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RATIONAL POINCARÉ DUALITY SPACES

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Manifolds play a particularly important role in topology. From the point of view of algebraic topology, their distinguishing feature is the Poincaré duality which exists in their homology and cohomology. Historically, Poincaré first observed the duality in terms of a dual cell decomposition which means that duality occurred at the chain level.

From the point of view of rational homotopy theory, simply connected spaces are equivalent to simply connected c.d.g.a.'s (commutative differential graded algebras). The role of manifold is then played by a c.d.g.a. which satisfies Poincaré duality at the rational cohomology level. Surprisingly, this turns out to imply an approximate chain level duality strong enough to yield higher order implications, e.g., in terms of Massey products. For example, one of the simplest manifolds with non-trivial Massey products is the homogeneous space $Sp(5)/SU(5)$. As first computed by Borel [2], $Sp(5)/SU(5)$ has rational cohomology algebra generated by classes $x \in H^6$, $y \in H^{10}$, $a \in H^{21}$, $b \in H^{25}$ with the only non-trivial products being $xb = ay = \mu$, the fundamental class. From Borel's calculations, it is easy to observe that $a = \langle x, x, y\rangle$ and $b = \langle x, y, y\rangle$, ordinary 3-fold Massey products. Another manifold with the same cohomology algebra is

$$(S^6 \times S^{25}) \# (S^{10} \times S^{21}),$$

the connected sum. Here both $\langle x, x, y\rangle$ and $\langle x, y, y\rangle = 0$. It turns out that Poincaré duality guarantees that $\langle x, x, y\rangle$ and $\langle x, y, y\rangle$ are simultaneously both zero or both non-zero. This aspect of Poincaré duality is part of the fall-out of the main topic of this paper: The classification of rational Poincaré duality spaces.

For general rational spaces, one approach to classification is given by the obstruction theory of Halperin-Stasheff and the machinery of Schlessinger-Stasheff. On a more elementary level, a variety of authors and techniques have shown, for example, that a cohomology algebra H is represented by a unique rational homotopy type if $H^i = 0$ for $0 < i < k$ and $i > 3k - 2$. However, Tim Miller [4] proved that if H was a Poincaré duality algebra, then $H^i = 0$ for $0 < i < k$ and $i > 4k - 2$ still guaranteed uniqueness of the rational homotopy type. J. Neisendorfer then asked if many of the

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results of Schlessinger and Stasheff on shallow spaces might not similarly be extended for Poincaré duality algebras. The following deeper result is in fact true:

THEOREM 1. *Let H be a Poincaré duality algebra of top dimension N and $H^1 = 0$. Let X be a simply connected rational space with $H(X) \approx H$ except $H^N(X) = 0$. If $Y = X \cup e^N$ with $H(Y) \approx H$, then the rational homotopy type of Y is unique. In other words, the rational homotopy type of a simply connected Poincaré duality space $Y = X \cup e^N$ is determined by X.*

In particular, the attaching map for e^N can be given in a particularly simple form, reflecting precisely the duality in H, namely as a linear combination of ordinary (non-iterated) Whitehead products with respect to some basis of $\pi_*(X) \otimes Q$.

The proofs are carried out not in terms of spaces and classifying maps, but rather in the equivalent rational homotopy category of d.g.l.'s (differential graded Lie algebras). Much of rational homotopy theory has followed Sullivan's [8] emphasis on c.d.g.a.'s which correspond to spaces via the Sullivan-de Rham rational forms $A^*(X)$ on a space X. Quillen's original approach to rational homotopy theory emphasizes differential graded Lie algebras in another way. With $d = 0$, the rational homotopy groups $\pi_*(\Omega X) \otimes Q$ form a graded Lie algebra under Samelson product. Moreover [6, p. 226] produces a non-trivial differential graded Lie algebra L_X which not only gives $H(L_X) \approx \pi_*(\Omega X) \otimes Q$ as graded Lie algebra under Samelson product, but also faithfully records the rational homotopy type of X. A simplistic way of characterizing such an L_X for nice X is as follows: There is a standard construction $\mathscr{A}$ such that for any d.g.l. L, we have $\mathscr{A}(L)$ as a c.d.g.a. and for L_X, we have $\mathscr{A}(L_X) \to A^*(X)$ as a model for X.

(For ordinary Lie algebras L, the construction $\mathscr{A}(L)$ is the standard complex of alternating forms used to define the Lie algebra cohomology.)

The model $\mathscr{L}(X)$. Thanks to Baues and Lemaire, we now have a particularly simple model of L_X as follows: For a commutative graded algebra H of finite type, we refer to the dual as the homology coalgebra and denote it by H_*. The underlying Lie algebra of $\mathscr{L}(X)$ is the free graded Lie algebra [6, p. 13] $L(H)$ on the desuspension $s^{-1}H_*$ of the reduced homology $(s^{-1}H_*)_n = (H_*)_{n+1}$.

In fact, $L(H)$ is bigraded, using bracket length as the second gradation. A typical bihomogeneous element of $\mathscr{L}(X)$ will be an iterated bracket

$$[x_i],\ [x_i, x_j],\ [x_i, [x_j, x_k]], \text{ etc.},$$

where x_α is a homogeneous element of H_*. If x_α has degree n_α, then $[x_i]$ has degree $n_i - 1$, $[x_i, x_j]$ has degree $n_i + n_j - 2$, etc. The differential d on $\mathscr{L}(X)$ can be written as $d = d_1 + d_2 + \dots$ where d_i increases bracket length by i. In particular, $d_1 : [H_*] \to [H_*, H_*]$ is dual to the multiplication $H \otimes H \to H$. If $d = d_1$, then $\mathscr{L}(X) = \mathscr{L}(H_*)$, Quillen's functor applied to the coalgebra H_* [6].

That H satisfies Poincaré duality is reflected in the differential of the fundamental class, $\mu \in H_N$. If $\{x_i, \mu\}$ is a basis for H_*, then $d_1[\mu] = \frac{1}{2}\Sigma[x_i, b_i]$ and the b_i are a dual basis for H_*, henceforth denoted x_i^*. In particular, we can choose a basis for H_* in dimensions less than or equal to $N/2$ and then complete this to a basis for all of H_* by using the corresponding duals in dimensions $>N/2$. Thus we have two bases $\{x_i, \mu\}$ and $\{x_i^*, \mu\}$ which, except possibly in dimension $N/2$ if N is even, are reindexings of each other.

We will take advantage of Poincaré duality by manipulating the x_i of low dimension and letting Poincaré duality do or verify the rest for us.

On the face of it, $d_i[\mu]$ could be non-zero, but we will show there is a choice of basis of $\mathscr{L}(X)$ so that $d_i[\mu] = 0$ for $i > 1$ (Theorem 2).

On other generators, d_i may well be non-zero. Because d_1 plays a very different role from the other d_i, we will denote d_1 by ∂ and let $p = d_2 + d_3 + \dots$. (The letter p reflects the point of view from which p is regarded as a perturbation of d_1 [3] and [7].) The relation $d^2 = 0$ can be written as $(\partial + p)^2 = 0$. By considering the second grading, this implies $\partial^2 = 0$ and $\partial d_2 + d_2\partial = 0$. Once we have $d_i(\mu) = 0$ for $i > 1$, this implies

$$d_2\partial(\mu) = 0 \quad \text{or} \quad [d_2 x_j, x_j^*] \pm [x_j, d_2 x_j^*] = 0$$

which is the first piece of the higher order duality claimed. In particular, this explains why $\langle x, x, y\rangle$ and $\langle x, y, y\rangle$ are both zero or both non-zero if

$$H \approx H(Sp(5)/SU(5)) \approx H(S^6 \times S^{25} \# S^{10} \times S^{21}).$$

We hope that the results together with their proofs will provide a more "geometric" insight into the algebraic Poincaré duality of rational homotopy theory. If so, we owe a debt of gratitude to the referee who reminded us that an excess of elegance may run counter to insight.

The main theorem recast. The main theorem can now be recast in the following form. Corresponding to a decomposition $Y = X \cup e^N$, we have a map of d.g.l.'s $\mathscr{L}(X) \to \mathscr{L}(Y)$ and in fact $\mathscr{L}(Y)$ can be described as $\mathscr{L}(X)[\mu]$. (Here we have abused the usual notation for adjoining a variable to carry it over to the Lie algebra setting. Since $\mathscr{L}(X)$ is free on the x_i, $\mathscr{L}(Y)$ is just the free Lie algebra on $\{x_i, \mu\}$.)

THEOREM 1′. *Given cofibrations $Y_i = X \cup e^N$, and an isomorphism $H(\mathscr{L}(Y_1)) \approx H(\mathscr{L}(Y_2))$ which restricts to the identity on $H(\mathscr{L}(X))$, there is an automorphism ϕ of $\mathscr{L}(X)$ which extends to an isomorphism*

$$\Phi : \mathscr{L}(Y_1) \to \mathscr{L}(Y_2)$$

taking μ to μ.

Proof. We will in fact construct an automorphism ϕ of the form $1 + \psi = 1 + \psi_1 + \psi_2 + \ldots$ where ψ_i increases bracket length by i and is non-zero only above the middle dimension.

Let $L_i = \mathscr{L}(Y_i)$. Under the given isomorphism of $H(L_1)$ with $H(L_2)$, the d_1 parts of the differentials for L_1 and L_2 agree. We therefore write those differentials as $\partial + p_1$ and $\partial + p_2$ and will consider the difference $p = p_2 - p_1$.

First, consider the terms of bracket length 3. In terms of our basis $\{x_i\}$, we can write

$$(p_2 - p_1)\mu = \sum[x_i, r_i] + \text{terms of greater bracket length}$$

where r_i is a linear combination of two-fold brackets. In fact, and this is key, by judicious use of the Jacobi identity, we can assume $r_i = 0$ unless x_i has degree (in H_*) $< N/2$. Now define $\psi_1(x_i^*) = r_i$ for all i, so $\psi_1 x_i = 0$ if deg $x_i \leqslant N/2$ and $\psi_1 x_i^* = 0$ if deg $x_i \geqslant N/2$. Thus we have

$$\begin{aligned}(1 + \psi_1)(\partial + p_1)(\mu) &= (1 + \psi_1)\left(\sum[x_i, x_i^*]\right) + p_1(\mu) + \psi_1 p_1(\mu)\\ &= \partial\mu + \sum[x_i, \psi_1 x_i^*] + p_1(\mu) + \psi_1 p_1(\mu)\\ &= \partial\mu + (p_2 - p_1)\mu + p_1(\mu) + \psi_1 p_1(\mu),\end{aligned}$$

but $\psi_1 p_1(\mu)$ has bracket length at least 4.

Thus we are able to proceed by induction. Assume ψ_j has been constructed for $j < n$. Write the terms of length $n + 2$ in

$$(p_2 - (1 + \psi_1 + \cdots + \psi_{n-1})p_1)$$

as

$$\sum[x_i, s_i]$$

where s_i is now a linear combination of brackets of length $n + 1$ and define

$$\psi_n(x_i^*) = s_i.$$

Of course we need ϕ to be a chain map on all of $\mathscr{L}(H)$, not just on μ. This is a little more subtle. Again consider terms of length 3. We need to show

$$(1 + \psi_1)(\partial + p_1)(x_i^*) = (\partial + p_2)(1 + \psi_1)(x_i^*)$$

modulo terms of length greater than 3. For the terms of length 2, we have

$$\partial x_i^* = \partial x_i^*$$

while for the terms of length 3, we need to verify

$$p_1 x_i^* + \psi_1 \partial x_i^* = p_2 x_i^* + \partial\psi_1 x_i^*.$$

Since $\mathscr{L}(H)$ is free, we can look instead at

$$[x_i, px_i^*] + [x_i, (\partial\psi - \psi\partial)x_i^*] \quad \text{where } \psi = \psi_1.$$

Applying $(1 + \psi_1)$ to $\partial^2\mu = 0$, we have, in length 4,

$$[\psi\partial x_i, x_i^*] + [\partial x_i, \psi x_i^*] + [\psi x_i, \partial x_i^*] + [x_i, \psi\partial x_i^*] = 0.$$

Since $\dim x_i \leqslant N/2$, $\psi x_i = 0$ and $\psi\partial x_i = 0$ so we have

$$[\partial x_i, \psi x_i^*] + [x_i, \psi\partial x_i^*] = 0.$$

On the other hand from $(\partial + p_i)^2(\mu) = 0$ in length 4 we have, with $p = p_2 - p_1$,

$$[px_i, x_i^*] + [x_i, px_i^*] + [\partial x_i, r_i] + [x_i, \partial r_i] = 0;$$

but p is non-zero only on μ. Since $\psi x_i^* = r_i$ by construction, we deduce

$$[x_i, px_i^*] + [x_i, \psi\partial x_i^*] + [x_i, \partial\psi x_i^*] = 0$$

as desired.

Finally, we wish to show that the top cell is attached in a particularly nice way.

THEOREM 2. *If Y is a simple connected rational space such that $H = H(Y)$ satisfies Poincaré duality, then there is a Lie algebra model $\mathscr{L}(H)$ with*

$$d(\mu) = \frac{1}{2}\sum [x_i, x_i^*];$$

there are no terms of higher order. Equivalently $Y = X \cup e^N$ where e^N is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(x) \otimes Q$.

Proof. We know there is a model $\mathscr{L}(H) = (L(H), \partial + p)$. We will in fact construct a new perturbation q such that $q(\mu) = 0$ and a map of d.g.l.'s

$$(L(H), \partial + q) \to (L(H), \partial + p)$$

of the form $1 + \psi = 1 + \psi_1 + \psi_2 + \ldots$ where ψ_j increases bracket length by j.

Again let $\partial(\mu) = \frac{1}{2}\Sigma[x_i, x_i^*]$ display a dual basis for H as in Theorem 1. Define

$$(1 + \psi) = 1 + \sum \psi_j : L(H) \to L(H)$$

by

$$p(\mu) = \sum [x_i, \psi_j(x_i^*)].$$

Define the derivation $q_n : L(H) \to L(H)$ increasing bracket length by n by

$$q_n(x_i^*) = p_n(x_i^*) + \sum_1^{n-1} (p_j \psi_{n-j} - \psi_{n-j} q_j)(x_i^*)$$

with $p_1 = q_1 = \partial$. It is then trivial to check that

$$(1 + \psi)(\partial + q) = (\partial + p)(1 + \psi)$$

on all x_i^* and

$$(1 + \psi)(\partial + q)(\mu) = (1 + \psi)(\partial\mu) = (\partial + p)(\mu)$$

by construction.

Bibliography

1. H. J. Baues and J. M. Lemare, *Minimal models in homotopy theory,* Math. Ann., vol. 225 (1977), pp. 219–242.
2. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts,* Ann. of Math., vol. 57 (1953), pp. 115–207.
3. S. Halperin and J. Stasheff, *Obstructions to homotopy equivalences,* Advances in Math., vol. 32 (1979), pp. 233–279.
4. T. J. Miller, *On the formality of (k-1) connected compact manifolds of dimension less than or equal to 4k-2,* Illinois J. Math., vol. 23 (1979), pp. 253–258.
5. J. Neisendorfer, *Lie algebras, coalgebras and rational homotopy theory of nil potent spaces,* Pacific J. Math., vol. 75 (1978), pp. 429–450.
6. D. Quillen, *Rational homotopy theory,* Ann. of Math., vol. 90 (1969), pp. 205–295.
7. M. Schlessinger and J. Stasheff, *Deformation theory and rational homotopy type,* Publ. Sci. l'IHES, France, to appear.
8. D. Sullivan, *Infinitesimal computations in topology,* Publ. Sci. l'IHES, France, vol. 47 (1978), pp. 269–331.

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