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FINE CONVERGENCE AND PARABOLIC CONVERGENCE FOR THE HELMHOLTZ EQUATION AND THE HEAT EQUATION

BY

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Introduction

Consider, for $\alpha > 0$ fixed, the Helmholtz equation $\Delta u - 2\alpha u = 0$ on $\mathbf{R}^n$. It is not hard to see that the corresponding Martin boundary is a sphere, and that every positive solution has an integral representation

$$u(x) = \int_{S^{n-1}} e^{\lambda\langle x,b\rangle}\mu(db)$$

where $\lambda = (2\alpha)^{1/2}$ and μ is a positive measure on the sphere. The rotationally invariant function given by this formula where μ is equal to Lebesque measure σ will be denoted by h.

O. Linden [12] proved a Fatou-type theorem recovering the values of $d\mu/d\sigma$ a.e. as limits at infinity of u/h along tubes of constant diameter. The present article proves a stronger result. It gives convergence through parabolic regions, which are in many ways more natural than tubes, and it does not require u to be globally defined. In other words this is an analogue of the well-known result of Privalov-Calderón-Carleson [2], [3] about harmonic functions in a half space of $\mathbf{R}^n$.

The method of proof is essentially that of Brelot and Doob [1], also used in [10]; it consists in deriving a geometric convergence result from fine convergence at the Martin boundary, which is guaranteed in a very general situation by the Fatou-Naïm-Doob theorem. There is however an essential difficulty to be surmounted: the natural version of the Harnack inequalities for the associated potential theory is not strong enough to permit the direct translation of the argument of Brelot and Doob to the Helmholtz equation (see remark following definition 2.1). In order to bypass this difficulty it is first useful to make a not entirely trivial reduction of the problem (Theorem 1.2) and then to use a strengthened one-sided Harnack type inequality (Proposition 2.4) which is obtained from the theory of the heat equation.

In Section 1 the reduction theorem is proved and section two gives the

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proof of the parabolic convergence result mentioned above (Theorem 2.9).

If one is interested only in global solutions then Theorem 2.9 can be strengthened in two ways, each one corresponding to a different method of proof. The first one, suggested by Doob, consists of a reduction via the Appell transform to a parabolic convergence result for the heat equation [6]. This gives parabolic convergence for quotients of arbitrary positive solutions to the Helmholtz equation. The second proof is based on the classical idea (cf. [20, Ch. XVII]) of using a Hardy-Littlewood type maximal function. This method was used by Linden [12] for tubes and is here shown to extend to parabolic regions. It has the advantage of working for all complex α that are not real negative and an appropriate class of solutions. These results are obtained in section three.

Finally, Section 4 is devoted to a discussion of the following problem which arises naturally in connection with the methods in earlier sections. Can Doob's parabolic Fatou-type theorem for the heat equation [6] be deduced using the theory of fine convergence? For positive solutions it is shown that this is easy to do, but for arbitrary quotients it appears more awkward.

The authors wish to thank J. L. Doob for suggesting they use the Appell transform to prove Theorem 3.1.

Let $f(s)$ and $g(s)$ be two functions defined for s near s_0. The notations $f(s) \sim g(s)$ and $f(s) \approx g(s)$ indicate respectively that $f(s)/g(s)$ has a nonzero limit as $s \to s_0$ and that there is a constant C with $1/C \leqslant f(s)/g(s) \leqslant C$ for s close enough to s_0.

1. Fine Versus Admissible Convergence: Reduction to a Special Case

With each of the two equations under consideration there is an associated potential theory on a state space X, the harmonic functions being the solutions. Furthermore, the positive globally harmonic functions h have a unique integral representation $h(x) = \int K_b(x)\mu(db)$ where b runs through a boundary B of the state space (for additional details in each case see Sections 2 and 4 respectively).

In both cases the potential theory is coupled with a Hunt process (exponentially killed Brownian motion in the case of one, and the heat process in the case of the other). Doob's results in [4] on conditional Brownian motion apply to both processes (this is explicitly worked out for processes in duality by Föllmer in [8] and stated by Doob in [5] for the heat process). As a result, for u, v any two harmonic functions *the theorem of Fatou-Naïm-Doob* is valid: the quotient u/v has fine limit at ν-a.e. (minimal) boundary point b equal to $(d\mu/d\nu)(b)$ where μ and ν are the measures that represent u and v respectively.

The reader unfamiliar with either the probabilistic proof of this theorem [4], [8] or the potential theoretic proof of [14], [15] may find it useful to consult [17] for an elementary exposition of [15]. Here X is a measurable space and a cone $\mathcal{S}$ of "superharmonic" functions is given on X for which certain hypotheses are verified. In particular each $s \in S$ has a Riesz Decomposition $s = p + h$ with p a "potential" and h "harmonic". Further there is a measurable boundary space such that each "harmonic" function $h \in \mathcal{S}$ has a unique integral representation as

$$h(x) = \int K_b(x)\mu(db),$$

μ a positive measure on B and K_b a minimal "harmonic" function for each $b \in B$.

Assume that $\hat{X} = X \cup B$ is a topological space.

DEFINITION 1.1. An *admissible system* A is a function on $\hat{X} \setminus X = B$ with $A(b) \subset X$ and b a limit point of $A(b)$, for all $b \in B$. A function f *converges A-admissibly* to λ at b if for all $\varepsilon > 0$ there exists a neighbourhood U of b such that $|f(x) - \lambda| < \varepsilon$, for all $x \in U \cap A(b)$. This will be indicated by writing

$$\lambda = (A\text{-lim } f)(b).$$

THEOREM 1.2. *Let A be an admissible system and let h be a positive harmonic function with representing measure μ. Assume that for any positive harmonic function u,*

$$[(\text{fine limit } u/h)(b) = 0] \Rightarrow [(A\text{-lim } u/h)(b) = 0] \quad \mu\text{-a.e.},$$

(*where the exceptional set depends on u and A*).

Then,

$$A\text{-lim } u/h = \frac{d\nu}{d\mu} \qquad \mu\text{-a.e.} \quad \textit{if } \nu \textit{ represents } u.$$

Proof. Let $\mathcal{M}$ denote the set of positive measures μ on $(B, \mathcal{B})$ such that

$$\int K_b(x)\mu(db) < \infty \quad \text{for all } x \in X.$$

Then $\mathcal{M}$ is a convex subcone of the set of positive Borel measures on $(B, \mathcal{B})$ which is closed under the lattice operations $\vee$ and $\wedge$. These operations are defined as follows:

$$\mu \vee \nu = 1/2\{\mu + \nu + |\mu - \nu|\} \quad \text{and} \quad \mu \wedge \nu = 1/2\{\mu + \nu - |\mu - \nu|\},$$

where $|\alpha|$ denotes the total variation of a signed measure α. The following lemma is easily established (cf. [17]).

LEMMA. *Let $\mu \geq 0$. Then*

$$\frac{d}{d\mu}|\alpha| = \left|\frac{d\alpha}{d\mu}\right|.$$

Hence,

$$\frac{d}{d\mu}(\eta \vee \nu) = \left(\frac{d\eta}{d\mu}\right) \vee \left(\frac{d\nu}{d\mu}\right) \quad and \quad \frac{d}{d\mu}(\eta \wedge \nu) = \left(\frac{d\eta}{d\mu}\right) \wedge \left(\frac{d\nu}{d\mu}\right).$$

Let u be a positive harmonic function with representing measure ν. Since u/h converges finely to $d\nu/d\mu$ μ-a.e. it suffices to consider what happens for $b \in \{d\nu/d\mu > 0\}$. Let q, p be positive rationals and $r = p + q$. It follows from the lemma that the function $u \rightsquigarrow$ (fine limit $u/h)(b)$ preserves the lattice operations $\wedge$, $\vee$ in the cone of positive harmonic functions (cf. (B) in [17] for the definitions of $\wedge$, $\vee$).

Assume that $q \leqslant (d\nu/d\mu)(b) \leqslant r$. Then $[qh - u \wedge (qh)]/h$ and $[rh - u \vee (rh)]/h$ converge finely to 0 at b.

By hypothesis this implies

(*) $\quad (A\text{-lim}[u \wedge (qh)]/h)(b) = q \quad$ and $\quad (A\text{-lim}[u \vee (rh)]/h)(b) = r$

except for $b \in E_{q,r}$, where $\mu(E_{q,r}) = 0$.

Since for u, v, w any three harmonic functions

$$\min\{u, v + w\} \leqslant \min\{u, v\} + \min\{u, w\}$$

it follows that $u \wedge (v + w) \leqslant u \wedge v + u \wedge w$ (cf. (3) in [17]). Consequently,

$$0 \leqslant [u \wedge (rh) - u \wedge (qh)]/h \leqslant p.$$

Combining this with the identity $u = u \vee (rh) + u \wedge (qh) + \{u \wedge (rh) - u \wedge (qh)\} - rh$ it follows from (*) that

$$q \leqslant (A\text{-}\underline{\lim}\ u/h)(b) \leqslant (A\text{-}\overline{\lim}\ u/h)(b) \leqslant r$$

(where these limits are defined in the obvious way) unless $b \in E_{q,r}$.

It therefore follows, for any positive rational number p, that there is a set E_p with $\mu(E_p) = 0$ such that, for all $b \notin E_p$,

$$|(d\nu/d\mu)(b) - (A\text{-}\overline{\lim}\ u/h)(b)| \leqslant p \quad \text{and} \quad |(d\nu/d\mu)(b) - (A\text{-}\underline{\lim}\ u/h)(b)| \leqslant p.$$

An obvious countability argument completes the proof.

Remark. The above argument holds with "fine limit" replaced by some other limit notion *providing* that for this limit notion $u/h \to d\nu/d\mu$ μ-a.e. For example, when considering classical harmonic functions on $\mathbf{R}^n \times \mathbf{R}^+$ that are Poisson integrals of positive functions in $L^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$ it is relatively easy to establish the existence of normal limits (cf. [16]). The above arguement shows that in order to establish the existence of non-tangential limits it suffices to verify that a.e. the non-tangential limit is zero if the normal limit is zero. As will be remarked later this is an immediate consequence of the Harnack inequalities.

2. A Local Fatou Theorem for the Helmholtz Equation

Let $\alpha > 0$. Consider the Helmholtz equation $\Delta u = 2\alpha u$ on $\mathbf{R}^n$. All potential-theoretic concepts in this section are to be understood as being relative to the potential theory associated with the operator $Lu = (1/2)\Delta u - \alpha u$. For convenience let $\sqrt{2\alpha}$ be denoted by λ.

The Martin compactification $\hat{X}$ of $X = \mathbf{R}^n$ is obtained by adjoining S^{n-1} at infinity. To show this one considers the Green function $G(x)$ with pole at 0. Then

$$\begin{aligned} G(x) &= \int_0^\infty e^{-\alpha t}(2\pi t)^{-n/2}e^{-\|x\|^2/2t}dt \\ &= \alpha^{n/2-1}\pi^{-n/2}(\lambda\|x\|)^{-\nu}K_\nu(\lambda\|x\|) \quad (\text{where } \nu = n/2 - 1) \\ &\sim \|x\|^{(1-n)/2}e^{-\lambda\|x\|} \quad ([18]\ 6.22\ (15) \text{ and } 7.23\ (1)). \end{aligned}$$

Let $G(x, y)$ denote the Green function with pole at y. Then $G(x, y) = G(x - y)$. Consequently, $G(x, y)/G(y)$ has a limit for all x as $\|y\| \to \infty$ if and only if $\|x - y\| - \|y\|$ has a limit. This is the case precisely when $y/\|y\|$ has a limit b on S^{n-1} which shows that $B = \hat{X}\setminus X$ can be identified with S^{n-1}. A basic neighbourhood U of $b \in S^{n-1}$ is

$$\{x \mid \|x\| > n, \langle x', b\rangle > 1 - 1/m\} \cup \{c \in S^{n-1} \mid \langle c, b\rangle > 1 - 1/m\},$$

where $x' = x/\|x\|$. The minimal harmonic functions, normalized to take the value 1 at 0, are the functions $K_b(x) = e^{\lambda\langle x, b\rangle}$, $b \in S^{n-1}$. Let

$$K_\mu(x) = \int K_b(x)\mu(db), \quad Kf(x) = K_{f\sigma}(x)$$

where σ is normalised Lebesgue measure and denote $K1(x)$ by $h(x)$ (the rotationally invariant solution of $\Delta u = 2\alpha u$ with $u(0) = 1$).

DEFINITION 2.1. For all $b \in S^{n-1}$ and $B > 0$ define the *admissible region* $A(b;B)$ to be

$$\{x \in \mathbf{R}^n \mid \|x - \|x\|b\| \leq B\|x\|^{1/2}\}$$

and the *truncated admissible region* $A^N(b;B)$ to be $A(b;B) \cap \{x \mid \|x\| \geq N\}$.

A function f is said *to converge admissibly at b* if for all $B > 0$ $f(x)$ has a limit as $\|x\| \to \infty$, $x \in A(b;B)$.

Remarks. 1. For admissible convergence the paraboloids

$$\{tb + y \mid t > 0, \langle y, b\rangle = 0, \|y\|^2 \leq Bt\}$$

could be used instead of the regions $A(b;B)$.

2. When $n \geq 4$ it is not hard to show that a tube of constant width about the ray determined by $b \in S^{n-1}$ is thin at b. A set *E is thin at b* if

there is a potential p that dominates K_b on E ([14, Théorème 5 p, 205], also [17]). In probabilistic terms this is equivalent to saying that for any x with probability 1 the paths of the K_b-process starting from x fail to meet E sufficiently near to b ([4, p. 455], [5, p. 4], and [8, lemma p. 140]). Consequently, such tubes cannot be used as admissible regions if one hopes to relate fine and admissible convergence. (See the proof of Theorem 2.9.)

This thinness comes from the fact that if $B_t = B(tb;r)$ then $R_{B_t}K_b(0) \approx t^k$ for $t \geq t(r)$ where $2k = 1 - n$ (where for any set E and superharmonic function u, $R_E u(x) = \inf\{v(x) | v \geq u$ on E, v superharmonic$\}$). Hence, it is not hard to construct a potential (as the sum of a series) dominating $\hat{R}_E K_b(\hat{R}_E u$ is the lower semicontinuous regularization of $R_E u$ and is superharmonic), if E is a tube of constant width "at" b. Another consequence of this estimate is that the standard "bubble" set used by Brelot and Doob [1] and also used in [10] is thin at the appropriate minimal point. From this fact comes the difficulty alluded to in the introduction.

PROPOSITION 2.2. *Let $b \in S^{n-1}$ and let $E = \cup_m E_m$, $E_m = A(b;B) \cap \{x \mid \|x\| = R_m\}$ with $R_m \uparrow +\infty$. Then E is not thin at b (i.e., if $v \geq 0$ is superharmonic and $v \geq K_b$ on E then $v \geq K_b$, which is equivalent to saying that no potential dominates K_b on E (cf., G) [17]).*

Proof. For $x \in A(b;B)$, $K_b(x) \approx e^{\lambda\|x\|}$ since

$$\|x\|\{1 - \langle x', b\rangle\} = 2\|x\|\,\|x' - b\|^2 \approx 1, \quad \text{where } x = \|x\|x'.$$

Further, the value at the origin of the harmonic function h_m on $\{x \mid \|x\| < R_m\}$ with boundary value 1_{E_m} is $h_m(0) \sim \sigma((1/R_m)E_m)/h(R_m)$ where $h = K1$ is the positive radial solution corresponding to σ.

Now

$$h(x) = \int e^{\lambda\langle x,b\rangle}\sigma(db) \sim e^{\lambda\|x\|}/\|x\|^{(n-1)/2} \quad [18, 6.15\ (2) \text{ and } 7.25\ (1)].$$

Hence, $h_m(0) \sim e^{-\lambda R_m}$ and so for some constant c, $c \leq k_m(0) \leq 1/c$ where k_m is the solution on $\mathring{B}(0;R_m)$ of the Dirichlet problem with boundary value $K_b 1_{E_m}$. The Perron-Wiener-Brelot method of solving the Dirichlet problem shows that $k_m \leq R_E K_b$ on $\mathring{B}(0;R_m)$. Therefore, if k is the limit on $\mathbf{R}^n$ of some subsequence (k_{m_n}) (which will exist by virtue of Harnack's inequalities) it follows that $k \leq R_E K_b$. Since $k > 0$, no potential dominates K_b on E; i.e., E is not thin at b.

DEFINITION 2.3. For every u satisfying $\Delta u = 2\alpha u$ in some domain of $\mathbf{R}^n$ let Du be defined (for all $t > 0$ and $x \in \mathbf{R}^n$ such that the formula is meaningful) by

$$Du(x, t) = (2\pi t)^{-n/2} e^{-\|x\|^2/2t} u(x/t) e^{-\alpha/t}.$$

Remark. For any solution $v(x, t)$ of the heat equation $\partial v/\partial t = \frac{1}{2}\Delta v$ on a domain where $t \neq 0$, the Appell transform [19]

$$Av(x, t) = (2\pi t)^{-n/2}e^{-\|x\|^2/2t}v(x/t, -1/t)$$

determines another solution. If $v(x, t) = u(x)e^{\alpha t}$ then v is a solution of the heat equation and $Du = Av$.

PROPOSITION 2.4. *Let $u > 0$ be a solution of $\Delta u = 2\alpha u$ defined on a neighbourhood of a set $A^N(b;B)$ and let $h = K1$. If $0 < B_1 < B$ there exists a constant C, independent of u, such that*

$$\frac{u(p)}{h(p)} \geq C\frac{u(q)}{h(q)}$$

for all $p, q \in A^{2N}(b;B_1)$ such that $2\|p\| = \|q\|$.

Proof. If W is the domain of Du then there exist constants M_0 and t_0 such that

$$W \supset \{(x, t) \in \mathbf{R}^n \times \mathbf{R}_+ \mid \|x - \lambda b\| \leq \sqrt{t}/M_0, \quad 0 < t \leq t_0\}$$

(for example, for sufficiently large t_0 and $B_0 < B$ take $B_0M_0\lambda^{1/2} = 2$).

Since Du satisfies the heat equation, Harnack's inequalities [13] imply that if $M_1 > M_0$ there is a constant C such that for all $t < t_0/2$,

$$(*) \qquad Du(x, 2t) \geq C\, Du(y, t)$$

provided $\|x - \lambda b\| \leq \sqrt{2t}/M_1$, $\|y - \lambda b\| \leq \sqrt{t}M_1$.

Assume $\|x\| = \|y\| = \lambda$ and let $p = x/2t$, $q = y/t$. Then $2\|p\| = \|q\|$ and substitution in the inequality (*) of these values gives $u(p)e^{-\lambda\|p\|} \geq C\,u(q)e^{-\lambda\|q\|}$. The result follows from the asymptotic behaviour of the radial function h (see the proof of 2.2).

Remark. For global positive solutions u this can be proved directly without using Harnack's inequality for the heat equation by considering the behaviour of the functions $K_b(x)/h(x)$.

LEMMA 2.5. *Let $E \subset S^{n-1}$. Assume that to each $b \in E$ there is associated the set*

$$A(b; B) \cap \{x \mid \|x\| > N\} = A^N(b; B)$$

(where N and B vary with b) and let $U = \cup_{b\in E} A^N(b; B)$.

Let $\varepsilon > 0$. Then for any given B_0 there exists a compact set D and N_0 such that

(1) $\cup_{b\in D} A^{N_0}(b; B_0) \subset U$

and

(2) $\sigma^*(E \setminus D) < \varepsilon$,

where σ^* *is the outer measure determined by* σ.

Proof. An obvious adaptation of Calderón's argument [2] using the regions $A(b; B)$ instead of cones proves the result. It is also possible to transform $\mathbf{R}^n$ into the interior of the unit ball by the map $x \to \|x\|(1 + \|x\|^2)^{-1}x$. This map transforms the region $A(b; B)$ into a region which is essentially a cone and Calderón's agument then applies.

COROLLARY 2.6. *For each* $B > 0$, *for almost all* $b \in E$ *there exists* $N = N(b, B)$ *such that* $A^N(b; B) \subset U$.

PROPOSITION 2.7. *Let* $E \subset S^{n-1}$ *and assume that for each* $b \in E$ *a truncated parabolic region* $A^N(b; B)$ *is given. Let* U *be the union of these regions. Then* U^c *is thin at almost every point of* E.

Proof. The argument of Constantinescu-Cornea used to prove Théorème 8 in [1] applies once the following result is established.

Note that if $D(x; B)$ denotes the complement of the union of all those $A(b; B)$ that do not contain x then the intersection of $\overline{D(x; B)}$ with the Martin boundary S^{n-1} is $C(x) = \{c \mid \|c - x'\| \leq B/\sqrt{\|x\|}\}$, where $x = \|x\|x'$.

LEMMA 2.8. *For all* $x \in \mathbf{R}^n$, $x \neq 0$, *there is a constant* $A = A(B) > 0$ *such that* $(K1_{C(x)})(x) \geq Ah(x)$.

Proof. It is clear that it suffices to consider x with $\|x\|$ large. If $c \in C(x)$ then $K_c(x) = e^{\lambda\langle c,x\rangle}$ and so $K_c(x)e^{-\lambda\|x\|} = e^{-\lambda\|x\|\{1-\langle c,x'\rangle\}}$. Now for large $\|x\|$,

$$1 - \langle c, x'\rangle = \|c - x'\|^2/2 \approx 1/\|x\| \quad \text{if} \quad c \in C(x).$$

Since the measure of $C(x) \sim \|x\|^{-(n-1)/2}$ and $h(x) \sim (\|x\|)^{-(n-1)/2} e^{\lambda\|x\|}$ this proves the lemma.

THEOREM 2.9. *Let* $E \subset S^{n-1}$ *and assume that for each* $b \in E$ *there is associated a region* $A^N(b; B)$. *Let* u *be a solution of the Helmholtz equation on* $U = \bigcup_{b\in E} A^N(b; B)$. *On each set* $A^N(b; B)$ *assume that* u/h *is either bounded above or below.*

Then u/h *converges admissibly at almost every point of* E.

Proof. As usual (cf. [1]) it suffices to consider the case of $u > 0$ and U connected. Since the operator $\frac{1}{2}\Delta u - \alpha u$ is strictly elliptic the points of S^{n-1} (where S^{n-1} is viewed as the Martin boundary of $\mathbf{R}^n$) at which U^c is thin can be identified with a Borel subset of the minimal points $\Delta_1(U)$ of Martin boundary of U (see the appendix of [10]). Denote this subset by S'. Let S be the subset of S' obtained by removing the exceptional sets of corollary 2.6 for an increasing unbounded sequence (B_n).

Let $B > 0$. Define an admissible system by setting, for each point d of the Martin boundary of U, $A(d) = A^N(d;B)$ for N suitably large if $d \in S$ (see Corollary 2.6) and the intersection with U of a neighbourhood of d in the Martin compactification of U otherwise.

By Theorem A.6 of [10], by the Fatou-Naïm-Doob theorem and by Theorem 1.2 it will suffice to prove that if u/h has fine limit zero at $b \in S$ then it has A-admissible limit zero at b.

Assume this to be false. Then there exists an $\varepsilon > 0$ and a sequence $(x_m) \subset A(b;B)$ such that $\|x_m\| \to \infty$ and $(u/h)(x_m) \geqslant \varepsilon$ for all m. Let $0 < B_1 < B$. Then by Proposition 2.4, $(u/h) \geqslant C\varepsilon$ on the union E of the sets

$$E_m = \{x \in A(b;B_1) \mid \|x\| = \|x_m\|/2\}.$$

By Proposition 2.2, E is not thin at b which contradicts the fact that the fine limit of u/h is zero at b.

Remark. The thinness of a tube of constant width about a half ray referred to earlier shows that the above argument cannot be applied when parabolic regions are replaced by such tubes.

3. The Fatou Theorem for Global Solutions of the Helmholtz Equation

For global solutions Theorem 2.9 can be strengthened in two ways. The first of these is stated as

THEOREM 3.1. *Let u and v be positive solutions of the equation $\Delta u = 2\alpha u$, $\alpha > 0$, defined on all of $\mathbf{R}^n$ and let μ, ν be the representing measures of u, v on the Martin boundary S^{n-1}. Then u/v converges admissibly ν-a.e. to $d\mu/d\nu$.*

Proof. Let Du and Dv be as in Definition 2.3. They are solutions of the heat equation on the upper half space $\mathbf{R}^n \times \mathbf{R}_+$ and with respect to the Gaussian kernel

$$\tilde{K}_c(x, t) = (2\pi t)^{-n/2} e^{-\|x-c\|^2/2t}$$

have the representing measures μ', ν' that correspond to μ and ν when S^{n-1} is identified with the sphere of radius $\lambda = \sqrt{2\alpha}$ in the boundary $\mathbf{R}^n \times \{0\}$ (this follows from the fact that $DK_b = \tilde{K}_{\lambda b}$).

By a theorem of Doob (Theorem 5.2 in [6] and for $n = 1$, Theorem 3.1 in [5]) Du/Dv has a parabolic limit $d\mu'/d\nu'$ ν'-a.e. Since $(Du/Dv)(x, t) = (u/v)(x/t)$ this implies the result.

THEOREM 3.2. *Let* λ *be a complex number with* Re $\lambda > 0$. *Consider solutions to* $\Delta u = \lambda^2 u$ *of the form*

$$u(x) = \int_{S^{n-1}} e^{\lambda\langle x,b\rangle}\mu(db),$$

μ *a positive measure. Let* $h(x) = \int_{S^{n-1}} e^{\lambda\langle x,b\rangle}\sigma(db)$, σ *normalized Lebesgue measure. Then,* u/h *converges admissibly to* $d\mu/d\sigma$ σ-a.e.

Proof. By standard arguments [20, Ch. XVII] it suffices to prove the following estimate.

Let $f \in L^1(S^{n-1}) = L^1(\sigma)$ and define

$$f^*(c) = \sup_{0<r\leq 2} \frac{1}{\sigma(B(c;r))}\int_{B(c;r)} |f(b)|\sigma(db).$$

Then there are constants $C = C(B)$ and $L(B)$ (independent of f) such that

$$\sup_{x\in A(c;B)} \int |f(b)||\overline{K}(x, b)|\sigma(db) \leq Cf^*(c) \quad \text{when} \quad \|x\| \geq L(B),$$

and where

$$\overline{K}(x, b) = e^{\lambda\langle x,b\rangle}/h(x),\ h(x) = \int e^{\lambda\langle x,b\rangle}\sigma(db).$$

Let $a = \operatorname{Re}\lambda > 0$.

Let $x = t\,c + y \in A(c;B)$ with $\langle c, y\rangle = 0$. Then $\|y\| \leq B\sqrt{t}$. Further, $t \leq \|x\| \leq 2t$ if $\|x\| \geq L$ for some constant $L = L(B)$. Assume from now on that $\|x\| \geq L$.

Since $|h(x)| \sim \|x\|^{-(n-1)/2}e^{a\|x\|}$ there is a constant C such that

$$\overline{K}(x, b) \leq C\,t^{(n-1)/2}e^{-at}e^{a\langle x,b\rangle}.$$

Now $\langle x, b\rangle = t\langle c, b\rangle + \langle y, b - c\rangle \leq t\langle c, b\rangle + B\sqrt{t}\|b - c\|$ implies

$$\begin{aligned}\overline{K}(x, b) &\leq C\,t^{(n-1)/2}e^{-at(1-\langle c,b\rangle)}e^{B\sqrt{t}\|b-c\|}\\ &= C\,t^{(n-1)/2}e^{-at\|b-c\|^2/2}\\ &= C\,t^{(n-1)/2}e^{-a[t\|b-c\|^2/2-B\sqrt{t}\|b-c\|]}.\end{aligned}$$

Fix $0 < \eta \leq 1$ (say $\eta = 1$). Then

$$\overline{K}(x, b) \leq \begin{cases} C\,t^{(n-1)/2}e^{-a[k^2/2-B(k+1)]} & \text{if } \dfrac{k}{\sqrt{t}} \leq \|b - c\| \leq \dfrac{k+1}{\sqrt{t}} \text{ and } \dfrac{k}{\sqrt{t}} \leq \eta\\ C\,t^{(n-1)/2}e^{-a[t\eta^2/2-2B\sqrt{t}]} & \text{for } \eta \leq \|b - c\|.\end{cases}$$

Hence,

$$\sup_{\substack{x\in A(c;B)\\ \|x\|\geq L}} \int |f(b)|\,|\overline{K}(x, b)|\sigma(db)$$

$$\leq \sum_{k=0}^{[\eta\sqrt{t}]} \int_{k/\sqrt{t}\leq\|b-c\|\leq(k+1)/\sqrt{t}} |f(b)||\overline{K}(x, b)|\sigma(db)$$

$$+ \int_{\|b-c\|\geq\eta} |f(b)|\,|\overline{K}(x, b)|\sigma(db)$$

$$\leq C \sum_{k=0}^{[\eta\sqrt{t}]} t^{(n-1)/2} e^{-a[k^2/2-B(k+1)]} \int_{\|c-b\|\leq(k+1)/\sqrt{t}} |f(b)|\sigma(db)$$

$$+ C\, t^{(n-1)/2} e^{-a[t\eta^2/2-2B\sqrt{t}]}\|f\|_1.$$

Now $\sigma(\{b \mid \|c - b\| \leq r\}) \sim r^{n-1}$ for small r and so the above expression is dominated by

$$C' \sum_{k=0}^{[\eta\sqrt{t}]} (k + 1)^{n-1} e^{-a[k^2/2-B(k+1)]} \frac{1}{\sigma(\{\|c - b\| \leq (k + 1)/\sqrt{t}\})}$$

$$\int_{\|c-b\|\leq(k+1)/\sqrt{t}} |f(b)|\sigma(db) + C'\|f\|_1$$

$$\leq \left[C' \sum_{k=0}^{\infty} (k + 1)^{n-1} e^{-a[k^2/2-B(k+1)]} + C'' \right] f^*(c)$$

$$= C f^*(c).$$

4. The Case of the Heat Equation

Let $X = \mathbf{R}^n \times (0, +\infty)$. The heat equation $\Delta u - 2(\partial u/\partial t) = 0$ determines a potential theory on X with the property that every positive solution u has the form

$$u(x) = \int (2\pi t)^{-n/2} e^{-\|x-b\|^2/2t} \mu(db),$$

where μ is a positive measure on $\mathbf{R}^n$. If $\hat{X} = \mathbf{R}^n \times \mathbf{R}_+$ then the boundary $B = \hat{X}\setminus X$ is identified with $\mathbf{R}^n$. An *admissible system* A is given by setting

$$A(b;B) = A(b) = \{(x, t) \mid \|x - b\| \leq Bt^{1/2}\}$$

for each $b \in \mathbf{R}^n$, where $B = B(b) > 0$ is arbitrary. A function f *converges admissibly* at b if for all $B > 0$ the limit of $f(x, t)$ exists as $t \to 0$, $(x, t) \in A(b;B)$.

Doob [6] proved that if u, v are any two positive solutions of the heat equation, then u/v converges admissibly ν-a.e. on B to $d\mu/d\nu$ where μ, ν are the respective representing measures. This proof uses direct estimates on the Gaussian kernel and is independent of the theory of fine convergence. An earlier proof of this result (when $n = 1$) is given in [5] where in addition Doob states that u/v converges finely to $d\mu/d\nu$ and indicates how to relate path-convergence to fine convergence. In this section it will be shown for the case $v = 1$ (and hence $d\nu = dx$) that Doob's fine convergence result implies his admissible (i.e., parabolic) result. The argument used breaks down for $v \neq 1$ because the nature of the Harnack inequality for a heat equation does not permit one to control the nearby values of a quotient by the value at one point (as is possible for a second order strictly elliptic equation).

For the potential theory on X associated with the heat equation a lower semicontinuous function u is superharmonic if (a) $u(x, t) > -\infty$ for all (x, t) and (b) for any (x, t) and cylinder $\{(y, s) \mid a_i < y_i < b_i, t_0 < s < t\} = C$ with $a_i < x_i < b_i$ the value $u(x, t) \geq \int u d\mu^c_{(x,t)}$ where $\mu^c_{(x,t)}$ is the "parabolic measure" on $\partial C \setminus (\mathbf{R}^n \times \{t\})$ that reproduces solutions of the heat equation defined on a neighbourhood of $\overline{C}$. See Doob [7] for details (where such a function is said to be "superparabolic").

Let $\mathcal{S}$ denote the convex cone of non-negative superharmonic functions on X which are finite on a dense set. Denote by $\mathcal{H}$ the convex subcone of positive solutions of the heat equation and by $\mathcal{P}$ the convex subcone of functions $u \in S$ which have the following property: $h \in \mathcal{H}$ and $h \leq u$ implies $h = 0$. Then it is well known that $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{S}$ satisfy the hypotheses (1)–(11) inclusive of [17] (cf. [7]) with

$$K_b(x, t) = \left(\frac{1}{2\pi t}\right)^{1/2} e^{-(1/2t)\|x-b\|^2}.$$

The parabolic admissible regions are related to thinness as shown in the next result.

PROPOSITION 4.1. *Let $b \in B$ and $E = \cup_m E_m$, where $E_m = \{(x, t_m) \mid \|x - b\| \leq B\sqrt{t_m}\}$ and $t_m \downarrow 0$. Then E is not thin at b (i.e., if $v \geq 0$ is superharmonic and $v \geq K_b$ on E then $v \geq K_b$, which is equivalent to saying that no potential dominates K_b on E (cf.* (G) [17])).

Proof. Let $\dot{x}_0 = (a, s) \in X$ and let A_m be the complement of the bounded cylinder

$$U_m = \mathring{B}(b;m) \times (t_m, m).$$

Then, for sufficiently large m, $\dot{x}_0 \notin A_m$.

Assume E is thin at b. This is equivalent to the existence of a potential $p \in \mathcal{P}$ with $p \geq K_b$ on E. Let p_m be the function on X obtained by replacing p on U_m by the solution of the Dirichlet problem with boundary value p

on ∂U_m, i.e., if $\dot{x} = (x, t) \in U_m$,

$$p_m(\dot{x}) = \int p d\mu^m_{(x,t)}$$

where $\mu^m_{(x,t)}$ is the parabolic measure of $\dot{x}$ determined by U_m. Then $p_m \in \mathcal{P}$, $p \geqslant p_m \geqslant p_{m+1}$ and $\lim_{m\to\infty} p_m = 0$ as it is in $\mathcal{H}$.

It is clear that $p_m \geqslant K_b$ on E_{m+1}. Hence, $R_E K_b \neq K_b$ implies

$$\lim_{m\to\infty} R_{E_m} K_b(\dot{x}_0) = 0.$$

Consequently, to prove the proposition it suffices to show that there exists a constant $C(\dot{x}_0) > 0$ with $R_{E_m} K_b(\dot{x}_0) \geqslant C(\dot{x}_0)$ for sufficiently large m. Let $t = t_m < s$ where $\dot{x}_0 = (a, s)$. Then,

$$R_{E_m} K_b(\dot{x}_0) = \int_{\|y-b\|\leqslant B\sqrt{t}} \frac{1}{[4\pi^2(s-t)t]^{n/2}} \exp\left(-\frac{1}{2}\left\{\frac{\|y-a\|^2}{s-t} + \frac{\|y-b\|^2}{t}\right\}\right) dy$$

since the probability (for the heat process associated with $\frac{1}{2}\Delta u - \partial u/\partial t$) of starting from $\dot{x}_0$ and hitting $A \subset \mathbf{R}^n \times \{t\}$ is the probability for Brownian motion of being in A at time $s - t$ after having started from a at time zero.

Now

$$\begin{aligned}\frac{\|y-a\|^2}{s-t} + \frac{\|y-b\|^2}{t} &= \frac{t\{\|y-b\|^2 + 2\langle b-y, a-b\rangle + \|a-b\|^2\} + (s-t)\|y-b\|^2}{t(s-t)} \\ &= \frac{s\|y-b\|^2}{t(s-t)} + \frac{\{2\langle b-y, a-b\rangle + \|a-b\|^2\}}{s-t}.\end{aligned}$$

Assume $0 < t < s/2$. Then $\|y - b\| \leqslant B\sqrt{t}$ implies

$$\frac{\|a-b\|^2}{2s} \leqslant \frac{\{2\langle b-y, a-b\rangle + \|a-b\|^2\}}{s-t} \leqslant \frac{3\|a-b\|^2}{s}$$

as long as $4B\sqrt{t} \leqslant \|a - b\|$.

Consequently, in computing a lower bound for $R_{E_m} K_b(\dot{x}_0)$ one can forget the corresponding term in the exponential.

Consider, for $t = t_m$,

$$I(m) = \left(\frac{1}{2\pi t}\right)^{n/2} \int_{\|y\|\leqslant B\sqrt{t}} e^{-s\|y\|^2/[2t(s-t)]}\, dy$$

for $0 < t < \min\{s/2, \|a - b\|^2/16B^2\}$.

Then

$$I(m) \geqslant \left(\frac{1}{2\pi t}\right)^{n/2} \int_{\|y\| \leqslant B\sqrt{t}} e^{-\|y\|^2/t}\, dy = c.$$

This non-thinness result is the key to proving Doob's theorem for u/v when $v = 1$.

THEOREM 4.2. *Let $u > 0$ be a solution of the heat euqation on $\mathbf{R}^n \times (0, +\infty)$. Then, for all $b \in \mathbf{R}^n$,*

$$[(\text{fine limit } u)(b) = 0] \Rightarrow [(A\text{-lim } u)(b) = 0].$$

Hence, u converges admissibly to $d\mu/dx$ a.e. on $\mathbf{R}^n$, where μ is the representing measure for u.

Proof. Assume $(y_m, t_m) \in A(b)$ with $u(y_m, t_m) \geqslant \lambda \geqslant 0$ and $t_m \downarrow 0$. Then by Harnack's inequality [10], for some $c > 0$, $u \geqslant \lambda c$ on

$$E = \bigcup_m E_m, \quad E_m = \{(x, 2t_m) \mid \|x - b\| \leqslant B\sqrt{t_m}\}.$$

Since E is not thin at b by Proposition 3.1, the fact that u converges finely to 0 implies $\lambda = 0$. The last statement is a consequence of Theorem 1.2.

Remark. This is the same argument as the one used to prove Theorem 2.9. Further, for classical harmonic functions u on $\mathbf{R}^n \times \mathbf{R}_+$ that are positive this argument shows that the nontangential limit is zero if the radial limit is zero. This is because by Harnack's inequality there is a constant c with $u(x, y) \leqslant C\, u(x_0, y)$ for all $y > 0$ if $\|x - x_0\| \leqslant ay$, $a > 0$ (cf. (3.17) on p. 63 of [15]).

These techniques can also be applied to the heat equation on

$$\mathbf{R}_+ \times \mathbf{R} = \{(x, t) \mid x > 0\}.$$

In this case the boundary $\mathbf{R} = \{0\} \times \mathbf{R} = B_1$ together with $B_2 = (0, +\infty]$ parametrizes the minimal functions K_b, where

$$K_b(x, t) = \begin{cases} (1/\sqrt{2\pi}) \dfrac{x}{(t-b)^{3/2}} e^{-x^2/2(t-b)} & \text{if } t > b \in B_1 \\ 0 & \text{if } t \leqslant b \in B_1, \end{cases}$$

and

$$K_b(x, t) = \sinh(bx) e^{\frac{b^2 t}{2}} \quad \text{if } 0 < b < +\infty \in B_2$$

$$K_b(x, t) = x \quad \text{if } b = +\infty \in B_2.$$

The positive solutions are all of the form $\int K_b(x, t)\mu(db)$ [9], and in [9] Kaufmann and Wu show that quotients of positive solutions converge admissibly in the usual way, where the admissible regions $A(b)$ at $b = (0, b) \in B_1$ are of the form

$$A(b) = \{(x, t) \mid 0 < B_1x^2 < t - b < B_2x^2\}.$$

These regions are "canonical" in view of the following result.

PROPOSITION 4.3. *Let* $0 < A < 1$ *and* $t_m \downarrow 0$. *Set* $E_m = [A\sqrt{t_m}, 1/A\sqrt{t_m}] \times \{t_m\}$. *Then* $E = \bigcup_m E_m$ *is not thin at* $(0, 0)$.

Proof. The argument of Proposition 4.1 can be imitated with U_m taken to be the rectangle $[A\sqrt{t_m}, m] \times [t_m, m]$. It will then be sufficient to verify that

$$R_{E_m}K_0(\dot{x}_0) \geq c(\dot{x}_0) > 0,$$

is independent of m, for each point $\dot{x}_0 = (a, s)$ with $s > 0$.

The probability for the heat process on $X = \mathbf{R}_+ \times \mathbf{R}$ of starting at $\dot{x}_0$ and being in $A \subset \mathbf{R} \times \{t\}$ at time $t < s$ is the probability for Brownian motion on $\mathbf{R}$ killed at zero of starting from a at time zero and being in A at time $s - t$. Therefore,

$$R_{E_m}K_0(\dot{x}_0) = \frac{1}{2\pi(s - t)^{1/2}t^{3/2}} \int_{A\sqrt{t}}^{\sqrt{t}/A} xe^{-x^2/2t}\{e^{-(x-a)^2/2(s-t)} - e^{-(x+a)^2/2(s-t)}\}\, dx$$

$$\geq c(1/t) \int_{A\sqrt{t}}^{\sqrt{t}/A} e^{-(1/2)[x^2/2t+(x-a)^2/(s-t)]}\{1 - e^{-2ax/(s-t)}\}\, dx$$

$$\text{if } 0 < t < \min\{s/2, 1\}.$$

Now

$$x^2/t + (x - a)^2/(s - t) = sx^2/t(s - t) + \{a^2 - 2xa\}/(s - t)$$

and for $0 < t < s/2$ and $x \leq \sqrt{t}/A$ the second term is essentially a constant as long as $4\sqrt{t}/A \leq |a| = a$. Therefore a lower bound may be obtained by estimating, for $0 < t < \min\{s/2, 1\}$,

$$(1/t) \int_{A\sqrt{t}}^{\sqrt{t}/A} e^{-sx^2/2t(s - t)}\{1 - e^{-2ax/(s - t)}\}\, dx$$

$$\geq (1/\sqrt{t}) \int_{A}^{1/A} e^{-y^2}\{1 - e^{-2a\sqrt{t}y/s}\}\, dy \geq C(\dot{x}_0).$$

Remarks. (1). Wu has pointed out to the authors that this result for X proves a special case of Kemper's "two-sided" parabolic convergence result for positive solutions [11]. This states (in the case under consideration) that if $u \geq 0$ is a solution of the heat equation on $\mathbf{R}_+ \times \mathbf{R} = \{(x, t) \mid x > 0\}$ then, dt – a.e. on $\{0\} \times \mathbf{R}$, $u(x, t)$ has a limit as $(x, t) \to (0, b)$ with $|t - b| < A(b)x^2$. It suffices to note that Moser's theorem [13] implies the following Harnack inequality for $u > 0$ a solution of the heat equation on $\mathbf{R}_+ \times \mathbf{R}$: let $B > 0$; then there is a constant $c > 0$ such that, for all $t > 0$,

$$\inf_{x(t) \leqslant x' \leqslant 2x(t)} u(x', 2t) \geqslant c \sup_{|s| \leqslant t} u(x(t), s) \quad \text{where } t = 2Bx^2(t).$$

If $t_0 > 0$, scaling by $(x, t) \rightsquigarrow (\lambda x, \lambda^2 t)$ for $\lambda = 1/x(t_0)$ maps

$$[x(t_0), 2x(t_0)] \times \{2t_0\} \quad \text{onto} \quad [1, 2] \times \{2B\}$$

and

$$\{x(t_0)\} \times [-t_0, t_0] \quad \text{onto} \quad \{1\} \times [-B, B].$$

If $(y_n, t_n) \to (0, 0)$ with $|t_n| \leqslant By_n^2$ and $u(y_n, t_n) \geqslant \varepsilon > 0 (\forall n)$ then in view of the Harnack inequality u cannot converge finely to 0 at (0, 0). This observation and the argument used in Theorem 4.2 prove Kemper's result in the case of $\mathbf{R}_+ \times \mathbf{R}$.

(2) Assume that the part of the kernel representing positive solutions to the heat equation on $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times \mathbf{R} = X$ corresponding to ∂X (with $t \in \mathbf{R}$) is given by

$$K_b(x, t) = (-1)\frac{\partial}{\partial x_n} W(x, t; (y', 0), s)$$

$$\text{where } b = (y', 0, s) \in \mathbf{R}^{n-1} \times \{0\} \times \mathbf{R}.$$

The proof of the proposition carries over when the admissible regions $A(b)$ are of the form

$$\{A^2(t - s) < \|x' - y'\|^2 + x_n^2 < (1/A^2)(t - s), \quad x_n > C\|x'\|\},$$

where the generic point in X is denoted by (x', x_n, t).

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