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TWO EXAMPLES OF SETS OF RANGE UNIQUENESS

BY

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To my teacher, F. Burton Jones

1. Introduction

A subset of K of R^2 is a set of range uniqueness (s.r.u.) for entire functions provided that if f and g are entire and $f(K) = g(K)$, then $f \equiv g$ on R^2. This concept was introduced by Diamond, Pomerance, and Rubel in [1] and, at the end of that paper, a number of problems concerning s.r.u.'s were posed. In this note, we answer some of these problems by giving two examples of s.r.u.'s, one an arc and the other an open topological disk.

The basic idea of our construction is to use the fact that entire functions preserve angles at most points. Using this fact, and the topological machinery developed in Section 2, it is then shown that an arc with a dense set of mutually distinct "kinks" is an s.r.u. In section three we construct such an arc and, in Section 4, we show how to construct the second example mentioned above.

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2. Images of Compact Sets Under Local Homeomorphisms

The results of this section are stated for a class $\mathscr{H}$ which properly contains the class E of non-constant entire functions. It takes no more effort to establish these results for $\mathscr{H}$, and it is hoped that they may be used to give examples of s.r.u.'s for the class $\mathscr{H}$ itself.

A function $f : R^2 \to R^2$ is called a local homeomorphism at $x \in R^2$ provided f maps some open neighborhood of x homeomorphically onto an open set in R^2. Given a map $f : R^2 \to R^2$, the *singular set*, S_f, is the set of points at which f is *not* a local homeomorphism.

We now define $\mathscr{H}$ to be the class of continuous maps $f : R^2 \to R^2$ such

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that

(a) S_f has no limit points in R^2 (i.e., the intersection of S_f with any compact set is finite); and
(b) for each $y \in R^2$, the set of $f^{-1}(y)$ has no limit points in R^2.

Clearly, the class $\mathscr{H}$ contains the class $\mathscr{E}$ of non-constant entire functions.

We begin by establishing a very useful property of functions in $\mathscr{H}$.

PROPOSITION 1. *Let f be in $\mathscr{H}$ and let K be a compact subset of R^2. Given any open set W which meets $f(K) - f(K \cap S_f)$, there exists an open set $V \subset W$ which meets $f(K)$ and an open set U such that f maps U homeomorphically onto V and $f(K \cap U) = f(K) \cap V$.*

Proof. Note that since K is compact and f is in $\mathscr{H}$, $K \cap f^{-1}(y)$ is finite for each point y in R^2. Choose a point y_0 in W which lies in $f(K) - f(K \cap S_f)$ and write

$$K \cap f^{-1}(y_0) = \{x_1, \dots, x_n\};$$

f is a local homeomorphism at each x_i and, therefore, we can find an open neighborhood $\tilde{W}$ of y_0, lying in W, and a collection of open neighborhoods $\tilde{U}_1, \dots, \tilde{U}_n$ of $x_1, \dots, x_n$ such that each $\tilde{U}_i$ is mapped homeomorphically onto $\tilde{W}$ by f.

A simple argument involving upper semi-continuity of point inverses in K shows that there is a still smaller neighborhood $\tilde{V}$ of y_0, lying in W, with the property that, if y is a point of $f(K) \cap \tilde{V}$, then every point of $K \cap f^{-1}(y)$ lies in one of the $\tilde{U}_i$. Let $U_i = \tilde{U}_i \cap f^{-1}(\tilde{V})$; then f maps each U_i homeomorphically onto $\tilde{V}$ and $f(K) \cap \tilde{V} = \cup f(K \cap U_i)$.

Let $K_i = K \cap U_i$ and $A_i = f(K_i)$, for $i = 1, \dots, n$, so that

$$f(K) \cap \tilde{V} = A_1 \cup \cdots \cup A_n.$$

Choose a minimal subcollection of the A_i which cover $f(K) \cap V$ and reorder subscripts so that $f(K) \cap V = A_1 \cup \cdots \cup A_m$ $(m \leq n)$ but no fewer of these A_i's will cover $f(K) \cap \tilde{V}$. It then follows that no A_i is contained in the union of the remaining ones. Thus $A_1 - (A_2 \cup \cdots \cup A_m) \neq \emptyset$.

We now define V to be set $\tilde{V} - (A_2 \cup \cdots \cup A_m)$. The A_i are relatively closed in $\tilde{V}$ hence V is an open subset of $\tilde{V}$ and therefore V is open in R^2. Also $f(K) \cap V$ is simply equal to $A_1 - (A_2 \cup \cdots \cup A_m)$. We let

$$U = U_1 \cap f^{-1}(V).$$

Then U is open in R^2 and f maps U homeomorphically onto V. It is not hard to check that $f(K \cap U) = f(K) \cap V$, so the proposition is established.

As a consequence, we have the following result.

PROPOSITION 2. *Let K be a compact perfect set and suppose $f(K) = g(K)$ where f and g are in $\mathscr{H}$. Then there exist open sets U, V, and W with* *the following properties:*

(a) *U meets K and is mapped homeomorphically onto W by f;*
(b) *V meets K and is mapped homeomorphically onto W by g;*
(c) *The function $h = (g|V)^{-1} \circ f$ maps U homeomorphically onto V and $h(K \cap U) = K \cap V$.*

Proof. Let $L = f(K) = g(K)$; L is perfect and $f(K \cap (S_f \cup S_g)$ is finite so we can certainly choose an open set $\hat{W}$ which meets L and misses

$$f(K \cap (S_f \cup S_g)).$$

By Proposition 1, we can pick open sets $\tilde{U}$, $\tilde{W}$ such that $\tilde{U}$ meets K, $\tilde{W}$ lies in $\hat{W}$, f maps $\tilde{U}$ homeomorphically onto $\tilde{W}$ and $f(K \cap \tilde{U}) = L \cap \tilde{W}$. Again by Proposition 1, we can pick open sets V and W such that V meets K, W lies in $\tilde{W}$, g maps V homeomorphically onto W and $g(K \cap V) = L \cap W$. Let

$$U = \tilde{U} \cup f^{-1}(W);$$

it is then easy to check that U, V and W are the required sets.

Let us say that a subset K of R^2 is *conformally rigid* provided that if U and V are open sets in R^2 with $U \cap K \neq \emptyset$ and h is an analytic homeomorphism of U onto V which takes $K \cap U$ onto $K \cap V$, then $K \cap U = K \cap V$ and h is the identity map on $K \cap V$.

By Proposition 2, a conformally rigid set is a set of range uniqueness for entire functions.

3. A Conformally Rigid Arc

Our construction can be described briefly as follows. We reflect the Cantor ternary function in the line $y = x$ and integrate. The graph of the resulting function is a conformally rigid arc.

Let I denote the unit interval and let $I_1, I_2, I_3, \ldots$ be the middle third sets used in the construction of the Cantor ternary set:

$$I_1 = [1/3, 2/3],\ I_2 = [1/9, 2/9],\ I_3 = [7/9, 8/9],\ \ldots .$$

We index sets so that all intervals of a given length are indexed consecutively with I_n lying to the right of I_{n-1}. Let D denote the set of diadic rationals in $(0, 1)$ and let D be indexed in a similar way:

$$D = \{d_1, d_2, d_3, \ldots\} \quad \text{where } d_1 = 1/2,\ d_2 = 1/4,\ d_3 = 3/4,\ \ldots$$

so that the elements of D with a given denominator are indexed consecutively with d_n lying to the right of d_{n-1} (and $d_i \neq d_j$ for $i \neq j$).

We define a function $f : [0, 1] \to [0, 1]$ as follows. For $x = d_n \in D$, $f(x) = a_n$ where a_n is the right hand endpoint of the interval I_n. Note that f is strictly increasing on D. It is easily verified that if $x \in I - D$ then $\lim \{f(d) | d \in D, d \to x\}$ exists and we define $f(x)$ to be this limit. The function

f thus defined, has these properties:

(a) f is monotone increasing on $[0, 1]$;
(b) f is continuous at each $x \in I - D$;
(c) f has a jump discontinuity at each $d_n \in D$, whose magnitude is precisely $b_n - a_n$ where b_n is the lefthand endpoint of I_n.

As alluded to earlier, the best way to picture f and to verify its properties is to take the standard Cantor ternary function, as described say on page 131 of [2], and restrict this function to $I - \cup(a_n, b_n]$. If the graph of this restriction is then inverted in the line $y = x$, the resulting set is the graph of our function f.

We now observe that since f is monotone, it is Riemann-Stieltjes integrable [4, p. 109], and we define $F : [0, 1] \to [0, 1]$ by $F(x) = \int_0^x f(s)ds$. It is easily verified that if $x \in I - D$ then F is differentiable at x and $F'(x) = f(x)$. Moreover if $x \in D$, then the right and left hand derivatives D^+f, D^-f exist at x. Indeed, if $x = d_n$, then $D^+f(x) = b_n$ and $D^-f(x) = a_n$. Let $z = (x, F(x))$ be a point on the graph of F with $x \in (0, 1)$ and let $\alpha(z)$ denote the acute angle between the right and left hand tangent lines to the graph of F at z. Then $\alpha(z) = 0$ at precisely those points where F is smooth. At the point $z = (x, F(x))$ where $x = d_n \in D$, $\alpha(z)$ is not zero and, in fact, $\tan \alpha(z)$ can be expressed in terms of the right and left hand derivatives by

$$\tan \alpha(z) = \frac{b_n - a_n}{1 + a_n b_n}.$$

We now note that these numbers can be computed explicitly and that they are all distinct (indeed, they form a strictly decreasing sequence).

Let K be the graph of the function F and let U, V be open subsets of R^2 each of which intersects K. Suppose h is an analytic homeomorphism of U onto V taking $K \cap U$ onto $K \cap V$. Using standard techniques, e.g., [3, p. 149], it is easy to check that for each $z \in K \cap U$, $\alpha(z) = \alpha(h(z))$. It follows that $z = h(z)$ for all $z \in K \cap U$ whose first coordinate lies in D and hence $z = h(z)$ identically on $K \cap U$. Thus, K is conformally rigid.

4. An Open Disk which Is an s.r.u.

The techniques of the preceding section can just as well be used to construct a conformally rigid simple closed curve.

PROPOSITION 3. *Let C be a conformally rigid simple closed curve and let Δ be the topological open disk bounded by C. Then Δ is an s.r.u. for entire functions.*

Proof. Suppose f and g are entire functions with $f(\Delta) = g(\Delta)$. This common image is an open set in R^2 and, if we let L denote its boundary, then L is contained in $f(C) \cap g(C)$.

Since L is the boundary of an open set in R^2 it has at least one component L' which is non-degenerate. By a slight elaboration of the proof of Proposition 2, we can find open sets U, V, W such that $C \cap U \neq \emptyset \neq C \cap V$, $L' \cap W \neq \emptyset$, f maps U homeomorphically onto W and takes $C \cap U$ onto $L' \cap W$, g maps V homeomorphically onto W and takes $C \cap V$ onto $L' \cap W$. Then $(g|V)^{-1} \circ f$ maps U homeomorphically onto V and takes $C \cap U$ onto $C \cap V$. Since C is conformally rigid, $f \equiv g$.

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