ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 1, Spring 1983

AN EXAMPLE OF AN ARAKELIAN GLOVE WHICH IS A WEAK ARAKELIAN SET

BY

MYRON GOLDSTEIN

Abstract

In 1976 at the London Mathematical Symposium on Potential Theory at Durham, England, L. A. Rubel posed the problem of characterising weak Arakelian sets or at least showing whether or not there exists an Arakelian glove which is weak Arakelian. We shall give an example of an Arakelian glove which is weak Arakelian and we shall give a necessary condition for a closed set F in the plane to be weak Arakelian. It seems likely that this necessary condition is also sufficient for F to be weak Arakelian but at the present time we are not able to prove the sufficiency.

1. Introduction

Let C denote the complex plane and let F be a closed subset of C. Let $\hat{C}$ denote the Riemann sphere. Arakelian showed that a necessary and sufficient condition that every function $f(z)$ continuous on F and holomorphic in F^o can be uniformly approximated on F by an entire function $f(z)$ is that $\hat{C} - F$ be connected and locally connected. We shall call a closed subset F of C Arakelian if $\hat{C} - F$ is connected and locally connected. We shall say that F is weak Arakelian if whenever $f(z)$ is continuous on F and holomorphic in F^o, there exists an entire function $g(z)$ such that whenever $\{z_n\}_{n=1}^{\infty}$ is any sequence in F, $|f(z_n)| \to \infty$ if and only if $|g(z_n)| \to \infty$. We shall say that F is Arakelian near ∞ if there exists a compact subset K of C such that $\hat{C} - (F \cup K)$ is connected and locally connected. Thus if F is Arakelian near ∞, all the bounded components of $C - F$ are contained in some open disk D. Therefore by Theorem 1 in [5], every function $f(z)$ continuous on $F - D$ and holomorphic in $(F - D)^o$ is the uniform limit on $F - D$ of meromorphic functions $g(z)$ having at most one pole in $C - F$. If we let $h(z)$ denote the entire function obtained by subtracting from $g(z)$ the singular part of $g(z)$ at the pole of $g(z)$, then if $\{z_n\}_{n=1}^{\infty}$ is any sequence in F, $|f(z_n)|$ tends to infinity if and only if $|h(z_n)|$ tends to infinity and therefore F is weak Arakelian. It follows that we can

Received March 23, 1981.

add or delete a compact set from a weak Arakelian set and we still have a weak Arakelian set. We shall use this fact together with a recent theorem from the theory of harmonic approximation and an estimate for harmonic measure to construct an Arakelian glove which is weak Arakelian and thus answer a question of Rubel in the affirmative. We shall give a necessary condition for a closed subset F of C to be weak Arakelian and use this result to construct examples of closed sets which are not weak Arakelian. Our necessary condition is probably also sufficient but we have not been able to prove this.

2. Characterisations of Weak Arakelian Sets

As in the introduction, let F be a closed subset of C. We shall now state and prove our necessary condition for F to be weak Arakelian.

THEOREM 1. *A necessary condition for F to be weak Arakelian is that for each compact set K, there exists a compact set $\tilde{K}$ such that if V is a bounded connected open set with $\partial V \subset F \cup K$, then either $V \subset F \cup \tilde{K}$ or $(V - \tilde{K}) \cap F = \emptyset$.*

Proof. Assume the contrary. Then for some compact set K and every positive integer n, there exists a bounded connected open set V_n with $\partial V_n \subset F \cup K$ but nevertheless $V_n \not\subset F \cup \overline{D}_n$ and $(V_n - \overline{D}_n) \cap F \neq \emptyset$ where $\overline{D}_n$ denotes the closed disk $|z| \leq n$. Hence there is a sequence $\{z_n\}_{n=1}^{\infty}$, z_n contained in V_n, z_n not in $F \cup \overline{D}_n$, and $\lim_{n\to\infty} z_n = \infty$. For each positive integer n, let w_n be a point in $(V_n - \overline{D}_n) \cap F$. Let $T = C - \cup_n V_n$. Hence by Theorem 1 in [5], there is a meromorphic function $f(z)$ whose poles lie in the set $\{z_1, z_2, z_3, \ldots\}$, such that $|f(z)| < 1$ on T and $|f(w_n) - n| < 1$. Since $f(z)$ is holomorphic on F, we have by hypothesis that there exists an entire function $g(z)$ with $g(z)$ bounded on $\cup_n \partial V_n$ and $\lim_{n\to\infty}|g(w_n)| = \infty$. Since this contradicts the maximum principle, the theorem is proved.

Theorem 1 is due to the author, P. M. Gauthier, and W. Hengartner.

Using this condition, we shall give some examples of closed sets F which are not weak Arakelian.

In Figure 1, F consists of the flattened spiral together with the limit rays L_1 and L_2 and the limit segment H. F is not weak Arakelian since if we take K equal to the dotted rectangle K, then for any compact set $\tilde{K}$, it is clear that we can find a bounded connected open set V such that $\partial V \subset F \cup K$ but neither is $V \subset F \cup \tilde{K}$ nor is $(V - \tilde{K}) \cap F = \emptyset$ since every point of the unbounded rays L_1 and L_2 are limit points of the flattened spiral and thus the spiral is unbounded. In this example, $\hat{C} - F$ whilst connected is not locally connected.

In Figure 2, F consists of rectangles R_n, open at the bottom, joined together as sketched in Figure 2, together with the limit ray L and a sequence of points z_n lying within the rectangles R_n and tending to infinity as n tends

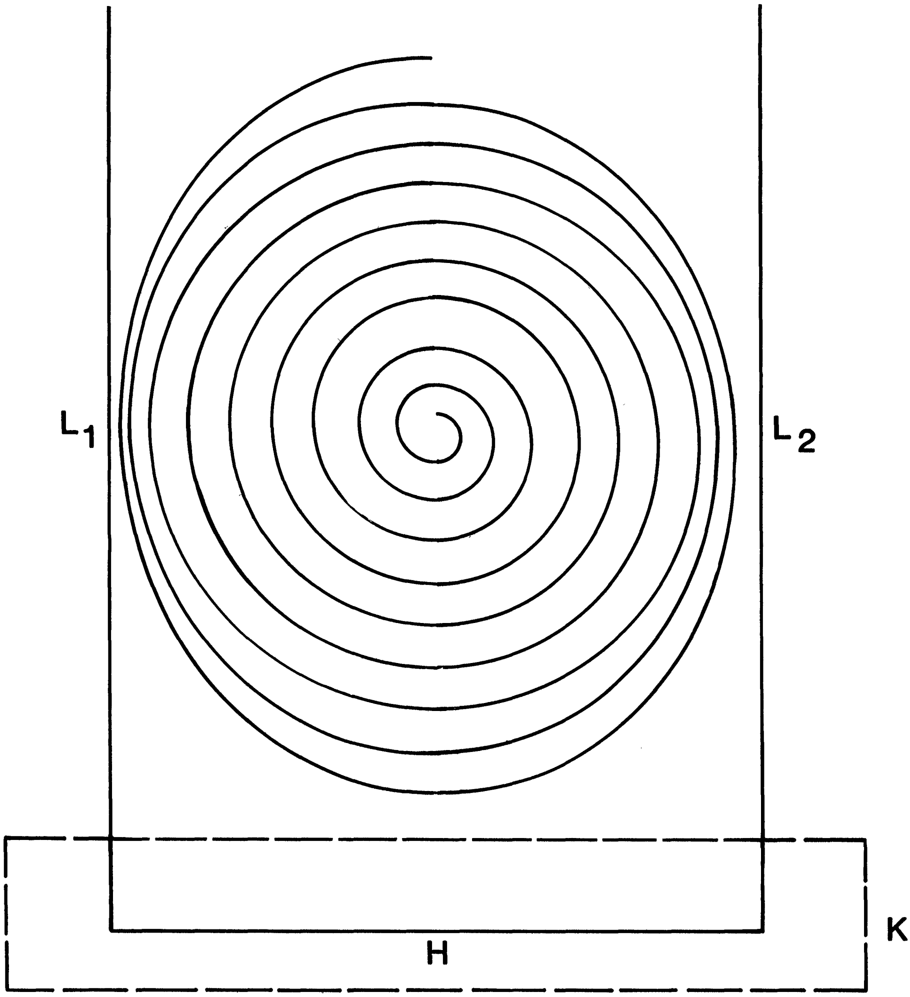

FIG. 1

to infinity. If we take K to be the dotted segment H, then if $\tilde{K}$ is any compact set, we can find a bounded connected open set V with $\partial V \subset F \cup K$ but neither is $V \subset F \cup \tilde{K}$ nor is $(V - \tilde{K}) \cap F = \emptyset$. All we have to do is to take V to be the interior of a closed rectangle R_n with n taken so large that $V - \tilde{K} \neq \emptyset$ and such that z_n does not lie in $\tilde{K}$. Again, $\hat{C} - F$ whilst connected is not locally connected.

The problem of characterising weak Arakelian sets appears to be closely connected with the theory of harmonic approximation. Thus if F is a closed set in the plane with empty interior and if F is a set of uniform approximation by entire harmonic functions [6], then F is weak Arakelian. To see this, let $f(z)$ be a continuous function on F. By assumption, there is an entire harmonic function $u(z)$ such that $u(z)$ approximates $1 + |f(z)|$ to within $\frac{1}{2}$

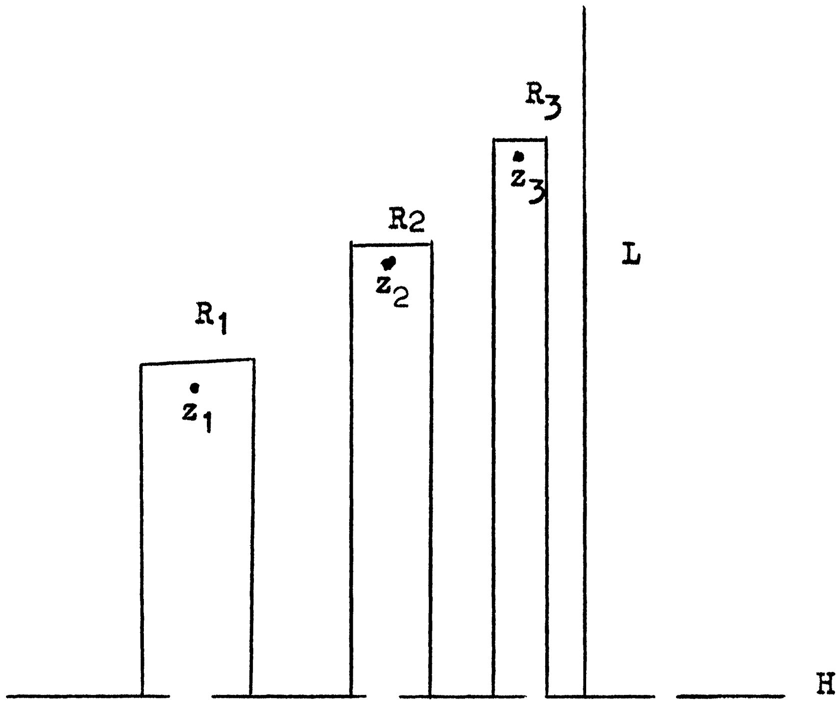

FIG. 2

on F. Let $h(z) = u(z) + iu^*(z)$. Then $|\exp(h(z)|$ and $|f(z)|$ go to infinity simultaneously on sequences of points in F.

3. The Arakelian Glove

Professor Rubel asked the question whether or not there exists an Arakelian glove which is weak Arakelian. We shall answer this question in the affirmative using a recent result from the theory of harmonic approximation.

In Figure 3, the heights of the rectangles are approaching infinity whilst the widths are approaching zero as the rectangles approach the limit line L. The closed set F sketched in Figure 3 is called an Arakelian glove. This terminology is due to Rubel. It should be noted that F consists of the rectangles and the limit line L. We shall now prove the main result of this paper which is that F is weak Arakelian. Of course F is not Arakelian near infinity since $\hat{C} - F$ although connected is not connected at infinity.

Let $f(z)$ be continuous on F. Since F is nowhere dense, we may and shall assume that $f(z) \geqslant 1$ on F. Since we may modify F on any compact subset of C, we may replace F by the closed set in Figure 4 which we shall again refer to as F.

By the Tietze extension theorem, we may extend f continuously to the

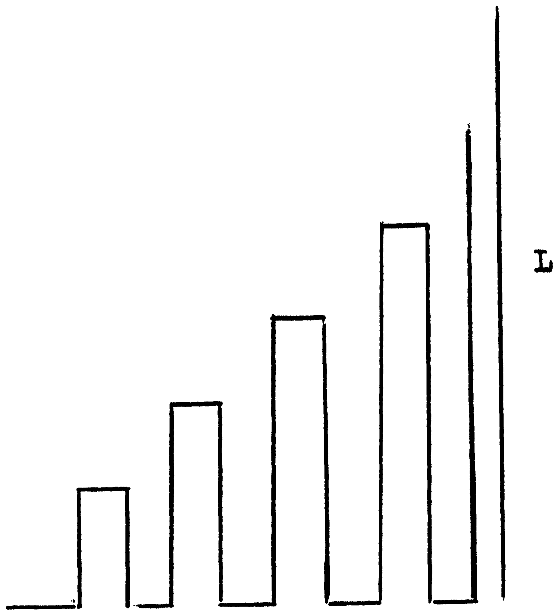

FIG. 3

new F of Figure 4 in such a way that the extended function, which we shall continue to denote by $f(z)$, is greater than or equal to one on the new F of Figure 4. Let M_n denote the maximum of $f(z)$ on the boundary of the rectangle R_n. If $\phi(x)$, $x \geqslant 0$, is strictly increasing, continuous, and tends to infinity as x tends to infinity, then $\phi(f)$ tends to infinity if and only if $f(z)$ tends to infinity. Since we may choose ϕ so that $\phi(x)$ tends to infinity arbitrarily slowly, and such that $\phi(x) \geqslant 1$ whenever $x \geqslant 1$, we may assume that $M_n w_n$ tends to zero as n tends to infinity by replacing f by $\phi(f)$ if necessary where w_n denotes the width of R_n. As Figure 4 indicates, w_n tends to zero as n tends to infinity. Also as Figure 4 indicates, as n approaches infinity, the heights of the rectangles R_n are approaching infinity and the rectangles R_n are approaching the limit line L.

Let a be a point on the limit line L and for each positive integer n, let a_n be a point inside or on the rectangle R_n such that a_n tends to a as n tends to infinity. We shall prove that $H_f^{(\hat{F})^o}(a_n)$ tends to $f(a)$ where $H_f^{(\hat{F})^o}$ denotes the solution of the Dirichlet problem for $(\hat{F})^o$ with boundary function f. Here $\hat{F}$ denotes the union of F with all the bounded components of $C - F$, and the superscript o denotes as usual the interior of the set under consideration.

Let $\delta > 0$ such that $|f(z) - f(a)| < \varepsilon/2$ if $z \in F$ and $|z - a| < 4\delta$. Let

$$\langle S_1 = \{z\colon \operatorname{Im} z \in (a - \delta, a + \delta)\} \quad \text{and} \quad S_2 = \{z\colon \operatorname{Im} z \in (a - 2\delta, a + 2\delta).$$

Thus S_1 and S_2 are horizontal strips symmetric about a horizontal line through a of widths 2δ and 4δ respectively. Since a_n tends to a as n tends

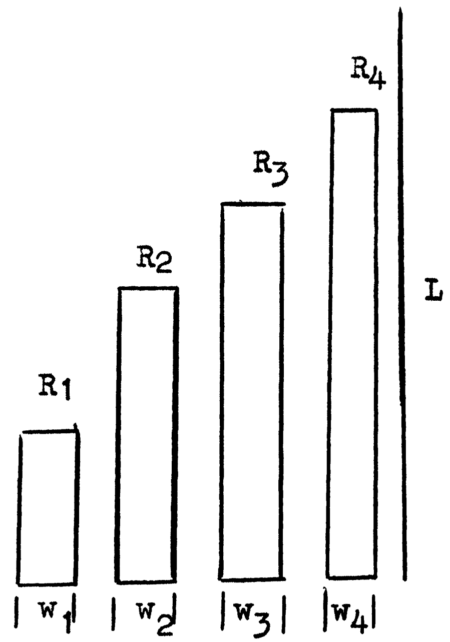

FIG. 4

to infinity, there is a positive integer n_1 such that $a_n \in S_1$ whenever $n \geqslant n_1$. Assume now that $n \geqslant n_1$. Let $\alpha_n = \{z \in \partial R_n \setminus S_2\}$ and let $\beta_n = \partial R_n - \alpha_n$. Then β_n is either a single arc or the union of two disjoint arcs β_n' and β_n''. If $\gamma_n \subset \partial R_n$, let $\omega(a_n, \gamma_n, R_n)$ denote the harmonic measure of γ_n with reference point a_n for the domain consisting of the interior of R_n. Since R_n is convex, it follows from Carleman's estimate for harmonic measure on page 73 of [4] and the condition $M_n w_n \to 0$ that

$$H_f^{(\widehat{F})^o}(a_n) = \int_{\partial R_n} f d\omega(a_n, \beta_n, R_n) + o(1)$$

as n tends to infinity. By Carleman's estimate for harmonic measure referred to in the previous sentence, we have that $\omega(a_n, \beta_n, R_n)$ tends to one as n tends to infinity. By the mean value theorem for integrals, there exist $a_n' \in \beta_n'$ and (if $\beta_n'' \neq \emptyset$) $a_n'' \in \beta_n''$ such that

$$\int_{\partial R_n} f d\omega(a_n, \beta_n, R_n) = f(a_n')\omega(a_n, \beta_n', R_n) + f(a_n'')\omega(a_n, \beta_n'', R_n).$$

Therefore

$$\begin{aligned} |H_f^{(\widehat{F})^o}(a_n) - f(a)| &= |f(a_n')\omega(a_n, \beta_n', R_n) \\ &\quad + f(a_n'')\omega(a_n, \beta_n'', R_n) - f(a)| + o(1) \\ &\leqslant |f(a_n')\omega(a_n, \beta_n, R_n) - f(a)| \\ &\quad + |f(a_n'')|\omega(a_n, \beta_n'', R_n) + o(1) \\ &= o(1) \end{aligned}$$

as n tends to infinity. Hence

$$\lim_{n\to\infty} H_f^{(\hat{F})^o}(a_n) = f(a).$$

Hence $H_f^{(\hat{F})^o}$ is continuous on F and harmonic in $(\hat{F})^o$. By Theorem 10.14, page 216 in [3], F is stable for the Dirichlet problem since $C - \hat{F}$ is not thin at any point of F. By a theorem due to Walter Hengartner and Martine Labreche which will appear in the proceedings of the 1981 séminaire de mathématiques supérieures, été 1981, held at the Université de Montréal, $\hat{F}$ is a set of uniform harmonic approximation by entire harmonic functions since $\hat{F}$ is locally a set of uniform harmonic approximation and since $\hat{C} - \hat{F}$ is connected and locally connected. Hence there exists an entire harmonic function u such that

$$|u - H_f^{(\hat{F})^o}| < 1$$

on $\hat{F}$ and hence on F since $F \subset \hat{F}$. Let h denote the analytic completion of u and let $g = \exp(h)$. Then $|g|$ and f go to infinity simultaneously on F and therefore F is weak Arakelian.

The author wishes to thank professors Walter Hengartner and Paul M. Gauthier with whom the author had many helpful discussions. The author also wishes to thank the referee for his many helpful suggestions.

References

1. J. M. Anderson, K. F. Barth and D. A. Brannan, *Research problems in complex analysis,* Bull. London Math. Soc., vol. 9, (1977), pp. 129–162.
2. P. M. Gauthier, M. Goldstein and W. H. Ow, *Uniform approximation by harmonic functions with logarithmic singularities,* Trans. Amer. Math. Soc., vol. 261 (1980), pp. 169–183.
3. L. L. Helms, *Introduction to potential theory,* Wiley, New York, 1969.
4. R. Nevanlinna, *Analytic functions,* Springer-Verlag, New York, 1970.
5. S. Scheinberg, *Uniform approximation by meromorphic functions having prescribed poles,* Technical Report, Dept. of Math., University of California, Irvine, 1979.
6. A. A. Shaginyan, *Uniform and tangential harmonic approximation on arbitrary sets,* Math. Notes, vol. 9 (1979), pp. 70–84.

Department of Mathematics, Arizona State University, Tempe, Arizona 85287.

Arizona State University
Tempe, Arizona