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CATEGORICAL QUOTIENTS OF CERTAIN ALGEBRAIC GROUP ACTIONS

BY

A. FAUNTLEROY

Introduction

Let G be a connected algebraic group acting on a normal variety X. If the stability group of each point of X is finite then Seshadri in [S] showed that there exists a finite G-morphism $p: Z \to X$ such that the action of G on Z is locally trivial and hence Z/G exists as an algebraic scheme over k. In effect, a quotient of X by G exists up to a finite extension of X'. We use Seshadri covers in this paper to show that when G is unipotent and X quasi-affine then a categorical quotient exists, provided that the action of G is AQA (see Definition 1) in the following sense. There is a quasi-affine variety Y and a surjective open morphism $q: X \to Y$ which is constant on G orbits and which satisfies the following universal mapping property:

Given any morphism ϕ from X to a variety V which is constant on the orbits of G, there exists a unique morphism $\psi: Y \to V$ such that $\psi \circ q = \phi$.

We also show that if X is normal and quasi-affine then there exists a nonempty open set X^{ss} of 'semi-stable' points such that the action of G on X^{ss} is AQA.

If G is not unipotent there are conditions under which a similar conclusion holds (Theorem 3). In general quotients of quasi-affine varieties by connected groups need not be quasi-affine so the hypothesis required are quite strong (see, however, Remark 4 below).

We now fix our terminology. All schemes will be reduced algebraic k-schemes with k a fixed algebraically closed field. A variety is a separated integral scheme. All algebraic groups are assumed to be affine. If X is an irreducible scheme we identify $\Gamma(X, O_X)$ with the subring of everywhere defined rational functions in $k(X)$—the function field of X. Unless otherwise stated "point" will mean closed point.

Let X be an irreducible algebraic scheme over k. We say that X is *almost quasi affine* if there exists a quasi-finite surjective morphism $f: X \to Y$ with Y a quasi-affine variety.

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THEOREM 1. *Let X be a normal irreducible algebraic scheme over k and assume that X is almost quasi-affine. Then there exist a normal quasi-affine variety Q and a birational surjective quasi-finite morphism q: $X \to Q$ satisfying the following universal mapping property:*

Given any morphism f from X to a variety V, there exists a unique morphism g: $Q \to V$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & V \\ & {}_{q}\searrow \quad \nearrow_{g} & \\ & Q & \end{array}$$

commutes. In particular, Q is unique up to isomorphism.

Proof. Let f: $X \to Y$ be a quasi-finite surjective morphism with Y quasi-affine. Then $k(X)/k(Y)$ is a finite algebraic extension so the normalization $\overline{Y}$ of Y in $k(X)$ is also quasi-affine. Now since f is surjective f_* induces an inclusion $f_*: \Gamma(Y, O_Y) \to \Gamma(X, O_X)$. Let R be a finitely generated k-subalgebra of $\Gamma(Y, O_Y)$ such that the canonical map $Y \to \operatorname{Spec} R$ is an open immersion (cf. [3, II. 5.1.9]). Let S be the integral closure of R in $k(X)$. Then we have a canonical open immersion $\overline{Y} \to \operatorname{Spec} S$ induced by the ring inclusion $S \subset \Gamma(\overline{Y}, O_Y)$. Since X is normal, $\Gamma(X, O_X)$ is integrally closed so $S \subset \Gamma(X, O_X)$ (via f_*: $R \to \Gamma(X, O_X)$ S is the integral closure of f_*R in $\Gamma(X, O_X)$). This gives a canonical map q: $X \to \operatorname{Spec} S$. Let Q denote the image of X. If $X_O \subset X$ is an open affine subvariety, then $q_O = q/X_O$ induces a quasi-finite morphism q_O: $X_O \to Q$. Since this map is also birational, q_O is an open immersion of X_O into Q. Applying this to a finite affine open cover of X we conclude that q is an open surjective quasi-finite birational morphism and that Q is open in Spec S hence quasi-affine.

Now suppose f: $X \to T$ is a morphism of X into a variety T. Let $\{X_i\colon 1 \leq i \leq n\}$ be an affine open cover of X. Consider the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow[qxq=\psi]{\phi=f\times f} & T \times T \\ \downarrow & & \\ Q \times Q & & \end{array}$$

Since Q and T are separated, $\phi^{-1}(\Delta(T))$ and $\psi^{-1}(\Delta(Q))$ are closed in $X \times X$. We claim $\psi^{-1}(\Delta(Q)) \subset \phi^{-1}(\Delta(T))$. Let

$$\Omega = \phi^{-1}(\Delta(T)) \quad \text{and} \quad \Lambda = \psi^{-1}(\Delta(Q)).$$

Now $\{X_i \times X_j | 1 \leq i, j \leq n\}$ is an affine open cover of $X \times X$ so

$$\Omega_{ij} = \Omega \cap (X_i \times X_j) \text{ (respectively } \Lambda_{ij} = \Lambda \cap (X_i \times X_j)), \quad 1 \leq i, j \leq n,$$

form an affine open cover of Ω (respectively Λ). If the claim were false we could find a regular function h on the affine variety $X_i \times X_j$ (for some pair (i, j)) with $h \equiv O$ on Ω_{ij} but $h \not\equiv O$ on Λ_{ij}. But $\psi_{ij} = \psi \mid X_i \times X_j$ is an open immersion of $X_i \times X_j$ into $Q \times Q$ and its image clearly meets $\Delta(Q)$ which is irreducible in $Q \times Q$. Thus our assumption implies that h is a rational function on Q, regular on $\psi_{ij}(\Omega_{ij})$ and $\psi_{ij}(\Lambda_{ij})$, which vanishes on the first but not the second. However,

$$\psi_{ij}^{-1}(\Delta Q) = \{(x, x') \mid q(x = q(x') \in q(X_i) \cap q(X_j)\}$$

and this set clearly contains $\Theta = \{(x, x) \mid x \in X_i \cap X_j\} \subset \Omega_{ij}$. Thus,

$$\psi_{ij}(\Omega_{ij}) \supset \overline{\psi_{ij}(\Theta)} \cap \psi_{ij}(X_i \times X_j) \supset \Delta(Q) \cap \psi_{ij}(X_i \times X_j) = \psi_{ij}(\Lambda_{ij}).$$

Thus, $h \equiv O$ in $\psi_{ij}(\Omega_{ij})$ but not on $\psi_{ij}(\Lambda_i)$, a contradiction. This shows $\Lambda_{ij} \subset \Omega_{ij}$ for all i, j so $\Lambda \subset \Omega$.

Now define the function $g\colon Q \to T$ as follows: If $p \in Q$ then $p \in q(X_i)$ for some i, so we put $g(p) = f \circ q_i^{-1}(p)$ where q_i is the isomorphism $X_i \to q(X_i) \subset Q$. If $p \in q(X_i) \cap q(X_j)$, then the above claim shows that $f \circ q_i^{-1}(p) = f \circ q_j^{-1}(p)$ so g is well defined. This establishes the universal mapping property. The uniqueness assertion follows by standard arguments from this.

Remark 1. Of course if X happens to be separated then X is isomorphic to Q.

We call the pair (Q, q) the *quasi-affine cokernel* of X.

We now apply the notion of almost quasi-affine variety to actions of unipotent algebraic groups.

DEFINITION 1. Let the connected unipotent group H act on the quasi-affine variety X with only finite stability groups. We say that the action is AQA (for almost quasi-affine) if there exists a Seshadri cover (Z, W, p) of X such that $W = Z/G$ is almost-quasi-affine.

THEOREM 2. *Let H be a connected unipotent group acting regularly on a normal quasi-affine variety X. Suppose the action of H on X is AQA. Then there exists a normal quasi-affine variety Y and a surjective open morphism $q\colon X \to Y$ satisfying the following universal mapping property:*

Given any variety Z and a morphism $f\colon X \to Z$ constant on the orbits of H, there exists a unique morphism $g\colon Y \to Z$ such that

$$\begin{array}{ccc} & f & \\ X & \to & Z \\ {}_q\searrow & & \nearrow_g \\ & Y & \end{array}$$

commutes. Moreover, if a geometric quotient Q of X modulo H exists and is separated then the canonical morphism $Y \to Q$ is an isomorphism.

Proof. Let (S, p) be a Seshadri cover of X and $T = S/H$ with T almost quasi-affine. Let $Q(T)$ be the quasi-affine cokernel of T and let B be a finitely generated normal k-subalgebra of $\Gamma(S, O_S)^H$ stable under the action of $\Gamma = \operatorname{Aut} k(S)/k(X)$ such that the canonical morphism $Q(T) \to \operatorname{Spec} B$ is an open immersion. Let $R = B \cap k(X)^H$. Then R is a normal k-algebra of finite type and B is integral over R. The image Y of $Q(T)$ in Spec R is therefore open hence quasi-affine. Now X is quasi-affine and X, Spec B and Spec R are normal so R is a subring of $\Gamma(X, O_X)^H$. Let q: $X \to \operatorname{Spec} R$ be the canonical map. Since the diagram

$$\begin{array}{ccc} S & \xrightarrow{p} & X \\ \downarrow & & \downarrow q \\ Q(T) & \longrightarrow & \operatorname{Spec} R \end{array}$$

commutes, the image of q is Y. Moreover since p is a finite morphism it is open. The morphism $S \to Q(T)$ factors as $S \to T \to Q(T)$, a composition of open morphisms, so $S \to Q(T)$ is an open map. Finally, $Q(T) \to Y$ is open so q: $X \to Y$ is an open map. The morphism q is clearly constant on H-orbits.

Now let f: $X \to Z$ be a morphism into a variety Z constant on the orbits of H. Then $f \circ p$ is constant on the orbits of H in S so we get a morphism g': $T = S/H \to Z$ such that

$$\begin{array}{ccc} S & \xrightarrow{f\circ p} & Z \\ & {}_{\pi}\searrow \quad \nearrow_{g'} & \\ & T & \end{array}$$

commutes (π being the quotient map). By Theorem 1, g' factors through $Q(T)$. Replacing Z by the closure of $f(X)$ if necessary we may assume without loss of generality that f is dominant. Suppose Z is affine. Then $f_*k(Z) \subset k(X)^H = k(Y)$. We now have a commutative diagram

$$\begin{array}{ccc} Q(T) & \xrightarrow{g'} & Z \\ & \searrow \quad \nearrow & \\ & Y & \end{array}$$

where the dotted arrow is a priori just a rational map. Let $\lambda \in k(Z)$ with $f_*(\lambda)$ regular at each point of $Q(T)$. We claim $f_*(\lambda)$ is regular on Y. Suppose not. Then since Y is normal there exists a discrete valuation ring θ of $k(Y)$ having a center of codimension one on Y with $f_*\lambda \notin \theta$. But $Q(T) \to Y$ is quasi-finite and surjective and $Q(T)$ is normal. Thus there exist a discrete valuation ring θ' of $k(Q(T))$ having a center of codimension one on $Q(T)$ with $\theta' > \theta$, i.e., $\theta = \theta' \cap k(Y)$. But $f_*(\lambda) \in \theta' \cap k(Y)$ a contradiction. Hence $f_*\lambda$ is regular on Y whenever it is regular on $Q(T)$. It follows that $f_*k[Z] \subset \Gamma(Y, O_Y)$ and the rational map g is regular on Y. In the general case we may replace Z by an affine open cover $\{Z_\alpha\}$ and consider

$$\begin{array}{ccc} g'^{-1}(Z_\alpha) & \longrightarrow & Z_\alpha \\ {}_p\searrow & & \nearrow_g \\ & p(g^{-1}(Z_\alpha)) & \end{array}$$

where $p\colon Q(T) \to Y$ is the natural map (induced by $R \subset B$). Then g will be regular on $p(g^{-1}(Z_\alpha))$. Since the open sets $p(g^{-1}(Z_\alpha))$ cover Y, g is regular on all of Y. This establishes the universal mapping property.

Finally if $Q = X/H$ exists (as a variety) then from the above construction of Y, we see that the canonical map $Y \to Q$ is a quasi-finite birational morphism so Y is an open subvariety of Q. Since $q\colon X \to Y$ is constant on orbits, $Y \to Q$ must also be surjective and hence an isomorphism. This completes the proof of the theorem.

If X is a variety on which the algebraic group G acts and Y satisfies the universal mapping property as above, i.e., is a categorial quotient for maps $X \to V$ constant on G orbits with V a variety, then we will call Y a *strict categorical quotient.*

Remarks 2. It may happen that X/H exists but is not separated (cf. [5, Example 2]). In that case Y is the quasi-affine cokernel of X/H.

3. Theorem 2 shows that a categorical quotient of X by H exists in the sense of [6] provided we restrict ourselves to the category of algebraic varieties rather than the category of k-schemes.

We shall call a quasi-affine variety Y over k *k-noetherian* if $\Gamma(Y, O_Y)$ is finitely generated over k. This concept has been studied by several authors [2], [4], [8]. We now give the generalization of Theorem 2 to arbitrary connected groups.

THEOREM 3. *Let G be a connected algebraic k-group and X a normal quasi-affine variety on which G acts k-morphically. Let H be the* (*connected*) *unipotent radical of G and put $G' = G/H$. Suppose the following conditions hold:*

(i) *For each x in X, the stability group of x in G is finite.*
(ii) *The action of H on X is AQA and the categorical quotient Y of X is k-noetherian.*
(iii) $\Gamma(Y, O_Y)$ *is a unique factorization domain.*
(iv) *There are no nontrivial homomorphisms from G' to G_m.*

Then the categorical quotient W of X by G exists and W is quasi-affine. If the orbits of G on X are closed, the fibers of $q\colon Y \to W$ are connected, and all have the same dimension, then W is the geometric quotient of Y mod G.

Proof. Let $R = \Gamma(Y, O_Y)$. Then (ii) and (iii) imply that R is factorial and of finite type as k-algebra. Since $R = \Gamma(X, O_X)^H$, R is stable under the natural action of G', and $X \to \operatorname{Spec} R$ is G equivariant and hence G' acts on Y. Let $B = R^G$. Since the character group $X(G) = \operatorname{morph}(G, G_m)$ is trivial by (iv), B is also factorial (cf. [7, Corollary 7]). Let W be the image of Y in Spec B under the canonical map q: Spec $R \to$ Spec B. By [6, Theorem 1.1], q is universally open so $q(Y) = W$ is open in Spec B hence quasi-affine. Since B is integrally closed, W is normal. The map q makes Spec B a universal categorical quotient of Spec R (cf. [6]) and thus W is a categorical quotient of $q^{-1}(W)$.

Now let f: $X \to Z$ be a morphism from Y into a variety Z which is constant on G' orbits. We must show f factors uniquely through a morphism g: $W \to Z$. It clearly suffices to show that for any affine open set $Z_0 \subset Z$, $f^{-1}(Z_0) = Y_0 \to Z_0$ factors through $q(Y_0) = W_0 \subset W$. Indeed, if this is shown then we can choose an affine open cover of Z, apply the result to each open set in the cover and define the map g locally. Uniqueness of g locally guarantees the resulting map will be well defined.

Now Y_0 is open in Y and $R = \Gamma(Y, O_Y)$ is factorial. Thus $\Gamma(Y_0, O_{Y_0}) = R_b$ for some G'-invariant function b. Since G' is reductive, $(R_b)^{G'}$ is finitely generated. If r/b is invariant then since R is factorial and $X(G') = \operatorname{morph}(G', G_m) = 1$, r must be invariant (cf. [7] and [9]). Thus $(R_b)^{G'} = B_b$. Now clearly $f_* k[Z_0] \subset B_b$ so $f_0 = f/Y_0$ factors through Spec $B_b \subset$ Spec B, i.e.,

$$\begin{array}{ccc} Y_0 & \xrightarrow{f_0} & Z_0 \\ & \pi\searrow \quad \nearrow & \\ & \operatorname{Spec} B_b & \end{array}$$

commutes. Since $\pi(Y_0) \subset W$ we get that f_0 factors through $\pi(Y_0)$. It follows that g_0: $W_0 = \pi(Y_0) \to Z_0$ is unique and this proves that W is a categorical quotient.

If the orbits of G on X are closed then $\pi(G \cdot x) = G' \cdot \pi(x)$ is closed. Thus the orbits of G' on Y are closed. Now q: $Y \to W$ is a surjective open map which sends closed G' invariant sets to closed subsets of W. Let $O(y)$ be an orbit and $p = \pi(O(y))$. Then $q^{-1}(p)$ is connected and of dimension K say. By the generic quotient theorem [10] there exist a nonempty open affine subset Y_0 of Y such that Y_0/G' exists. Then $k[Y_0] = R_b$ for some $b \in B$ and $k[Y_0/G'] = B[b^{-1}]$. Thus Y_0/G' can be identified with an open subset of W via

$$\begin{array}{ccc} Y_0 & \longrightarrow & \operatorname{Spec} B \\ & \searrow \quad \nearrow & \\ & \operatorname{Spec} B_b & \end{array}$$

The stability groups of $x \in X$ being finite implies the stability group of $y = q(x)$ is also finite and so $K = \dim G'$. Thus each component of $q^{-1}(p)$

is an irreducible closed G' stable subvariety of Y of dimension less than or equal to K. It follows that each component must be an orbit. Since orbits are disjoint and $q^{-1}(p)$ is connected, $q^{-1}(p)$ must be a single orbit. Thus q: $Y \to W$ is a surjective separable open orbit map. By [1, Proposition 6.6], $W = Y \bmod G$.

Remark 4. The hypothesis of the theorem may appear too strong but in fact are crucial. For example let $X = SL(n, k)$ and G a Borel subgroup of X acting on X by right translation. Then H is the unipotent radical of G. The quasi-affine variety Y is known to satisfy (i) and (ii). (If $n = 2$, $Y \cong A^2 - (0, 0)$.) The condition (iv) however does not hold since G' is a torus. Indeed, in this case X/G is the flag manifold of $\mathbf{P}^{n-1}$ so $Y/G' \cong X/G$ is not quasi-affine.

The hypothesis given in the last assertion of the theorem are clearly necessary for W to be the quotient of $Y \bmod G'$.

Let X be a normal quasi-affine variety on which the connected unipotent group G acts. As usual we assume that the stability group of each point in X is finite. Let $B = \Gamma(X, O_X)$ and $A = B^G$. A point x in X will be called *semi-stable* if $\dim c^{-1}(c(x)) = \dim G$ where c: $X \to \operatorname{Spec} A$ is the canonical map. Let X^{ss} denote the set of semi-stable points of X.

LEMMA 4. *X^{ss} is open, non-empty and G-stable.*

Proof. Let $\{A_\alpha\}$ be a directed system of subalgebras of A with A as limit and such that (*i*) each A_α is a finitely generated k-subalgebra of A, and (*ii*) each A_α is normal and has the same quotient field as A. The existence of such a directed system follows from [3, II, 5.1.9] and [9]. Let $r = \dim G$ and let c_α: $X \to \operatorname{Spec} A_\alpha$ be the canonical map. By [1; A.G. 10.1, 10.3] the subset Y_α^r of $\operatorname{Spec} A_\alpha$ defined by

$$Y_\alpha^r = \{y \in \operatorname{Spec} A_\alpha\colon \dim c_\alpha^{-1}(y) \geq r + 1\}$$

is a closed subset of $\operatorname{Spec} A_\alpha$. Clearly $c_\alpha^{-1}(Y_\alpha^1) = X_\alpha^r$ is closed and G-invariant in X; hence so is $X^r = \cap_\alpha X_\alpha^r$. But $x \in X^{ss}$ if and only if

$$\dim c_\alpha^{-1}(c_\alpha(x)) = r$$

for some α and hence for all $\beta > \alpha$, $\dim c_\beta^{-1}(c_\beta(x)) = r$. Since $X_\beta^r \subset X_\alpha^r$ for $\beta > \alpha$, $X^{ss} = X - X^r$ is nonempty, open and G-stable as claimed.

Remark. Let $X^r = X - X^{ss}$. Then X^r is closed in X so can be defined by an ideal in $B = \Gamma(X, O_X)$. Replacing B by a suitable finitely generated G-stable k-subalgebra we can assume X^r is defined by a finite set of elements. Also since $X^r = \cap X_\alpha^r$ we can find a finite type k-algebra A_α as in the proof of the lemma such that $X^r = q_\alpha^{-1}(Y^r)$. Then

$$X^{ss} = q_\alpha^{-1}(\operatorname{Spec} A_\alpha - Y_\alpha^r).$$

Evidently if $A_\alpha \subset A' \subset A$ with A' of finite type over k then

$$q' : X \to \operatorname{Spec} A'$$

has the same property as A_α, that is, $X^r = q'^{-1}(Y)$ for a suitable closed subset Y of Spec A.

THEOREM 5. *Let X be a normal quasi-affine variety on which the connected unipotent group G acts. Assume that the stability group in G of each point of x is finite. Then the action of G on X^{ss} is AQA. In particular, a strict categorical quotient Q of X^{ss} by G exists and is quasi-affine.*

Proof. We may as well replace X by X^{ss} and so assume that $X = X^{ss}$. Let (Z, W, p) be a Seshadri cover of X. We claim that W is almost-quasi-affine. For this consider the commutative diagram

$$(*) \qquad \begin{array}{ccc} Z & \xrightarrow{p} & X \\ q\downarrow & & \downarrow c \\ W & \xrightarrow{p} & \operatorname{Spec} A' \end{array}$$

where A' is a normal finite type k-algebra as in the remark above.

Let $w \in W$ and let T be a component of $p^{-1}(p(w))$. Then $q^{-1}(T)$ is a G-stable closed subset of W each component of which has dimension $\dim T + \dim G$ since q is locally trivial. Let $S \subset Z$ be one of these components. Then $\dim p(S) = \dim S$. By our assumption,

$$\dim c^{-1}(c(x)) = \dim G \quad \text{for each } x \in p(S).$$

Hence

$$\dim c(p(S)) = \dim p(S) - \dim G = \dim S - \dim G.$$

But $\dim S = \dim G + \dim T$; thus $\dim cp(S) = \dim T$. By $(*)$, $c(p(S)) = p(q(S)) = p(w)$, $\dim c(p(S)) = 0$ and T consists of a finite set of points. It follows that p is quasi-finite so W is almost quasi-affine.

A quasi-affine variety will be called quasi-factorial if $\Gamma(X, O_X)$ is a factorial ring.

PROPOSITION 6. *Let X be a quasi-factorial variety on which the connected unipotent group G acts with only finite stability groups. Let Q be the strict categorical quotient of X^{ss} by G. Let U be a G-stable open set in G. If a geometric quotient $Y = U$ mod G exists (as an algebraic scheme) then $U \subset X^{ss}$. If, further, Y is separated then the natural map $Y \to Q$ is an open immersion.*

Proof. Let $Y = U$ mod G and $Y_0 \subset Y$ be an open affine subset. Then $k[Y_0] = A[a^{-1}]$ by [7] or [9]. It follows that the natural map $Y_0 \to \operatorname{Spec} A$

is an open immersion. It is then clear that for $q\colon U \to Y$ the quotient map,

$$q^{-1}(y) = c^{-1}(y) \cap U \quad \text{for all } y \in Y_0.$$

But $q^{-1}(Y_0) = U_0 \subset U$ is G-stable. If $x \in U_0$ then $a(x) \neq 0$ and the natural map $U_0 \to \operatorname{Spec} A$ factors as

$$\begin{array}{ccc} U_0 & \longrightarrow & Spec\ A \\ & \searrow \quad \nearrow & \\ & \operatorname{Spec} A[a^{-1}] & \end{array}$$

and the fibers of c/U_0 are then clearly of the correct dimension so $U_0 \subset X^{ss}$. Since Y is covered by open affines and the inverse images of these under q cover U, $U \subset X^{ss}$ as claimed.

The canonical map $Y \to \operatorname{Spec} A$ is quasi-finite because Y can be covered by finitely many open affines and c restricted to each of these is an open immersion. By the Main Theorem, if Y is separated then c is an open immersion and Y is quasi-affine.

It is not hard to show that if a unipotent group acts on the normal quasi-affine X and the action is AQA then $X = X^{ss}$. We end with two examples to show the limitations of Proposition 6. Let G_a act on affine 3-space by

$$t \cdot (x, y, z) = (x + ty + (t^2/2)z, y + tz, z)$$

(take $k = \mathbf{C}$). Then $X = A^3 - \{(1, 0, 0)\}$ is quasi-factorial, the action of G_α on X is AQA but no geometric quotient exists by [5, Example 2].

The second example is obtained by letting G_α act on 4-space with coordinates w, x, y, z. The action on x, y, z is the same as above and

$$w(t\,p) = w(p) + tx(p) + (t^2/2)y(p) + (t^3/6)z(p).$$

Then using the computations given in [3; 3.1 and remark on p. 204] it can be seen that A^4 − {fixed points} does not consist of semi-stable points; there are fibers of dimension two. For this case X^{ss}/G_a is a universal geometric quotient.

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University of Illinois
Urbana, Illinois