) \cong G/K_i$ is of finite abelian subgroup rank, which completes our induction. Hence, in particular, $G/N = G/K_t$ is of finite abelian subgroup rank.

LEMMA 4.4. *If every primary elementary abelian factor of the radical group G is finite, then every epimorphic image, not* 1, *of G possesses an abelian characteristic subgroup, not* 1 (*which is either finite or torsionfree of finite rank*).

Proof. If $H \neq 1$ is an epimorphic image of G, then we deduce from the radicality of G the existence of a locally nilpotent normal subgroup, not 1. The Hirsch-Plotkin radical R of H is consequently different from 1. We recall that the Hirsch-Plotkin radical is the unique maximal locally nilpotent normal subgroup and hence characteristic; see Schenkman [p. 205, VI.7.b. Theorem]. The elementary abelian primary subgroups of R are finite by hypothesis. If F is a free abelian subgroup of R, then F and F/F^p (for p a prime) are factors of G. By hypothesis F/F^p is finite so that the rank of F is finite too. It follows that R is of finite abelian subgroup rank. Apply Baer [6; p. 98, Theorem] to see that R is hypercentral. Hence $\mathfrak{z}R \neq 1$. Since R is a characteristic subgroup of H, so is $\mathfrak{z}R$. The existence of a characteristic subgroup, not 1, of $\mathfrak{z}R$, and hence of H, which is either finite or torsionfree of finite rank is now readily verified.

Discussion of Lemma 4.4. (A) The first hypothesis of our lemma is certainly satisfied whenever G is of bounded abelian factor rank.

(B) The property that every epimorphic image, not 1, possesses an abelian characteristic subgroup, not 1, implies hypercommutativity. But it is a stronger requirement, as is shown by a famous example, due to McLain, which is characteristically-simple but the product of its abelian normal subgroups.

PROPOSITION 4.5. *If the radical torsion group G is of finite abelian subgroup rank, then*

(a) *G is hyperabelian of bounded abelian factor rank;*

(b) *every epimorphic image, not* 1, *of G possesses a finite abelian characteristic subgroup, not* 1;

(c) *G is locally finite* (*and locally soluble*);

(d) *the subgroup $\mathfrak{a}G$ of G, generated by all the subgroups of G without proper subgroups of finite index, is an abelian and radicable characteristic subgroup of G; and $G/\mathfrak{a}G$ is residually finite with finite Sylow subgroups and all p-Sylow subgroups of $G/\mathfrak{a}G$ are conjugate in $G/\mathfrak{a}G$;*

(e) *all p-Sylow subgroups of G are conjugate in G;*
(f) *G is countable.*

Proof. Since G is of finite abelian subgroup rank,

(1) every primary abelian subgroup of G is of finite rank and artinian.

Consider a primary subgroup P of G. Since G is radical, P is radical. It is a consequence of (1) that every abelian subgroup of P is artinian and of finite rank. Thus condition (5) of Baer [7; p. 359/360, Hauptsatz 8.15, A] is satisfied by P; and it follows that P is soluble and artinian. Since simple soluble groups are cyclic of order a prime, every simple factor of P is finite. Application of Baer [5; p. 7/8, Satz 2.1] shows the existence of an abelian subgroup of finite index in P. Since P is artinian, this implies that the intersection of all the subgroups of finite index in P is a characteristic subgroup P^* of P such that P/P^* is finite and P^* is abelian and radicable. Since P^* is of finite rank, P too is of finite rank; and since P^* and P/P^* are both locally finite p-groups, every finitely generated subgroup of P is a finite p-subgroup and hence nilpotent. Since P is of finite abelian subgroup rank, we deduce from Baer [6; p. 98, Theorem] that P is hypercentral. Thus we have shown:

(2) Every primary subgroup P of G is artinian, soluble, hypercentral, of finite rank; and there exists an abelian, radicable, characteristic subgroup P^* of P with finite P/P^*.

Denote by A the subgroup $\mathfrak{a}G$ of G, generated by all the subgroups of G without proper subgroups of finite index. This is a well determined characteristic subgroup of G (the hyporesiduum of G); and it is clear that A is free of proper subgroups of finite index. As G is radical and of finite abelian subgroup rank, so is A. Assume by way of contradiction that A is not abelian. Then $\mathfrak{z}A \subset A$. By radicality $A/\mathfrak{z}A \neq 1$ possesses a locally nilpotent normal subgroup, not 1. So there exists a normal subgroup B of A with $\mathfrak{z}A \subset B \subseteq A$ and locally nilpotent $B/\mathfrak{z}A$. But then B itself is locally nilpotent; and since G is a torsion group, so is B. Application of Lemma 4.3 shows that B is hypercentral; and as a hypercentral torsion group B is the direct product of its primary components B_p ; see Specht [p. 380, Satz 11]. As every B_p is a characteristic subgroup of the normal subgroup B of A, every B_p is a normal subgroup of A. It is a consequence of (2) that B_p is artinian, soluble, hypercentral, of finite rank; and there exists an abelian radicable characteristic subgroup B_p^* of B_p with finite B_p/B_p^*. As a characteristic subgroup of the normal subgroup B_p the subgroup B_p^* is normal in A. Consequently torsion groups of automorphisms are induced by A in B_p^* and in B_p/B_p^*. The first of these is finite, since torsion groups of automorphisms of abelian artinian groups are finite by Baer [4; p. 521, Theorem]. The second of these is finite since B_p/B_p^* is finite. But 1 is the only finite epimorphic image of A. Hence A induces only the 1-automorphism in B_p^* and in B_p/B_p^* so that in particular

$B_p^* \subseteq \mathfrak{z}A$. The group of automorphisms, induced by A in B_p, is consequently isomorphic to a subgroup of Hom $(B_p/B_p^*, B_p^*)$, the group of homomorphisms of B_p/B_p^* into B_p^*. But the abelian artinian group B_p^* contains only a finite number of elements of an order not exceeding the (finite) order of B_p/B_p^*; see Fuchs [p. 65, Theorem 19.2]. It follows that the group of automorphisms, induced by A in B_p, is finite; and since 1 is the only finite epimorphic image of A, we conclude that A induces only the 1-automorphism in B_p. Hence $B_p \subseteq \mathfrak{z}A$ for every prime p; and this implies the contradiction that $B = \prod_p B_p \subseteq \mathfrak{z}A \subset B$. Thus we have shown:

(3) The subgroup $A = \mathfrak{a}G$ of G which is generated by all the subgroups without proper subgroups of finite index is an abelian, radicable, characteristic subgroup of G.

Consider a subgroup X of G with $A \subseteq X$ such that X/A is free of proper subgroups of finite index. If Y is a subgroup of X with finite index $[X:Y]$, then $[X:AY]$ is likewise finite, implying $X = AY$. Hence

$$[X:Y] = [AY:Y] = [A:A \cap Y] \quad \text{is finite.}$$

But A is free of proper subgroups of finite index so that $A = A \cap Y \subseteq Y$ and $X = AY = Y$: we have shown that X too is free of proper subgroups of finite index. This implies $X = A$; and we have shown:

(4) 1 is the only subgroup of G/A without proper subgroups of finite index.

Since A is by (3) an abelian characteristic subgroup of the radical group G of finite abelian subgroup rank, we deduce from Lemma 4.1 that G/A is of finite abelian subgroup rank. Hence G/A is a radical torsion group of finite abelian subgroup rank so that (2) may be applied to G/A. It follows that every primary subgroup of G/A (is artinian, soluble, hypercentral and) possesses a subgroup of finite index without proper subgroups of finite index. Application of (4) shows that every primary subgroup of G/A is finite. Hence we have shown:

(5) G/A is of finite abelian subgroup rank; every primary subgroup of G/A is finite.

Every finitely generated hypercentral torsion group is finite and nilpotent; see Baer [9; p. 207, Corollary]. This implies that locally hypercentral torsion groups are locally finite. Since G is radical, it follows from Specht [p. 141, Satz 40*] that

(6) G is locally finite and locally soluble.

By (5), every p-Sylow subgroup of G/A is finite. Denote by S, T two p-Sylow subgroups of G/A. The quotient group G/A of G is locally finite by (6); so $\{S, T\}$ is a finite group and the p-Sylow subgroups S, T of G/A are

p-Sylow subgroups of the finite subgroup $\{S, T\}$ as well; thus they are conjugate in $\{S, T\}$. We have shown:

(7) Every Sylow subgroup of G/A is finite and all p-Sylow subgroups of G/A are conjugate in G/A.

It is a consequence of (6) and (7) that G/A is locally finite and locally soluble with finite Sylow subgroups. Hence it follows from Baer [9; p. 129, Satz 6.1, (b)] that G/A is residually finite: we have verified (c) and (d).

Consider a prime p. By (7), all p-Sylow subgroups of G/A are finite and have the same order $p^{e(p)} = q$. It follows that every p-subgroup of G/A is finite of order a divisor of q. Consider subgroups U, V of G/A such that $U' \subseteq V \subset U$ and U/V is a p-group. If L and K are p-Sylow subgroups of U, then L and K are both finite of order a divisor of q. Since G is locally finite, so is G/A; and this implies the finiteness of $\{L, K\}$. Since L and K are p-Sylow subgroups of U, they are p-Sylow subgroups of the finite group $\{L, K\}$ and hence they are conjugate in $\{L, K\}$. Since U/V is abelian, and since LV/V and KV/V are conjugate subgroups of the abelian group U/V, they are equal; and this implies $LV = KV$. Since G and G/A and hence U are torsion groups, every element in the p-group U/V is represented by a p-element in U; and every p-element in U is contained in a p-Sylow subgroup X of U. But all XV for X a p-Sylow subgroup of U are equal. Hence $U = XV$ for every p-Sylow subgroup X of U. Since such an X is of order a divisor of q, it follows that U/V is finite of order a divisor of q, and we have shown:

(8) If $q = p^{e(p)}$ is the common order of all the p-Sylow subgroups of G/A and if the factor P of G/A is an abelian p-group, then P is finite of order a divisor of q.

From (8) we conclude in particular that G/A is of bounded abelian factor rank. The abelian torsion group A of finite abelian subgroup rank is by (1) likewise of bounded abelian factor rank. Apply (4.2) to show that G is of bounded abelian factor rank. Since G is radical, application of Lemma 4.3 shows the validity of (b). This implies that G is in particular hyperabelian, proving the validity of (a).

The simple proof that (e) is a consequence of (d) may be left to the reader.

Every primary component of the abelian torsion group A is of finite rank, since G is of finite abelian subgroup rank. It follows from Fuchs [p. 66, Theorem 19.2 and p. 68, Exercise 19] that every primary component of A is a direct product of finitely many abelian groups of rank 1. Hence the primary components of A are countable so that A itself is countable. Every finitely generated subgroup of G/A is finite and soluble by (c); and the Sylow subgroups of G/A are finite by (d). Since G and hence G/A is radical by hypothesis, we may apply Baer [9; p. 123, Folgerung 4.6] to G/A, showing the countability of G/A. Since G/A and A are countable, so is G; and we have shown (f).

5. The group G is said to be of *uniformly bounded abelian subgroup rank*, if there exists a positive integer r such that

(I) every abelian primary subgroup of G has rank r and
(II) every torsionfree abelian subgroup of G has rank r.

Clearly condition (I) is equivalent to the following requirement:

(I′) every elementary abelian p-subgroup of G has order dividing p^r.

It is easily seen that G has uniformly bounded abelian subgroup rank if, and only if, G has bounded abelian subgroup rank and there exists a positive integer r such that for almost all primes p every elementary abelian p-subgroup of G has order dividing p^r.

Finally we note the almost evident fact that groups of finite rank have uniformly bounded abelian subgroup rank.

Similarly we say that the group G is of *uniformly bounded abelian factor rank*, if there exists a positive integer r such that

(I) every abelian primary factor of G has rank r.

From this we deduce (proceeding as in section 4)

(II) every torsionfree abelian factor of G has rank r.

(5.1) *If, for some normal subgroup N of G, the groups N and G/N are of uniformly bounded abelian factor rank, then G is of uniformly bounded abelian factor rank.*

We leave the proof to the reader, since it is quite analogous to the proof of (4.2).

If G is of uniformly bounded abelian factor rank, then G is of uniformly bounded abelian subgroup rank. We will prove the equivalence of these two properties now for locally finite and for radical groups.

LEMMA 5.2. *If G is locally finite and of uniformly bounded abelian subgroup rank, then G is of uniformly bounded abelian factor rank.*

Proof. Assume that all primary abelian subgroups of G have rank r. If S/T is a finite elementary abelian p-factor of G, there is a finite subgroup U of the locally finite group G such that $UT = S$; and if P is a p-Sylow subgroup of U, then $U = P(U \cap T)$ and $PT = S$ so that $S/T \cong P/(P \cap T)$. If N is any maximal abelian normal subgroup of the finite p-group P, then P/N is essentially the same as the group of automorphisms induced by P in N; and as N has rank r, we deduce from (2.4) that P has rank $\frac{1}{2}r(5r + 1)$. So $S/T \cong P/(P \cap T)$ has rank $\frac{1}{2}r(5r + 1)$, proving Lemma 5.2.

Before we come to our next result we want to give a generalization of Zassenhaus' celebrated theorem which will be needed in the sequel. To make the argument more transparent we begin with a remark which is probably well known. We add a proof for the convenience of the reader.

(5.3) *If G is a locally radical group, if n is a positive integer with $S^{(n)} = 1$ for every soluble subgroup S of G, then $G^{(n)} = 1$.*

Proof. It is well known that a group T satisfies $T^{(n)} = 1$ if, and only if, $X^{(n)} = 1$ for all subgroups X of T which are generated by at most 2^n elements. Thus

(1) locally soluble subgroups of G are soluble.

Consider a radical subgroup A of G. If the set $\mathfrak{S}$ of soluble normal subgroups of A is [linearly] ordered by inclusion, the set-theoretical union of all members of $\mathfrak{S}$ is a locally soluble normal subgroup of A, hence soluble by (1). Thus we may apply the Maximum Principle of Set Theory: There exists a maximal soluble normal subgroup T of A. If $T \subset A$, then $A/T \neq 1$ and there exists a locally nilpotent normal subgroup $K/T \neq 1$ of A/T. If U is a finitely generated subgroup of K, then $UT/T \cong U/(U \cap T)$ is finitely generated and as a subgroup of K/T nilpotent (and soluble), and $U \cap T$ is soluble, since T is soluble. Thus U is soluble; and this shows that K is locally soluble. Then, by (1), K is soluble, contrary to the maximality of T. Hence $A = T$. This implies that

(2) radical subgroups of G are soluble.

Thus local radicality of G implies local solubility and, by (1), solubility of G; and the soluble group G satisfies $G^{(n)} = 1$ by hypothesis.

Proposition 5.4. *To every positive integer n there exists a positive integer $\lambda(n)$ with the following property:*

If Γ is a locally radical group of linear transformations of an at most n-dimensional vector space over a commutative field, then,

$$\Gamma^{(\lambda(n))} = 1.$$

Note that this result may be applied to automorphism groups of primary elementary abelian groups and of torsionfree abelian groups of finite rank, since the former may be considered as groups of linear transformations over the p-adic numbers, the latter over the rational numbers. According to Huppert [p. 494, Satz 9] one may choose $\lambda(n) = 2n$.

Proof. It is the content of Zassenhaus' Theorem that there exists to every positive integer n a positive integer $\lambda(n)$ with the following property:

(+) If θ is a soluble group of linear transformations of an at most n-dimensional vector space over a commutative field, then $\theta^{(\lambda(n))} = 1$.

For this see Suprunenko [p. 32, Theorem 11].

Now Proposition 5.4 is an immediate consequence of (+) and (5.3).

The Hirsch-Plotkin-radical of a group X is the product $\mathfrak{hp}X$ of all locally hypercentral normal subgroups of X. This is a well determined characteristic subgroup of X; and it is well known that $\mathfrak{hp}X$ is locally nilpotent; see Schenkman [p. 205, VI.7.b. Theorem].

PROPOSITION 5.5. *Every radical group G with $\mathrm{t}G = 1$ whose torsionfree abelian subgroups are of finite rank has the following properties:*

(a) *G is soluble of finite rank.*

(b) *Torsion subgroups of G are finite and their orders are bounded.*

(c) *$\mathfrak{hp}G$ is nilpotent and every $\mathfrak{hp}G/\mathfrak{z}_i\,\mathfrak{hp}G$ is torsionfree.*

(d) *$G/\mathfrak{hp}G$ is an extension of a free abelian group of finite rank by a finite group [is noetherian and almost abelian].*

(e) *If n is the [finite] rank of $\mathfrak{hp}G$ and $\lambda(n)$ the function, introduced in Proposition* 5.4, *then*

$$G^{(\lambda(n)+n)} = 1.$$

(f) *The compositum $\mathfrak{a}G$ of all the subgroups of G without proper subgroups of finite index is torsionfree and nilpotent; and $G/\mathfrak{a}G$ is residually finite.*

Proof. We may assume without loss of generality that $G \neq 1$; and this implies the infinity of G because of $\mathrm{t}G = 1$. The Hirsch-Plotkin radical $H = \mathfrak{hp}G$ of G is a locally nilpotent characteristic subgroup of G which contains every locally nilpotent normal subgroup of G; see Schenkman [p. 205, VI.7.b. Theorem]. Since G is radical, there exists a locally nilpotent normal subgroup, not 1, of G which naturally is part of H so that $H \neq 1$. The set of all torsion elements of H is exactly $\mathrm{t}H$, since H is locally nilpotent; see Specht [p. 382, Satz 12]. Since H is characteristic, so is $\mathrm{t}H$. Hence $\mathrm{t}H \subseteq \mathrm{t}G = 1$ so that H is torsionfree.

Thus every abelian subgroup of H is torsionfree and as such of finite rank so that we may apply Baer [6; p. 98, Theorem and p. 96, Lemma] on H. It follows that

(1) H is torsionfree, nilpotent and of finite positive rank n, and H contains every locally nilpotent normal subgroup of G.

Since H is torsionfree and nilpotent, we may use Baer [2; p. 200, Corollary 1] to show that

(1′) $H/\mathfrak{z}_i\,H$ is torsionfree for every i.

By Plotkin [p. 513, Lemma 4] the Hirsch-Plotkin radical of a radical group contains its centralizer. Thus

(2) $\mathfrak{c}_G\,H = \mathfrak{z}H$.

Denote by Γ the group of automorphisms, induced in H by G. Then it is a consequence of (2) that

(3) Γ is essentially the same as $G/\mathfrak{z}H$.

Suppose that the finitely many normal subgroups X_i of G meet the following requirements:

(a) $X_0 = 1, H \circ X_{i+1} \subseteq X_i \subset X_{i+1}, X_s = H$.
(b) Every X_{i+1}/X_i is torsionfree (abelian).

Then we shall term this series $\mathfrak{X} = [\cdots, X_i, \cdots]$ for the purposes of this proof *an admissible chain of* H.

It is a consequence of (1) that $H = \mathfrak{z}_i H$ for almost all i; and we deduce from (1′) that every $\mathfrak{z}_{i+1} H/\mathfrak{z}_i H$ is torsionfree. Since $H = \mathfrak{h}\mathfrak{p}G$ is a characteristic subgroup of G and every $\mathfrak{z}_i H$ is a characteristic subgroup of H, every $\mathfrak{z}_i H$ is a characteristic subgroup of G. It follows that the set of the $\mathfrak{z}_i H$ is an admissible chain of H; and we have shown that

(4) the ascending central chain of H is admissible.

Since H is, by (1), of finite rank n, admissible chains contain at most $n + 1$ terms. Consequently there exists among the admissible chains, containing the ascending central chain of H, one $\mathfrak{M} = [M_i; 0 \leq i \leq m]$ of maximal length. Because of the maximality of the admissible chain $\mathfrak{M}$ we have:

(5) If Y is a normal subgroup of G with $M_i \subset Y \subseteq M_{i+1}$, then M_{i+1}/Y is a torsion group.

It is clear that every M_i is a Γ-admissible subgroup of H. Denote by θ the set of all those automorphisms in Γ which induce the 1-automorphism in every M_{i+1}/M_i. It is clear that θ is a normal subgroup of Γ.

Denote by T the set of all elements in G which induce in H an automorphism belonging to θ. Since $H \circ M_{i+1} \subseteq M_i$ for all i by construction, we find that $H \subseteq T$. On the other hand, since every automorphism in θ induces the 1-automorphism in every M_{i+1}/M_i, we obtain $T \circ M_{i+1} \subseteq M_i$ for all i.—Define inductively T_i by the rules:

$$T_1 = T, \qquad T_{i+1} = T \circ T_i.$$

Then we are going to prove by complete induction that

(+) $T_i \circ M_j \subseteq M_{j-i}$ for every j and $i \leq j$.

This is certainly true for $i = 1$, since

$$T_1 \circ M_j = T \circ M_j \subseteq M_{j-1} \quad \text{for } 1 \leq j.$$

Thus we may assume that (+) has been verified for every j and every $k < i$ with $1 < i \leq j$. Then

$$\begin{aligned} T_i \circ M_j &= (T \circ T_{i-1}) \circ M_j \\ &\subseteq [T \circ (T_{i-1} \circ M_j)][T_{i-1} \circ (T \circ M_j)] \qquad \text{by Zassenhaus [p. 120]} \\ &\subseteq [T \circ M_{j-i+1}][T_{i-1} \circ M_{j-1}] \\ &\subseteq M_{j-i}; \end{aligned}$$

and this completes the inductive proof of $(+)$. It follows in particular that

$$T_m \circ H = T_m \circ M_m \subseteq M_0 = 1.$$

Hence $T_m \subseteq \mathfrak{c}_G H = \mathfrak{z}H \subseteq H = M_m$; and now one verifies by a simple complete induction that

$$T_{m+i} \subseteq M_{m-i} \quad \text{for every } i \leq m.$$

In particular we have

$$T_{2m} \subseteq M_0 = 1,$$

proving that T is nilpotent of class $2m$. Since H is the Hirsch-Plotkin radical of G, it contains all locally nilpotent normal subgroups of G so that $T \subseteq H$ and hence

(6) $T = H$ and $\theta \cong H/\mathfrak{z}H$.

We noted before that $T_m \subseteq \mathfrak{z}H$. This implies by (6)

(6*) $H_m \subseteq \mathfrak{z}H$ and $H_{m+1} = 1$.

Denote by θ_i for $0 \leq i < m$ the set of automorphisms in Γ which induce the 1-automorphism in $M_{i+1}/M_i = A_i$. It is clear that θ_i is a normal subgroup of Γ, and that

(7) $\Gamma_i = \Gamma/\theta_i$ is essentially the same as the group of automorphisms, induced in A_i by Γ; and that

(8) $\theta = \bigcap_{i=0}^{m-1} \theta_i$.

Because of (1) and the definition of an admissible chain A_i is a torsionfree abelian group of finite rank n. Since Γ_i is an epimorphic image of Γ, it is by (3) an epimorphic image of the, by hypothesis, radical group G. Hence Γ_i is a radical group. Thus we may apply the generalized Theorem of Zassenhaus (Proposition 5.4) to show

(9) $\Gamma_i^{(\lambda(n))} = 1$.

This is equivalent to $\Gamma^{(\lambda(n))} \subseteq \theta_i$ for every i by (7); and this implies by (8) that

(10) $\Gamma^{(\lambda(n))} \subseteq \theta$.

Combine this with (6) to see that

(11) $G^{(\lambda(n))} \subseteq H$.

It is a consequence of (6*) that $H_{m+1} = 1$ and this implies $H^{(m)} = 1$. But $m \leq n$ so that

(12) $H^{(n)} = 1$;

and combining (11) and (12) we obtain

(13) G is soluble with $G^{(\lambda(n)+n)} = 1$.

It is a consequence of (9), (1) and the definition of admissible chains that $A_i = M_{i+1}/M_i$ is a torsionfree abelian group of finite rank n and that the group Γ_i of automorphisms, induced in A_i by G, is by (13) soluble. Application of (5) shows furthermore:

If $Y \neq 1$ is a Γ_i-admissible subgroup of A_i, then A_i/Y is a torsion group.

It follows that Γ_i is a primitive group of automorphisms of A_i, according to Baer [3; p. 142, Definition 1]; and a fortiori a semiprimitive group of automorphisms of A_i, according to Baer [3; p. 144, Definition 2]. Thus we may apply Baer [3; p. 164, Hauptsatz 1] on the pair Γ_i, A_i; and it follows that

(14) every torsion subgroup of Γ_i is finite and every maximal abelian normal subgroup of Γ_i has finite index in Γ_i.

It is a consequence of the Maximum Principle of Set Theory that every group possesses a maximal abelian normal subgroup. Application of (14) shows therefore the existence of an abelian normal subgroup Δ of Γ_i with finite Γ_i/Δ. The torsion subgroup $t\Delta$ of the abelian group Δ is finite by (14) and therefore it is a direct factor of Δ; see Fuchs [p. 187, Theorem 50.3].

Consequently there exists a torsionfree subgroup Δ^* of Δ with finite Δ/Δ^*. Since Γ_i/Δ is finite, the index $[\Gamma_i : \Delta^*]$ is likewise finite. It is well known that every subgroup of finite index contains a normal subgroup of finite index. Thus we have shown that

(15) Γ_i contains a torsionfree abelian normal subgroup of finite index.

By combination of (7), (8) and (15) we conclude that Γ/θ contains a torsionfree abelian normal subgroup of finite index. Combination of this result with (6) and (1) shows:

(16) There exists a torsionfree normal subgroup Σ of Γ, containing θ, with finite Γ/Σ and Σ/θ abelian and torsionfree.

Denote by S the set of all elements in G which induce in H an automorphism belonging to Σ. Then $\Sigma \cong S/\mathfrak{z}H$ and we have shown:

(17) There exists a torsionfree normal subgroup S of G, containing H, with finite G/S and torsionfree abelian S/H.

Since S is torsionfree and all torsionfree abelian subgroups of G are by hypothesis of finite rank, S is of finite abelian subgroup rank. Since H is nilpotent, application of Lemma 4.3 shows that S/H is of finite abelian subgroup rank. Since S/H is torsionfree abelian by (17), we conclude that S/H is of finite rank. H is of finite rank by (1); thus

(18) S is of finite rank.

We consider now finite chains of subgroups X_i of H with the following properties:

(A) Every member of $\mathfrak{M}$ is an X_i.

(B) Every X_i is a normal subgroup of S [or equivalent: every X_i is Σ-admissible].

(C) $1 = X_0$, $X_i \subset X_{i+1}$, $X_s = H$ and every X_{i+1}/X_i is torsionfree abelian.

We note that $\mathfrak{M}$ itself is such a chain and that $s \leq n =$ rank of H; note that the rank of H is finite by (1). This implies the existence of such a chain $\tilde{\mathfrak{M}} = [\tilde{M}_i ; 0 \leq i \leq \tilde{m}]$ of maximal length $\tilde{m}$. If Y is a Σ-admissible subgroup of H with $\tilde{M}_i \subset Y \subseteq \tilde{M}_{i+1}$, then we deduce from the commutativity of $\tilde{M}_{i+1}/\tilde{M}_i$ and the maximality of $\tilde{m}$ that $\tilde{M}_{i+1}/Y$ is a torsion group. If Σ_i is the group of automorphisms, induced in $\tilde{M}_{i+1}/\tilde{M}_i$ by Σ, then Σ_i is an epimorphic image of S/H, since $\mathfrak{M}$ is part of $\tilde{\mathfrak{M}}$ and since therefore

$$H \circ \tilde{M}_{i+1} \subseteq \tilde{M}_i .$$

But S/H is abelian by (17) so that Σ_i is abelian. Since $(\tilde{M}_{i+1}/\tilde{M}_i)/Y$ is a torsion group for every Σ-admissible $Y \neq 1$, and since $\tilde{M}_{i+1}/\tilde{M}_i$ is torsionfree abelian of finite rank [by (C) and (18)], we may apply Baer [3; p. 143, Folgerung 2 + p. 141, Hilfssatz 2]. Hence

(D) the ring of endomorphisms of $\tilde{M}_{i+1}/\tilde{M}_i$, spanned by Σ_i, is a subring of a finite algebraic number field [= field of finite degree over the rationals].

The multiplicative group of a finite algebraic number field is the direct product of a finite cyclic group and a free abelian group; see Fuchs [p. 297, Theorem 76.2]. Subgroups of free abelian groups are free abelian; see Fuchs [p. 45, Theorem 12.1]. Hence it follows from (D) that Σ_i is the direct product of a finite cyclic group and a free abelian group. Since Σ_i is an epimorphic image of S, and since S is of finite rank, Σ_i is of finite rank, and we have shown:

(E) The group Σ_i of automorphisms of $\tilde{M}_{i+1}/\tilde{M}_i$, induced by S [or Σ], is the direct product of a finite cyclic group and a free abelian group of finite rank. In particular Σ_i is finitely generated.

If we denote by Λ_i the totality of automorphisms in Σ which induce in $\tilde{M}_{i+1}/\tilde{M}_i$ the 1-automorphism, then Λ_i is a normal subgroup of Σ with $\Sigma/\Lambda_i \cong \Sigma_i$. If we let

$$\Lambda = \bigcap_{i=0}^{m-1} \Lambda_i ,$$

then Λ is likewise a normal subgroup of Σ and Σ/Λ is isomorphic to a subgroup of the direct product of the groups Σ_i. Since the Σ_i are finitely generated abelian groups by (E), it follows that

(F) Σ/Λ is a finitely generated abelian group.

Denote by L the totality of elements in S which induce in H automorphisms from Λ. Since Λ is a normal subgroup of Σ, [and since Σ is the group of automorphisms of H, induced by elements in S], L is a normal subgroup of S. Since $H \circ \tilde{M}_{i+1} \subseteq \tilde{M}_i$ [as we noted before], $H \subseteq L$. Since $L \circ \tilde{M}_{i+1} \subseteq \tilde{M}_i$

for every i by our definition of L, it follows that $H = \tilde{M}_{\tilde{m}}$ is part of the hypercenter of L; and since $L/H \subseteq S/H$ and the latter group is abelian by (17), and since H is nilpotent by (1), it follows that L is a nilpotent normal subgroup of S. Hence

$$L \subseteq \mathfrak{hp}S.$$

As a characteristic subgroup of the normal subgroup S, the Hirsch-Plotkin-radical of S is a normal subgroup of G. Hence

$$\mathfrak{hp}G = H \subseteq L \subseteq \mathfrak{hp}S \subseteq \mathfrak{hp}G$$

so that in particular $H = L$. Hence

$$S/H = S/L \cong \Sigma/\Lambda$$

is by (F) and (17) a finitely generated torsionfree abelian group, proving that

(19) S/H is a free abelian group of finite rank.

Since $H = \mathfrak{hp}G$ and G/S is finite by (17), this completes the proof of (d).

Since S is torsionfree and G/S finite by (17), every torsion subgroup of G is isomorphic to a subgroup of G/S and hence finite of an order dividing $[G:S]$. This proves (b). That G is of finite rank, is contained in (17), (18); and that G is soluble, is contained in (1), (17). This proves (a). The properties (c) and (e) are contained in (1), (1′) and (13).

Since $\mathfrak{a}G$ is a characteristic subgroup of G, so is $\mathfrak{ta}G$. As a characteristic torsion subgroup $\mathfrak{ta}G \subseteq \mathfrak{t}G = 1$. It follows that $\mathfrak{a}G$ is a radical group of finite abelian subgroup rank with $\mathfrak{ta}G = 1$. Thus we may apply (c) and (d). Hence $\mathfrak{hpa}G$ is torsionfree and nilpotent; and $\mathfrak{a}G/\mathfrak{hpa}G$ is an extension of a free abelian group of finite rank by a finite group and as such it is residually finite. But $\mathfrak{a}G$ and hence $\mathfrak{a}G/\mathfrak{hpa}G$ are free of proper subgroups of finite index. Consequently $\mathfrak{a}G/\mathfrak{hpa}G = 1$ so that $\mathfrak{a}G = \mathfrak{hpa}G$ is torsionfree and nilpotent.

It is a consequence of (a) that $G/\mathfrak{a}G$ is soluble of finite rank; and it is an immediate consequence of the definition of $\mathfrak{a}G$ that 1 is the only subgroup of $G/\mathfrak{a}G$ which is free of proper subgroups of finite index. Thus we may apply Robinson [3; p. 496, Theorem A] to show the residual finiteness of $G/\mathfrak{a}G$, completing the proof of (f).

6. *The residuum* of a group X is the intersection $\mathfrak{res}\ X$ of all the subgroups Y of X with finite index $[X:Y]$. It is clear that $\mathfrak{res}\ X$ is a well determined characteristic subgroup of X with residually finite $X/\mathfrak{res}\ X$.

The hyporesiduum of a group X is the compositum $\mathfrak{a}X$ of all the subgroups of X without proper subgroups of finite index. This is a well determined characteristic subgroup of X without any proper subgroup of finite index. It is clear that

$$Y \subseteq X \quad \text{implies} \quad \mathfrak{a}Y \subseteq \mathfrak{a}X.$$

It is likewise clear that

$$\mathfrak{a}X \subseteq \mathfrak{res}\, X.$$

It follows that $\mathfrak{a}X$ is the terminal member of the descending sequence of transfinitely iterated residua of X. One derives either from this fact or deduces directly that

$$\mathfrak{a}[X/\mathfrak{a}X] = 1.$$

Finally it is easily seen that

$$\mathfrak{a}[G/\mathfrak{res}\, G] = 1;$$

and that

$$[K\ \mathfrak{res}\ G]/K \subseteq \mathfrak{res}\ (G/K) \quad \text{for every normal subgroup } K \text{ of } G.$$

THEOREM 6.1. *If G is a radical group, if $\mathfrak{t}G$ is of finite abelian subgroup rank, and if every torsionfree abelian subgroup of G is of finite rank, then G has the following properties:*

(a) (1) *G is countable and of bounded abelian factor rank.*
(2) *Every epimorphic image, not* 1, *of G possesses an abelian characteristic subgroup, not* 1, *which is either finite, elementary and primary or torsionfree of finite rank.*

(b) $\mathfrak{a}G = \mathfrak{res}\, G$ *is nilpotent.*

(c) (1) *$\mathfrak{t}G$ is locally finite-soluble.*
(2) *$\mathfrak{at}G$ is abelian and radicable.*
(3) *$\mathfrak{t}G/\mathfrak{at}G$ is residually finite; its Sylow subgroups are finite and its Sylow subgroups of equal characteristic are conjugate in $\mathfrak{t}G/\mathfrak{at}G$.*

(d) (1) *$G/\mathfrak{t}G$ is soluble of finite rank and its torsion subgroups are finite of bounded order.*
(2) *$\mathfrak{hp}[G/\mathfrak{t}G]$ is nilpotent and every $\mathfrak{hp}[G/\mathfrak{t}G]/\mathfrak{z}_i\, \mathfrak{hp}[G/\mathfrak{t}G]$ is torsionfree.*
(3) *$[G/\mathfrak{t}G]/\mathfrak{hp}[G/\mathfrak{t}G]$ is an extension of a free abelian group of finite rank by a finite group [is noetherian and almost abelian].*

Proof. Since $\mathfrak{t}G$ is radical of finite abelian subgroup rank, application of Proposition 4.5 shows the validity of (c) and of the following facts:

(a′) Every epimorphic image, not 1, of $\mathfrak{t}G$ possesses an elementary abelian, primary, finite characteristic subgroup, not 1; $\mathfrak{t}G$ is countable and of bounded abelian factor rank.

Because of (c) and (a′) and the fact that characteristic subgroups of normal subgroups are normal subgroups, we may apply Lemma 3.6 on G and its characteristic subgroup $\mathfrak{t}G$. It follows that $G/\mathfrak{t}G$ is a radical group with $\mathfrak{t}(G/\mathfrak{t}G) = 1$ whose torsionfree abelian subgroups are of finite rank. Consequently we may apply Proposition 5.5 to $G/\mathfrak{t}G$, proving the validity of (d). Thus $G/\mathfrak{t}G$ is countable as is $\mathfrak{t}G$ by (a′), proving the countability of G. Since

$G/\mathfrak{t}G$ and $\mathfrak{t}G$ are of bounded abelian factor rank, application of (4.2) shows that G itself is of bounded abelian factor rank. Since $G/\mathfrak{t}G$ is soluble of finite rank, the same is true of every epimorphic image of $G/\mathfrak{t}G$. It follows that every epimorphic image, not 1, of $G/\mathfrak{t}G$ possesses an abelian characteristic subgroup, not 1, of finite rank. Combining this with the corresponding fact about $\mathfrak{t}G$—see (a′)—one derives easily the validity of (a). Thus we have shown the validity of (a), (c) and (d). But by (a) every factor of G is a radical group of bounded abelian factor rank; and thus it follows that

(1) every factor of G has properties (a), (c) and (d).

Consider a factor A of G with $A = \mathfrak{a}A$. Then we deduce from (1), (d) that $\mathfrak{hp}[A/\mathfrak{t}A]$ is nilpotent and torsionfree; and that $[A/\mathfrak{t}A]/\mathfrak{hp}[A/\mathfrak{t}A]$ is an extension of a free abelian group of finite rank by a finite group and so in particular residually finite. But every epimorphic image of $A = \mathfrak{a}A$ is free of proper subgroups of finite index so that $[A/\mathfrak{t}A]/\mathfrak{hp}[A/\mathfrak{t}A] = 1$. Hence $A/\mathfrak{t}A = \mathfrak{hp}[A/\mathfrak{t}A]$; and we have shown:

$A/\mathfrak{t}A$ is torsionfree and nilpotent.

Assume by way of contradiction that $\mathfrak{t}A \not\subseteq \mathfrak{z}A$. Then $\mathfrak{t}A \cap \mathfrak{z}A \subset \mathfrak{t}A$. These groups are characteristic subgroups of A; and application of (1), (a) to the factor $\mathfrak{t}A$ of A and G shows the existence of a characteristic subgroup C of A with

$$\mathfrak{t}A \cap \mathfrak{z}A \subset C \subseteq \mathfrak{t}A \text{ and finite abelian } C/[\mathfrak{t}A \cap \mathfrak{z}A].$$

Clearly A induces a finite group of automorphisms in $C/[\mathfrak{t}A \cap \mathfrak{z}A]$. But $A = \mathfrak{a}G$ does not possess proper subgroups of finite index. Hence A induces the 1-automorphism in $C/[\mathfrak{t}A \cap \mathfrak{z}A]$. Naturally A induces the 1-automorphism in $\mathfrak{t}A \cap \mathfrak{z}A$. It follows that the group Σ of automorphisms, induced by A in C, stabilizes $\mathfrak{t}A \cap \mathfrak{z}A$. It follows consequently from Schlette [p. 406, Proposition 2.1] that Σ is isomorphic to a subgroup of

$$\mathrm{Hom}\,(C/[\mathfrak{t}A \cap \mathfrak{z}A], \mathfrak{t}A \cap \mathfrak{z}A).$$

We recall that $C/(\mathfrak{t}A \cap \mathfrak{z}A)$ is of finite order and consequently of finite exponent e. Since $\mathfrak{t}A \cap \mathfrak{z}A$ is an abelian torsion group of finite abelian subgroup rank, there exists only a finite number of solutions of the equation $x^e = 1$ in $\mathfrak{t}A \cap \mathfrak{z}A$. But there exists only a finite number of homomorphisms of a finite group into a finite group so that in particular Σ is finite. Recall that Σ is an epimorphic image of $A = \mathfrak{a}G$ and that the latter group is free of proper subgroups of finite index. Hence $\Sigma = 1$ and A induces in C only the 1-automorphism. Consequently $C \subseteq \mathfrak{z}A$ so that

$$\mathfrak{t}A \cap \mathfrak{z}A \subset C \subseteq \mathfrak{t}A \cap \mathfrak{z}A,$$

a contradiction proving that $\mathfrak{t}A \subseteq \mathfrak{z}A$. But we have shown already that

$A/\mathrm{t}A$ is torsionfree and nilpotent. Hence A itself is nilpotent; and we have shown:

(2) If A is a factor of G with $A = \mathfrak{a}A$, then A is nilpotent and $A/\mathrm{t}A$ is torsionfree.

Consider a factor F of G of finite exponent e [so that $F^e = 1$]. Application of (1), (c) shows that $\mathfrak{a}F = 1$ and that F is locally finite-soluble with finite Sylow subgroups. Since only the divisors of e can be orders of elements in F, we may apply Baer [9; p. 112, Lemma 1.2] to show the finiteness of F. This we note:

(3) A factor F of G is finite if [and only if] F is of finite exponent.

Consider next a factor F of G with $\mathfrak{a}F = 1$. Suppose that S is a subgroup of $\mathrm{t}F$ with finite index $[\mathrm{t}F:S]$. Then there exists a normal subgroup T of $\mathrm{t}F$ with $T \subseteq S$ and finite $\mathrm{t}F/T$. There exists a positive integer e with

$$[\mathrm{t}F/T]^e = 1;$$

and this is equivalent to $E = \{(\mathrm{t}F)^e\} \subseteq T$. Clearly E is a characteristic subgroup of $\mathrm{t}F$ and F; and $\mathrm{t}F/E$ is a factor of G whose exponent is finite. It follows from (3) that $\mathrm{t}F/E$ is finite. On the factor F/E of G we may apply (1). We note first that

$$\mathrm{t}(F/E) = \mathrm{t}F/E \text{ is finite and soluble}$$

by (1), (c); and it is a consequence of (1), (d) that $F/\mathrm{t}F$ is soluble of finite rank and is an extension of a torsionfree nilpotent group whose central factors are torsionfree too by a noetherian almost abelian group. Since $\mathrm{t}F/E$ is finite and soluble, there exists a finite series of normal subgroups of F/E whose factors are either finite abelian or torsionfree abelian. Thus we may apply Robinson [3; p. 501, Lemma 2.31] on the finite normal subgroup $\mathrm{t}F/E$ of F/E. Consequently there exists a subgroup L of F with finite index $[F:L]$ and $E = L \cap \mathrm{t}F$. It follows that

$$\mathfrak{res}\, F \cap \mathrm{t}F \subseteq L \cap \mathrm{t}F = E \subseteq T \subseteq S$$

for every subgroup S of $\mathrm{t}F$ with finite index $[\mathrm{t}F:S]$. Hence

$$\mathfrak{res}\, F \cap \mathrm{t}F \subseteq \mathfrak{res}\,(\mathrm{t}F).$$

Next we note that

$$\mathfrak{a}\mathrm{t}F \subseteq \mathfrak{a}F = 1.$$

Hence $\mathfrak{a}\mathrm{t}F = 1$; and we deduce from (1), (c) that $\mathrm{t}F$ is residually finite. Consequently $\mathfrak{res}\,(\mathrm{t}F) = 1$; and we have shown:

(4) $\mathfrak{res}\, F \cap \mathrm{t}F = 1$ for every factor F of G with $\mathfrak{a}F = 1$.

Consider next a factor F of G with finite $\mathfrak{res}\, F$. Since clearly $\mathfrak{a}F \subseteq \mathfrak{res}\, F$, we deduce $\mathfrak{a}F = 1$ from the finiteness of $\mathfrak{res}\ F$. Combine this with (4) to see that

$$\mathfrak{res}\ F = \mathfrak{res}\ F \cap \mathfrak{t}F = 1;$$

and thus we have shown:

(5) If F is a factor with finite $\mathfrak{res}\ F$, then $\mathfrak{res}\ F = 1$ and F is residually finite.

Consider again a factor F of G with $\mathfrak{a}F = 1$. Application of (1), (d) shows that

$\mathfrak{hp}[F/\mathfrak{t}F]$ is nilpotent and torsionfree and
$[F/\mathfrak{t}F]/\mathfrak{hp}[F/\mathfrak{t}F]$ is noetherian and almost abelian.

Denote by H the uniquely determined characteristic subgroup of F with $\mathfrak{t}F \subseteq H$ and $H/\mathfrak{t}F = \mathfrak{hp}[F/\mathfrak{t}F]$. Then

$$F/H \cong [F/\mathfrak{t}F]/\mathfrak{hp}[F/\mathfrak{t}F]$$

is noetherian and almost abelian and consequently residually finite so that

$$\mathfrak{res}\ F \subseteq H.$$

Application of (4) shows now that

$$\mathfrak{res}\ F \cong \mathfrak{t}F\ \mathfrak{res}\ F/\mathfrak{t}F \subseteq H/\mathfrak{t}F = \mathfrak{hp}[F/\mathfrak{t}F]$$

is nilpotent and torsionfree. From

$$\mathfrak{a}\ \mathfrak{res}\ F \subseteq \mathfrak{a}F = 1$$

we deduce that

$$\mathfrak{a}\ \mathfrak{res}\ F = 1.$$

Thus $\mathfrak{res}\ F$ is torsionfree, nilpotent with $\mathfrak{a}\ \mathfrak{res}\ F = 1$; and because of (1), (a) we may apply Proposition 5.5, (f) to the factor $\mathfrak{res}\ F$ of G, proving that

$$\mathfrak{res}\ F \text{ is residually finite.}$$

Consider a subgroup S of $\mathfrak{res}\ F$ with finite index $[\mathfrak{res}\ F{:}S]$. Then there exists a normal subgroup T of $\mathfrak{res}\ F$ with $T \subseteq S$ and finite $[\mathfrak{res}\ F]/T$. If e is the exponent of this finite group and $E = \{(\mathfrak{res}\ F)^e\}$, then E is a characteristic subgroup of $\mathfrak{res}\ F$ and F. Furthermore $E \subseteq T$ and $[\mathfrak{res}\ F]/E$ is of finite exponent. But factors of finite exponent are finite by (3). Hence

$$\mathfrak{res}\ (F/E) = [\mathfrak{res}\ F]/E \text{ is finite;}$$

and it follows from (5) that

$$[\mathfrak{res}\ F]/E = 1.$$

Hence

$$\mathfrak{res}\ F = E \subseteq T \subseteq S \subseteq \mathfrak{res}\ F,$$

proving $S = \mathfrak{res}\ F$. Thus we have shown that $\mathfrak{res}\ F$ is a residually finite group without proper subgroups of finite index. Consequently $\mathfrak{res}\ F = 1$;

and we have shown:

(6) If F is a factor with $\mathfrak{a}F = 1$, then $\mathfrak{res}\, F = 1$ and F is residually finite.

If $F = G/\mathfrak{a}G$, then $\mathfrak{a}F = 1$. It follows from (6) that $\mathfrak{res}\, F = 1$ and that F is residually finite. Hence

$$\mathfrak{res}\, G \subseteq \mathfrak{a}G \subseteq \mathfrak{res}\, G,$$

[the second inequality being obvious]. Hence

$$\mathfrak{res}\, G = \mathfrak{a}G \text{ is nilpotent}$$

by (2), completing the proof of (b) and of Theorem 6.1.

Discussion of Theorem 6.1. (A) Denote by $\mathfrak{R}$ the class of all radical groups of finite abelian subgroup rank. It is well known and easily verified that factors of radical groups are radical and that extensions of radical groups by radical groups are radical; see Plotkin. Combine this with Theorem 6.1, (a) and (5.1) to show that

factors of $\mathfrak{R}$-groups are $\mathfrak{R}$-groups and
extensions of $\mathfrak{R}$-groups by $\mathfrak{R}$-groups are $\mathfrak{R}$-groups.

(B) In the Discussion of Theorem 6.3, (A) we shall construct an example of an $\mathfrak{R}$-group which is not soluble. The class of $\mathfrak{S}_0$-groups, introduced by Robinson [1; p. 148], is exactly the class of soluble groups of bounded abelian quotient rank: it follows that $\mathfrak{S}_0$ is a proper subclass of $\mathfrak{R}$.

(C) Our proof of property (b) of Theorem 6.1 depended heavily on Robinson [3; p. 496, Theorem A and p. 501, Lemma 2.31]. But our property (b) is not a special case of Robinson's results, since—as mentioned ad (B)—the class $\mathfrak{S}_0$ is a proper subclass of $\mathfrak{R}$.

(D) It is a consequence of Theorem 6.1, (b) that the residuum of an $\mathfrak{R}$-group G is nilpotent and free of proper subgroups of finite index. If $0 < i$ and

$$x \in \mathfrak{z}_{i+1}\mathfrak{a}G, \quad x^p \in \mathfrak{z}_i \mathfrak{a}G \text{ for some prime } p,$$

then $x\mathfrak{z}_{i-1}\mathfrak{a}G$ is centralized by $(\mathfrak{a}G/\mathfrak{z}_{i-1}\mathfrak{a}G)^p = \mathfrak{a}G/\mathfrak{z}_{i-1}\mathfrak{a}G$ so that $x \in \mathfrak{z}_i\mathfrak{a}G$. Consequently

$\mathfrak{z}_i\mathfrak{a}G/\mathfrak{z}_{i-1}\mathfrak{a}\, G$ is radicable for $0 < i$;
$\mathfrak{z}_{i+1}\mathfrak{a}G/\mathfrak{z}_i\mathfrak{a}G$ is torsionfree and radicable for $0 < i$.

Consequently

$$\mathfrak{t}\mathfrak{a}G = \mathfrak{t}\mathfrak{z}\mathfrak{a}G = \mathfrak{a}\mathfrak{t}\mathfrak{z}\mathfrak{a}G \subseteq \mathfrak{a}\mathfrak{t}G \subseteq \mathfrak{a}G \cap \mathfrak{t}G \subseteq \mathfrak{t}\mathfrak{a}G;$$

and this implies

$$\mathfrak{t}\mathfrak{a}G = \mathfrak{a}\mathfrak{t}G = \mathfrak{t}G \cap \mathfrak{a}G \subseteq \mathfrak{z}\mathfrak{a}G;$$

see also Kurosh [p. 234–239].

LEMMA 6.2. *Assume that the torsion group G has the following properties*:

(A) *Primary subgroups of G are artinian.*

(B) *If p is a prime and S a subgroup of G, then any two p-Sylow subgroups of S are conjugate in S.*

Then G has the following properties:

(1) *If X is a normal subgroup of the subgroup Y of G, and if S is a p-Sylow subgroup of X, then $Y = X\mathfrak{n}_Y S$.*

(2) *If X is a normal subgroup of the subgroup Y of G, if Y/X is an abelian p-group, and if S is a p-Sylow subgroup of Y, then $Y = XS$.*

(3) *If S is a subgroup of G, then there exists a direct product D of primary groups with $S = DS'$.*

Proof. If X is a normal subgroup of the subgroup Y of G, if S is a p-Sylow subgroup of X, and if y is an element in Y, then y induces an automorphism in X so that S^y too is a p-Sylow subgroup of X. Apply (B) to show the existence of an element x in X with $S^x = S^y$. Hence $S = S^{yx^{-1}}$ so that yx^{-1} belongs to $\mathfrak{n}_Y S$ and $y = (yx^{-1})x$ belongs to $\mathfrak{n}_Y SX = X\mathfrak{n}_Y S$, proving (1).

Suppose next that X is a normal subgroup of the subgroup Y of G, that Y/X is an abelian p-group and that S is a p-Sylow subgroup of Y. If C is an element in Y/X, then C is a p-element in Y/X. Since furthermore G and hence Y is a torsion group, there exists a p-element c with $C = Xc$. Naturally c is contained in a p-Sylow subgroup T of Y. It is a consequence of (B) that there exists an element y in Y with $T^y = S$. From the commutativity of Y/X we deduce

$$C = C^y = Xc^y.$$

Hence c^y belongs to $C \cap S$ so that every coset of Y modulo X contains an element in S, proving $Y = XS$.

Consider a subgroup S of G and some enumeration

$$p_1, p_2, \cdots, p_i, p_{i+1}, \cdots$$

of the prime numbers. We are going to define by complete induction a descending chain of subgroups $S(i)$ of S as follows:

Let $S = S(0)$. If $S(i)$ is already defined, then choose a p_{i+1}-Sylow subgroup P_{i+1} of $S(i)$ and let

$$S(i + 1) = \mathfrak{n}_{S(i)} P_{i+1}.$$

It is clear that

$$\cdots \subseteq S(i + 1) \subseteq S(i) \subseteq \cdots \subseteq S(0) = S.$$

Clearly

$$S = S(0) = S(0)S'.$$

Assume that we have shown already the validity of $S = S(i)S'$. Clearly $S(i)/[S(i) \cap S']$ is abelian so that $[S(i) \cap S']P_{i+1}$ is a normal subgroup of $S(i)$. Since P_{i+1} is a p_{i+1}-Sylow subgroup of $S(i)$, it is a fortiori a p_{i+1}-Sylow subgroup of $[S(i) \cap S']P_{i+1}$. Application of (1) shows that

$$S(i) = [S(i) \cap S']P_{i+1}\mathfrak{n}_{S(i)} P_{i+1}.$$

Consequently

$$S = S(i)S' = S'[S(i) \cap S']P_{i+1}\mathfrak{n}_{S(i)} P_{i+1} = S'\mathfrak{n}_{S(i)} P_{i+1} = S'S(i+1),$$

completing the inductive proof of

(a) $S = S'S(i)$.

Every subgroup is a normal subgroup of its normalizer. Hence P_i is a normal subgroup of $S(i)$ so that $S(j) \cap P_i$ is a normal subgroup of $S(j) \subseteq S(i)$ whenever $i \leq j$. As the $S(j)$ form a descending sequence of subgroups, so do the $S(j) \cap P_i$ [for fixed i and $i \leq j$]. Since P_i is artinian by (A), this descending chain of subgroups of P_i terminates after a finite number of steps. If T_i is this terminal member, then

(b) $T_i = S(j) \cap P_i$ for almost all j;

and since almost all j exceed i, if follows that

(c) T_i is a normal subgroup of $S(j)$ for almost all j.

If $0 < i$, then P_i is a p_i-Sylow subgroup of $S(i-1)$. From

$$P_i \subseteq S(i) \subseteq S(i-1)$$

it follows that P_i is likewise a p_i-Sylow subgroup of $S(i)$. Since P_i is a normal subgroup of $S(i)$, it follows from (B) that P_i is the totality of p_i-elements of $S(i)$; and this implies that $S(j) \cap P_i$ is for $i \leq j$ the totality of p_i-elements of $S(j)$ and hence the one and only one p_i-Sylow subgroup of $S(j)$. Combine this with (b) and (c) to see that

(d) T_i is the totality of p_i-elements of $S(j)$ for almost all j and hence the one and only p_i-Sylow subgroup of $S(j)$ for almost all j.

S/S' is an abelian torsion group and hence the direct product of its primary components. The p_i-component of S/S' has the form C_i/S'. Select j large enough so that T_i is the totality of p_i-elements in $S(j)$; this is possible by (d). Since

$$S' \subseteq C_i \subseteq S = S'S(j)$$

by (a), we deduce

$$C_i = S'[S(j) \cap C_i].$$

Since every p_i-element in S belongs to C_i, it follows that

$$T_i \subseteq S(j) \cap C_i.$$

Thus T_i is the totality of p_i-elements in $S(j) \cap C_i$, its one and only p_i-Sylow subgroup. From $S = S'S(j)$ we deduce that every p_i-element in S/S' is represented by an element in $S(j)$ and hence by a p_i-element in $S(j)$; and this implies that every p_i-element in S/S' is represented by an element in T_i. Thus $C_i = S'T_i$; and we have shown that

(e) $S'T_i/S'$ is the totality of p_i-elements in S/S'.

If h and k are positive integers, then we deduce from (d) the existence of a positive integer n such that

T_h is the totality of p_h-elements in $S(n)$ and
T_k is the totality of p_k-elements in $S(n)$.

Thus T_h and T_k are characteristic subgroups of $S(n)$; and $h \neq k$ implies

$$T_h \circ T_k \subseteq T_h \cap T_k = 1$$

so that T_h and T_k commute elementwise. It follows that the compositum D of the T_i is their direct product. In particular D is the direct product of primary subgroups. From (e) we deduce that $S'D/S'$ contains all p_i-elements in S/S' for every i. But the abelian torsion group S/S' is the direct product of its primary components, proving $S/S' = S'D/S'$ and $S = S'D$.

Note that hypothesis (A) is used only in the proof of (3) ; and that (2) remains valid, if we substitute "soluble" for "abelian".

THEOREM 6.3. *The following properties of the group G are equivalent*:

(i) (a) *G is of finite rank.*

(b) $\mathfrak{hpt}G$ *is a hypercentral torsion group with artinian, soluble, almost abelian primary components.*

(c) *$G/\mathfrak{hpt}G$ is soluble.*

(ii) *G is a radical group of uniformly bounded abelian subgroup rank.*

(iii) *G is a radical group; and the abelian subgroups of* tG *as well as the torsionfree abelian subgroups of G are of finite rank.*

Proof. Extensions of hypercentral groups by soluble groups are certainly radical; and since G is of finite rank n, every abelian subgroup of G is of finite rank n. Hence (ii) is a consequence of (i).

If G is a radical group of uniformly bounded abelian subgroup rank, then there exists a positive integer r such that every primary abelian subgroup and every torsionfree abelian subgroup of G has rank r. This implies firstly that every abelian torsion subgroup [as a direct product of primary subgroups] has rank r. If secondly A is a torsionfree abelian subgroup of G, then A contains a free abelian subgroup F with A/F a torsion group; and this implies that A and F have the same finite rank r. Thus we have shown that (iii) is a consequence of (ii).

Assume finally the validity of (iii). Then G is in particular a radical group such that tG is of finite abelian subgroup rank and such that every torsionfree abelian subgroup of G is of finite rank. Hence we may apply Theorem 5.6 to prove that

(1) G is of bounded abelian factor rank;
(2) G/tG is soluble of finite rank.

Every subgroup S of tG is radical of finite abelian subgroup rank. Thus we may apply Proposition 4.5 to show:

(3) Every subgroup S of tG has the following properties:
(a) The p-Sylow subgroups of S are conjugate in S.
(b) S is locally finite and locally soluble.
(c) Every epimorphic image, not 1, of S possesses a finite, primary, elementary abelian characteristic subgroup, not 1.

Consider a locally hypercentral subgroup H of tG. Since tG is of finite abelian subgroup rank, so is H. Application of Baer [6; p. 98, Theorem] shows the hypercentrality of H. Apply Specht [p. 380, Satz 11] to show that H is the direct product of its primary components H_p. Every H_p possesses a maximal abelian normal subgroup A_p [Maximum Principle of Set Theory]. The compositum A of the A_p is their direct product; and it is easily seen that A is a maximal abelian normal subgroup of H. By hypothesis A is of finite rank n. From the hypercentrality of H and hence H_p and from the maximality of A_p one deduces readily that $A_p = \mathfrak{c}_{H_p} A_p$. Thus we may apply (2.4) to show that [H_p/A_p is finite and] H_p is of rank $\frac{1}{2}n(5n + 1)$, since A and A_p are of rank n. Since H is the direct product of its primary components H_p, and since direct products of cyclic groups of relatively prime order are of rank 1, the rank of H is likewise $\frac{1}{2}n(5n + 1)$. Thus we have shown:

(4) Locally hypercentral subgroups of tG are hypercentral of finite rank.

If P is a primary subgroup of tG, then P is by (3.b) locally nilpotent; and P is hypercentral of finite rank by (4). There exists a maximal abelian normal subgroup A of P [maximum Principle of Set Theory]. Since P is hypercentral, $A = \mathfrak{c}_P A$. Since P and A are of finite rank, we may apply (2.4) to show the finiteness of P/A. Since A is a primary abelian group of finite rank, A is artinian; see Fuchs [p. 65, Theorem 19.2]. It follows that

(5) primary subgroups of tG are hypercentral, artinian and almost abelian.

Let $T = tG$ and $H = \mathfrak{h}\mathfrak{p}tG$. Since the Hirsch-Plotkin-radical is locally hypercentral, we may apply (4) to show that

(4*) H is hypercentral of finite rank m.

Denote by λ the Zassenhaus function, appearing in Proposition 5.4. Since H and $T^{(\lambda(m))}$ are characteristic subgroups of T, so is $HT^{(\lambda(m))}$. Consider an epimorphism σ of T onto S with $H^\sigma \neq 1$. Since H is hypercentral by (4*), $\mathfrak{z}H^\sigma \neq 1$. Thus $\mathfrak{z}H^\sigma$ contains elements of prime number order, say p. Denote by P the totality of all elements x in $\mathfrak{z}H^\sigma$ with $x^p = 1$. This is a characteristic elementary abelian p-subgroup of $\mathfrak{z}H^\sigma$. Since H is by (4*) of finite rank m, so is P. Hence P is a characteristic subgroup, not 1, of H^σ of finite order dividing p^m. As a characteristic subgroup of the normal subgroup H^σ of S the subgroup P is a normal subgroup of S. From $P \subseteq \mathfrak{z}H^\sigma$ we deduce $H^\sigma \subseteq \mathfrak{c}_S P$. Application of Proposition 5.4 shows that the group of automorphisms, induced in P by S, is an at most $\lambda(m)$-step soluble group. Hence $S^{(\lambda(m))} \subseteq \mathfrak{c}_S P$; and we have shown that

$$[HT^{(\lambda(m))}]^\sigma = H^\sigma S^{(\lambda(m))} \subseteq \mathfrak{c}_S P.$$

Recall that $1 \subset P \subseteq H^\sigma$. It follows therefore that

(6) H is part of the hypercenter of $HT^{(\lambda(m))}$.

Assume by way of contradiction that $H \subset HT^{(\lambda(m))}$. Then we deduce from (3.c) the existence of an elementary abelian, finite, primary, characteristic subgroup, not 1, of $HT^{(\lambda(m))}/H$. Consequently there exists a normal subgroup N of T with

$$N' \subseteq H \subset N \subseteq HT^{(\lambda(m))}.$$

From the commutativity of N/H and (6) it follows that N is a hypercentral normal subgroup of T, implying the contradiction

$$\mathfrak{hp}T = H \subset N \subseteq \mathfrak{hp}T.$$

Thus we have shown that $H = HT^{(\lambda(m))}$; and this is equivalent to

(7) $(\mathfrak{t}G)^{(\lambda(m))} \subseteq \mathfrak{hpt}G$.

Consider a normal subgroup X of a subgroup Y of $\mathfrak{t}G$ with abelian Y/X. Because of (3.a) and (5) we may apply Lemma 6.2, (3) to the subgroup Y of $\mathfrak{t}G$. Consequently there exists a direct product D of primary groups with $Y = DY'$. It is a consequence of (5) that primary subgroups of $\mathfrak{t}G$ are hypercentral. Hence D is hypercentral; and it follows from (4) that D is of finite rank. The epimorphic image Y/X of $Y/Y' = DY'/Y' \cong D/(D \cap Y')$ is therefore an epimorphic image of D and hence of finite rank. Thus we have shown that

(8) every abelian factor of $\mathfrak{t}G$ is of finite rank.

It is a consequence of (4*) that $\mathfrak{hpt}G = H$ is hypercentral of finite rank. It is a consequence of (7) that $\mathfrak{t}G/\mathfrak{hpt}G$ is soluble. Hence it follows from (8) that $\mathfrak{t}G/\mathfrak{hpt}G$ is soluble of finite rank. Combine this with (2) to show that $G/\mathfrak{hpt}G$ is soluble of finite rank. Combination with (4*) shows that G is of finite rank; and combination of (4*) and (5) shows that $\mathfrak{hpt}G$ is a hypercentral group whose primary components are artinian, soluble and almost abelian. Thus we have derived (i) from (iii) and shown the equivalence of (i)–(iii).

Discussion of Theorem 6.3. (A) For every odd prime p there exists a finite

p-group P_p with the following properties:

$P_p^{(p)} \neq 1$ and the abelian subgroups of P_p have rank 3 ;

see Baer [8; p. 27, Bemerkung 4.5]. Denote by G the [restricted] direct product of these groups P_p. This group is a hypercentral torsion group with $G = \mathfrak{z}_\omega G$ and all abelian subgroups of rank 3. But G is not soluble.

Thus the class of radical groups whose abelian subgroups are of finite rank contains non-soluble groups; and it is impossible to prove solubility of the groups in the classes, discussed in Theorem 6.3 and a fortiori in Theorem 6.1.

(B) The Theorem of Kargapolov [2] that a soluble group is of finite rank if, and only if, its abelian subgroups are of finite rank is clearly a special case of our Theorem 6.3 and ad (A) we have shown that it is a proper special case.

(C) Robinson [2 p. 244] has shown that locally soluble groups of finite rank belong to the class of groups, discussed in Theorem 6.3.

7. It is customary to term the prime p *relevant for the group* X, if X contains elements of order p; and the set of all primes relevant for X has often been termed *the characteristic* $\mathfrak{ch}X$ *of* X. We shall refine this concept.

If the prime p is the order of an element in $\mathrm{t}X$, then we say that p *is essential for* X. The set of all primes essential for X shall be termed *the essence* $\mathfrak{e}X$ *of* X.

Clearly essential primes are relevant; but it is easy to construct groups with empty essence, possessing an infinity of relevant primes. Furthermore there exist groups with empty essence, possessing subgroups and epimorphic images with infinite essence.

THEOREM 7.1. *The following properties of the group G are equivalent*:

(i) *G is soluble of finite rank and of finite characteristic.*

(ii) *G is radical of finite abelian subgroup rank and the essence of G is finite.*

(iii) (a) *G is radical.*

(b) *Every elementary abelian subgroup of $\mathrm{t}G$ is finite.*

(c) *Every torsionfree abelian subgroup of G is of finite rank.*

(iv) (a) *$\mathrm{t}G$ is an extension of an artinian, abelian group by a finite soluble group.*

(b) *$G/\mathrm{t}G$ is soluble of finite rank, possesses a torsionfree subgroup of finite index whose commutator subgroup is nilpotent; and the torsion subgroups of $G/\mathrm{t}G$ are finite of bounded order.*

(v) (a) *G is radical.*

(b) *If A is an abelian subgroup of G, then $\mathrm{t}A$ is artinian and $A/\mathrm{t}A$ is of finite rank.*

Proof. It is clear that (i) implies (ii). —If (ii) is satisfied by G, then G is radical and every torsionfree abelian subgroup of G is of finite rank. Every elementary abelian subgroup of $\mathrm{t}G$ is the direct product of only a finite number of elementary abelian primary subgroups, since the essence of G is finite, so that the characteristic of $\mathrm{t}G$ and its subgroups is finite too. But primary elementary abelian subgroups of G are finite, since G is of finite abelian subgroup rank. Hence (iii) is a consequence of (ii).

Assume the validity of (iii). Consider an abelian subgroup A of tG. Its elements of squarefree order form an elementary abelian subgroup of tG which is finite by (iii.b). It follows that A is a torsion group with only a finite number of elements of squarefree order: A is artinian by Fuchs [p. 68, (19)]. Since tG is radical by (iii.a) and since the abelian subgroups of tG are artinian, tG is artinian and soluble by Baer [7; p. 359/360, Hauptsatz 8.15, **A**]. It is well known that artinian soluble groups possess abelian characteristic subgroups of finite index; see Baer [5 ; p. 7/8, Satz 2.1]. Thus (iv.a) is satisfied by G. It is a consequence of (iii) that the hypotheses of Theorem 6.1 are satisfied by G. Thus (iv.b) is a consequence of Theorem 6.1, (e)–(g).

Assume the validity of (iv). Artinian abelian groups are of finite rank by Fuchs [p. 66, Theorem 19.2]. It follows from (iv.a) that tG is soluble and of finite rank. Since G/tG is likewise soluble and of finite rank, we conclude that G is soluble and of finite rank. Artinian abelian groups are of finite characteristic by Fuchs [p. 66, Theorem 19.2]. It follows from (iv.a) that tG is of finite characteristic. By (iv.b) there exists a positive integer m such that every torsion subgroup of G/tG is finite of order a divisor of m. Denote by π the set of primes which belong either to the characteristic of tG or else are divisors of m. If T is a torsion subgroup of G, then TtG is likewise a torsion subgroup of G and TtG/tG is a torsion subgroup of G/tG, hence finite of order a divisor of m. It follows that the characteristic of T is contained in π. Thus the characteristic of G is part of the finite set π of primes; and we have derived (i) from (iv), proving the equivalence of (i)–(iv).

The equivalence of conditions (v) and (iii) is a fairly immediate consequence of Fuchs [p. 65, Theorem 19.2].

Remarks on Theorem 7.1 (A) Mal'cev has termed the group G of class A_3 if it possesses a finite normal chain with the following properties:

$$1 = K_0, \quad K_i \subseteq K_{i+1}, \quad K_s = G,$$

K_{i+1}/K_i is abelian and an extension of an artinian group by a (torsionfree) group of finite rank.

Čarin [1; Theorem 4] has shown that a soluble group is of class A_3 if, and only if, it meets our requirement (v.b.). Thus the class of groups discussed in Theorem 6.1 is identical with the Mal'cev class A_3 (which coincides with Robinson's class $\mathfrak{S}_1$; see Robinson [1; p. 159]), and Čarin's Theorem is contained in ours.

(B) It is quite obvious that condition (i) is subgroup inherited. On the other hand: the additive group of rationals belongs to our class whereas the rationals mod 1 do not. This shows that our class is not epimorphism inherited and is a proper subclass of the class of soluble groups of finite rank.

(C) Suppose that G is a radical group of finite abelian subgroup rank. If firstly G is a primary group, then condition (ii) of Theorem 6.1 is satisfied by G. It follows that G is artinian and almost abelian; and one verifies easily that

G is hypercentral. If secondly $tG = 1$, then again condition (ii) of Theorem 6.1 is satisfied by G. It follows that G is soluble of finite rank, that its torsion subgroups are finite of bounded order, and that there exists a subgroup of finite index whose commutator subgroup is nilpotent.

8. The group G shall be termed *a generalized radical group*, if it meets the following requirement:

Every epimorphic image, not 1, *of* G *possesses a normal subgroup, not* 1, *which is radical or finite.*

If X is a radical group, not 1, then the Hirsch-Plotkin radical of X is a locally hypercentral [and hence locally nilpotent] characteristic subgroup of X which is different from 1. Since characteristic subgroups of normal subgroups are normal subgroups, it follows that the group G is a generalized radical group if, and only if, every epimorphic image, not 1, of G possesses a normal subgroup, not 1, which is finite or locally nilpotent.

THEOREM 8.1. *The following properties of the group* G *are equivalent*:

(i) G *is a generalized radical group of finite abelian subgroup rank.*

(ii) G *is of bounded abelian factor rank and possesses a radical characteristic subgroup of finite index.*

(iii) (a) G *is of bounded abelian factor rank.*

(b) *Every infinite epimorphic image of* G *possesses a non-trivial, abelian, characteristic subgroup of finite rank.*

Proof. We assume first that G is a generalized radical group of finite abelian subgroup rank. Since every subgroup of G is likewise of finite abelian subgroup rank, we deduce from Theorem 6.1 (a) that

(1) every radical subgroup S of G is of bounded abelian factor rank.

Denote by R *the product of all radical normal subgroups of* G. This is a well determined characteristic subgroup of G; and it is well known and easily verified—see e.g. Plotkin [p. 14–16]—that

(2) R is a radical characteristic subgroup of G and 1 is the only radical normal subgroup of G/R.

We note that extensions of radical groups by radical groups are radical. If A/R is an abelian subgroup of G/R, then this implies together with (2) that A is a radical group. Application of (1) shows now that A is of bounded abelian factor rank. This shows that

(3) G/R is of finite abelian subgroup rank.

Let $H = G/R$. If N is a normal subgroup of H, then its center $\mathfrak{z}N$ is as characteristic subgroup of a normal subgroup an abelian normal subgroup of H; and we deduce $\mathfrak{z}N = 1$ from (2). Thus we have shown that

(4) $1 = \mathfrak{z}N = N \cap \mathfrak{c}_H N$ for every normal subgroup N of $H = G/R$.

Assume by way of contradiction *that H is infinite*. If F is a finite normal subgroup of H, then H/F is infinite. Since $H/\mathfrak{c}_H F$ is essentially the same as the group of automorphisms, induced in F by H, it follows that $H/\mathfrak{c}_H F$ is finite; and we deduce from (4) that $1 = F \cap \mathfrak{c}_H F$. If X is a subgroup of H with $1 = X \cap \mathfrak{c}_H F$, then

$$X \cong X\mathfrak{c}_H F/\mathfrak{c}_H F \subseteq H/\mathfrak{c}_H F$$

so that X is finite of order a divisor of the finite order n of $H/\mathfrak{c}_H F$. Among the normal subgroup X of H with $F \subseteq X$ and $1 = X \cap \mathfrak{c}_H F$ there exists consequently a maximal one V. It is clear that V is finite and that H/V is consequently infinite. Since H/V is an infinite epimorphic image of the generalized radical group G, there exists a normal subgroup W of H with $V \subset W$ and radical or finite W/V. Because of the maximality of V we have $1 \neq W \cap \mathfrak{c}_H F$. From $1 = V \cap \mathfrak{c}_H F$ we deduce now that

$$1 \neq W \cap \mathfrak{c}_H F = (W \cap \mathfrak{c}_H F)/(V \cap \mathfrak{c}_H F) \cong V(W \cap \mathfrak{c}_H F)/V \subseteq W/V$$

so that $W \cap \mathfrak{c}_H F$ is a normal subgroup, not 1, of H which is either radical or finite. But the first of these alternatives is ruled out by (2), proving the finiteness of $W \cap \mathfrak{c}_H F$. Furthermore

$$F \cap (W \cap \mathfrak{c}_H F) = 1$$

by (4). Thus we have shown:

(5) To every finite normal subgroup F of H there exists a finite normal subgroup F^* of H with $F^* \neq 1 = F \cap F^*$.

On the basis of (5) one constructs by complete induction an infinite sequence of finite normal subgroups $F(i) \neq 1$ of H with

$$1 = F(i+1) \cap \textstyle\prod_{j=0}^{i} F(j).$$

It follows in particular that the product of these normal subgroups $F(i)$ is their direct product. It is a consequence of (2) that none of the finite normal subgroups $F(i) \neq 1$ is soluble; and an application of the celebrated Theorem of Feit-Thompson shows that every $F(i)$ is of even order. There exists therefore a cyclic subgroup $C(i)$ of order 2 of $F(i)$. Since the product of the $F(i)$ is their direct product, the subgroup E, generated by the $C(i)$, is the direct product of the $C(i)$. It follows that E is an infinite elementary abelian 2-group; and this contradicts (3). Thus we have shown that

(6) G/R is finite.

It is a consequence of (2) and (6) that G is an extension of the radical characteristic subgroup R by the finite group G/R; and we deduce from (1) that R is of bounded abelian factor rank. Hence G too is of bounded abelian factor rank: (ii) is a consequence of (i).

Assume next that G is of bounded abelian factor rank and that C is a radical characteristic subgroup of G with finite G/C. Since C is of finite abelian subgroup rank, it follows from Theorem 6.1, (a), (b), (e) that C is hyperabelian and that every epimorphic image, not 1, of C possesses a non-trivial, abelian characteristic subgroup of finite rank. Now it is clear how to derive (iii) from (ii). —It is almost obvious that (i) is a consequence of (iii), proving the equivalence of our properties (i)-(iii).

Remark 8.2. It is easy to construct generalized radical groups which are infinite, though 1 is their only radical normal subgroup; any direct product of infinitely many, simple, finite, nonabelian groups will do.

Remark 8.3. Combining condition (ii) and Theorem 6.1 further structural properties of generalized radical groups of finite abelian subgroup rank may be obtained. To mention only one of them: all these groups are countable. We leave the enumeration of these properties to the reader.

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