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MAXIMAL IDEAL SPACES OF U-ALGEBRAS

BY

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1. Introduction

For $n > 1$ let B_n be the open unit ball in $\mathbf{C}^n$ and S_{2n-1} be its boundary. The group of $n \times n$ unitary matrices, $U(n)$, acts on $\mathbf{C}^n$ by multiplication on the right and this action takes B_n onto itself and S_{2n-1} onto itself. A subspace, X, of $C(S_{2n-1})$ is called a U-space if for each $f \in X$ and $V \in U(n)$, $f \circ V \in X$. A U-space which is closed under multiplication is called a U-algebra. In this paper the maximal ideal spaces are found for every closed U-algebra which contains the constant functions.

Let Z represent the natural numbers $\{0, 1, 2, \ldots\}$ and Z_+, the positive natural numbers. For $p, q \in Z$, let $H_{p,q}$ be the set of restrictions to S_{2n-1} of the harmonic polynomials in z and $\bar{z}$ which are homogeneous of degree p in z and q in $\bar{z}$.

A. Nagel and W. Rudin show in [2] that if X is a closed subspace of $C(S_{2n-1})$ and

$$Y = \{(p, q) \mid X \cap H_{p,q} \neq 0\},$$

then X is a U-space if and only if X is the closure in $C(S_{2n-1})$ of the direct sum $X^* = \sum_{(p,q) \in Y} H_{p,q}$. Thus, each closed U-space is associated with a set of lattice points, $Y \subseteq Z^2$, and a dense subspace, X^*, equal to a direct sum of $H_{p,q}$ spaces. If X is a closed U-algebra, its associated set of lattice points is called an *algebra pattern*. Note that $H_{p,q}$ is a U-space and is spanned by the unitary translates of the function $z_1^p \bar{z}_2^q$.

Define $H_{p,q} \cdot H_{r,s}$ to be the subspace of $C(S_{2n-1})$ spanned by $\{f \cdot g \mid f \in H_{p,q}, g \in H_{r,s}\}$ and $(H_{p,q})^m = (H_{p,q})^{m-1} \cdot H_{p,q}$ for $m > 1$. Nagel and Rudin prove in [2] and [3] the results:

PROPOSITION 1.1. (a) *$H_{p,q} \cdot H_{r,s} \subseteq \sum H_{p+r-j,q+s-j}$ where $j = 0, 1, \ldots, \min(p + q, r + s, p + r, q + s)$.*
(b) *If $n \geq 3$, $(H_{p,q})^m = \sum H_{mp-j,mq-j}$ where $j = 0, 1, \ldots, \min(mp, mq)$.*
(c) *If $n = 2$, $(H_{p,q})^2 = \sum H_{2p-2j,2q-2j}$ where $j = 0, 1, \ldots, \min(p, q)$.*
(d) *If $n = 2$ and $m > 2$, $(H_{p,q})^m = \sum H_{mp-j,mq-j}$ where $j = 0, 2, 3, 4, \ldots, \min(mp, mq)$.*
(e) *If $n = 2$, $H_{p+r-1,q+s-1} \subseteq H_{p,q} \cdot H_{r,s}$ if and only if $ps \neq qr$.*
(f) *$H_{p,q} \cdot H_{r,o} = \sum H_{p+r-j,q-j}$ where $j = 0, 1, 2, \ldots, \min(p + r, q)$.*

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It follows that a sufficient condition for $Y \subseteq Z^2$ to be an algebra pattern is that for all (p, q) and (r, s) in Y,

$$(p + r - j, q + s - j)$$

is also in Y for each $j = 0, 1, \ldots, \min(p + q, r + s, p + r, q + s)$. For $n \geq 3$ this condition is also necessary. For example, if $\mu \in (0, 1)$, define

$$Y_\mu = \{(p, q) \mid p = q = 0 \quad \text{or } q < \mu p\}.$$

Then Y_μ satisfies the sufficient condition for being an algebra pattern. Let E_μ be the closed U-algebra with algebra pattern Y_μ.

The central step in finding the maximal ideal spaces for all U-algebras is finding the maximal ideal spaces for the algebras E_μ. This is done in Theorem 2.1. The maximal ideal spaces of a few special U-algebras are found in Theorems 3.1 through 3.4. The remaining maximal ideal spaces are found in Theorems 4.1 and 4.2.

2. The algebras E_μ

For $r \in (0, 1]$ and $\mu \in (0, 1)$ define

$$\sigma(r, \mu) = \frac{\mu\lambda^{-1} + \lambda^{-\mu}}{1 + \mu}$$

where λ is given implicitly by $r = (\mu\lambda + \lambda^\mu)/(1 + \mu)$. σ is real analytic and is decreasing in both r and μ. For $w \in \mathbf{C}^{2n}$ let

$$r(w) = \sum_{j=1}^{n} |w_j|^2 \quad \text{and} \quad s(w) = \sum_{j=1}^{n} |w_{n+j}|^2.$$

Let

$$K_\mu = \left\{ w \in \mathbf{C}^{2n} \,\middle|\, \sum_{j=1}^{n} w_j w_{n+j} = 1, \; r(w) \in (0, 1). \quad \text{and} \quad s(w) < \sigma(r(w), \mu) \right\}.$$

Note. (a) K_μ is contained in the algebraic variety $A = \{w \mid \sum_{j=1}^{n} w_j w_{n+j} = 1\}$.

(b) $K_\mu \subseteq K_{\mu'}$ if $\mu \geq \mu'$.

(c) If $\pi: \mathbf{C}^{2n} \to \mathbf{C}^n$ is the projection onto the first n coordinates, $\pi(K_\mu) = B_n \backslash \{0\}$.

(d) For $z \in B_n \backslash \{0\}$, the set $\{w \in K_\mu \mid \pi(w) = z\}$ is an $(n - 1)$-dimensional ball.

(e) For $\{w_j\} \subseteq K_{\mu'}$, $\pi(w_j) \to 0$ if and only if $|w_j| \to \infty$.

Let $S = \{w \in \mathbf{C}^{2n} \mid r(w) = 1 \text{ and } w_j = \bar{w}_{n+j} \text{ for } j \leq n\}$. Then S is the collection of all the points w in the closure of K_μ, cl (K_μ), with $r(w) = 1$.

THEOREM 2.1. *For $\mu \in (0, 1)$ the maximal ideal space of E_μ is the one point compactification of the space* cl (K_μ). *Moreover, if $f \in E_\mu$, $\hat{f}$ is homolomorphic on K_μ and $\hat{f}(z) = f(\pi(z))$ for each $z \in S$.*

Define H_μ to be the algebra of functions continuous on the one point compactification of cl (K_μ) and holomorphic on K_μ. Theorem 2.1 will be proved after establishing the following facts.

PROPOSITION 2.2. *E_μ and H_μ are isomorphic Banach algebras.*

PROPOSITION 2.3. cl (K_μ) *is polynomially convex.*

Proof of Proposition 2.2. If $w \in \mathbf{C}^{2n}$, let

$$J_w = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \\ \bar{w}_{n+1} & \bar{w}_{n+2} & \cdots & \bar{w}_{2n} \end{bmatrix}.$$

For $V \in U(n)$ define $T_V: \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ by $T_V(z) = w$ if $J_w = J_z V$. Then T_V is biholomorphic with $T_V^{-1} = T_{V^{-1}}$. Note that for each $V \in U(n)$ and $z \in \mathbf{C}^{2n}$, $r(T_V(z)) = r(z)$, $s(T_V(z)) = s(z)$, and $z \in A$ if and only if $T_V(z) \in A$. Therefore, T_V maps K_μ onto itself. If $z \in S$, $\pi(T_V(z)) = \pi(z)V$. If $z \in A$ there are $u, v \geq 0$ and $V \in U(n)$ so that

$$J_Z V = \begin{bmatrix} u & 0 & 0 & \cdots & 0 \\ u^{-1} & v & 0 & \cdots & 0 \end{bmatrix}.$$

Here $r(z) = u^2$ and $s(z) = u^{-2} + v^2$.

For $p, q \in Z_+$, $x \in [0, 1]$, and $V \in U(n)$ define $\gamma_{p,q,x,V}: S_1 \to S$ by

$$\gamma_{p,q,x,V}(\zeta) = w \quad \text{if } J_w = \begin{bmatrix} x\zeta^p & \sqrt{1-x^2}\,\zeta^q & 0 & \cdots & 0 \\ x\zeta^{-p} & \sqrt{1-x^2}\,\zeta^{-q} & 0 & \cdots & 0 \end{bmatrix} V.$$

Then $\gamma_{p,q,x,V}$ is a continuous function of ζ and $\gamma_{p,q,x,V}$ extends to be holomorphic for $\zeta \in$ cl $(B_1)\backslash\{0\}$.

LEMMA 2.4. *$K_\mu = \{w \mid w = \gamma_{p,q,x,V}(\zeta)$ for some p and q with $q/p \in (\mu, 1)$, $x \in [0, 1]$, $V \in U(n)$, and $\zeta \in B_1\backslash\{0\}\}$.*

Proof of Lemma 2.4. Clearly, if $q/p = \mu' \in (\mu, 1)$, $x \in [0, 1]$, $V \in U(n)$, and $\zeta \in B_1\backslash\{0\}$, then $\gamma_{p,q,x,V}(\zeta) \in A$. Therefore, for some $u, v \geq 0$ and $V_1 \in U(n)$, $\gamma_{p,q,x,V}(\zeta) = w$ where

$$J_w V_1 = \begin{bmatrix} u & 0 & 0 & \cdots & 0 \\ u^{-1} & v & 0 & \cdots & 0 \end{bmatrix}.$$

Here

$$r(w) = u^2 = x^2 |\zeta|^{2p} + (1 - x^2)|\zeta|^{2q}$$

and

$$s(w) = u^{-2} + v^2 = x^2|\zeta|^{-2p} + (1 - x^2)|\zeta|^{-2q}.$$

Let $\lambda = |\zeta|^{2p} \in (0, 1)$ so

$$r(w) = x^2\lambda + (1 - x^2)\lambda^{\mu'} \quad \text{and} \quad s(w) = x^2\lambda^{-1} + (1 - x^2)\lambda^{-\mu'}.$$

As x varies in $[0, 1]$ and λ varies in $(0, 1)$, $r(w)$ ranges over $(0, 1)$. Fix a value of $r(w)$ in $(0, 1)$, that is fix $r(w) = x^2\lambda + (1 - x^2)\lambda^{\mu'}$, and consider what values can be obtained for

$$s(w) = x^2\lambda^{-1} + (1 - x^2)\lambda^{-\mu'}.$$

If $x = 0$ or 1, $s(w) = r(w)^{-1}$ which corresponds to having $u = \sqrt{r(w)}$ and $v = 0$. As x ranges over $(0, 1)$, $s(w)$ reaches a maximum at

$$x = \sqrt{\frac{\mu'}{1 + \mu'}}.$$

This gives

$$r(w) = \frac{\mu'\lambda + \lambda^{\mu'}}{1 + \mu'}$$

and

$$s(w) = \frac{\mu'\lambda^{-1} + \lambda^{-\mu'}}{1 + \mu'} = \sigma(r(w), \mu') < \sigma(r(w), \mu).$$

This shows that $s(w)$ can take on any value with $u^{-2} \leq s(w) < \sigma(r(w), \mu)$. This proves that $\gamma_{p,q,x,V}(\zeta)$ is in K_μ. Moreover, $z \in K_\mu$ only if there are $u, v \geq 0$ and $V_2 \in U(n)$ with

$$J_z V_2 = \begin{bmatrix} u & 0 & 0 & \cdots & 0 \\ u^{-1} & v & 0 & \cdots & 0 \end{bmatrix}.$$

Selecting p, q, x, V, V_1 and ζ so $T_{V_1}(\gamma_{p,q,x,V}(\zeta)) = T_{V_2}(z)$ shows $z = \gamma_{p,q,x,VV_1V_2^{-1}}(\zeta)$. This proves the lemma.

Note that if $q/p = \mu$ and $x = \sqrt{\mu/(1 + \mu)}$, $\gamma_{p,q,x,V}(\zeta)$ is in the boundary of K_μ.

$\pi\colon S \to S_{2n-1}$ is a homeomorphism so let $h\colon S_{2n-1} \to S$ be its inverse. Define

$$G\colon H_\mu \to C(S_{2n-1})$$

by $G(f) = f \circ h$. G is an algebra homomorphism and for $V \in U(n)$, $G(f) \circ V = f \circ T_V \circ h = G(f \circ T_V)$ so $G(H_\mu)$ is a closed U-algebra. Suppose $z_1^r \bar{z}_2^s = G(f)$ for $r, s \in Z$ and $f \in H_\mu$. Then, by Lemma 2.4, whenever $q/p \in [\mu, 1)$, $x \in [0, 1]$, and $V \in U(n)$, $f \circ \gamma_{p,q,x,V}$ is continuous on $\operatorname{cl}(B_1)$ and holomorphic on $B_1 \backslash \{0\}$. But then $f \circ \gamma_{p,q,x,V}$ is holomorphic on all of B_1 and $f \circ \gamma_{p,q,x,V}(0) = f(\infty)$ is independent of p, q, x, and V. One gets

$$f \circ \gamma_{p,q,x,V}(\zeta) = f \circ h \circ \pi \circ \gamma_{p,q,x,V}(\zeta)$$

when $|\zeta| = 1$ so $z_1^r \bar{z}_2^s \circ \pi \circ \gamma_{p,q,x,V}$ extends to be holomorphic on B_1. If V has entries $(\alpha_{j,k})$, then for $|\zeta| = 1$,

$$z_1^r \bar{z}_2^s \circ \pi \circ \gamma_{p,q,x,V}(\zeta) = (\alpha_{1,1} x\zeta^p + \alpha_{2,1}\sqrt{1-x^2}\zeta^q)^r$$

$$\overline{(\alpha_{1,2} x\zeta^p + \alpha_{2,2}\sqrt{1-x^2}\zeta^q)^s} = (\alpha_{1,1} x\zeta^p + \alpha_{2,1}\sqrt{1-x^2}\zeta^q)^r$$

$$(\overline{\alpha_{1,2}} x\zeta^{-p} + \overline{\alpha_{2,2}}\sqrt{1-x^2}\zeta^{-q})^s = \zeta^{qr-ps}(\alpha_{1,1} x\zeta^{p-q} + \alpha_{2,1}\sqrt{1-x^2})^r(\overline{\alpha_{1,2}}x + \overline{\alpha_{2,2}}\sqrt{1-x^2}\zeta^{p-q})^s$$

which extends to be holomorphic on B_1 only if $s/r \le q/p$. Moreover, $z_1^r \bar{z}_2^s \circ \pi \circ \gamma_{p,q,x,V}(0)$ is independent of x and V only if $r = s = 0$ or $s/r < q/p$. This can hold for each choice of p and q with $q/p \in [\mu, 1)$ only if $r = s = 0$ or $s < \mu r$. Thus, $G(H_\mu) \cap H_{p,q} \ne (0)$ if and only if $r = s = 0$ or $s < \mu r$ so $G(H_\mu) = E_\mu$.

If $z \in K_\mu$, $z = \gamma_{p,q,x,V}(\zeta)$ for some p, q, x, V, and ζ. If $f \in H_\mu$, $f \circ \gamma_{p,q,x,V}$ is holomorphic on B_1 so

$$|f(z)| = |f \circ \gamma_{p,q,x,V}(\zeta)| \le \sup_{\zeta \in S_1} |f \circ \gamma_{p,q,x,V}(\zeta)| \le \sup_{w \in S} |f(w)|.$$

Thus,

$$\|f\| = \sup_{w \in K_\mu} |f(w)| = \sup_{w \in S} |f(w)| = \sup_{z \in S_{2n-1}} |G(f(z))| = \|G(f)\|,$$

so G is a Banach algebra isomorphism.

Proof of Proposition 2.3. Since $K_\mu = \bigcap_{\mu' < \mu} K_{\mu'}$ it will be sufficient to prove Proposition 2.3 for rational $\mu = q/p$. Let $w \in \mathbf{C}^{2n} \backslash \mathrm{cl}\,(K_\mu)$. Let $P_1(z) = \sum_{j=1}^n z_j z_{n+j} - 1$. If $|P_1(w)| > 0$, then

$$|P_1(w)| > \sup_{z \in \mathrm{cl}\,(K_\mu)} |P_1(z)| = 0.$$

If $r(w) > 1$, let $P_2(z) = \sum_{j=1}^n z_j \bar{w}_j$. Then

$$|P_2(w)| = r(w) > \sqrt{r(w)} = \sup_{z \in \mathrm{cl}\,(K_\mu)} |P_2(z)|.$$

So assume $P_1(w) = 0$ and $r(w) \le 1$ but $s(w) > \sigma(r(w), \mu)$. Select $u, v > 0$ and $V_1 \in U(n)$ so

$$J_w V_1 = \begin{bmatrix} u & 0 & 0 & \cdots & 0 \\ u^{-1} & v & 0 & \cdots & 0 \end{bmatrix}$$

where $r(w) = u^2$ and $s(w) = u^{-2} + v^2$. For some $t < v$, $u^{-2} + t^2 = \sigma(r(w), \mu)$ so there is a $V_2 \in U(n)$ and $\zeta \in \mathrm{cl}\,(B_1)$ so

$$\begin{bmatrix} u & 0 & 0 & \cdots & 0 \\ u^{-1} & t & 0 & \cdots & 0 \end{bmatrix} V_2 = \begin{bmatrix} \sqrt{\dfrac{\mu}{1+u}}\,\zeta^p & \sqrt{\dfrac{1}{1+\mu}}\,\zeta^q & 0 & \cdots & 0 \\ \sqrt{\dfrac{\mu}{1+\mu}}\,\overline{\zeta^{-p}} & \sqrt{\dfrac{1}{1+\mu}}\,\overline{\zeta^{-q}} & 0 & \cdots & 0 \end{bmatrix}.$$

Let $P_3(z) = z_2^p z_{n+1}^q$. For $\lambda \in \mathrm{cl}\,(B_1)$ define w_λ by

$$J_{w_\lambda} V_1 = \begin{bmatrix} u & 0 & 0 & \cdots & 0 \\ u^{-1} & \bar{\lambda} v & 0 & \cdots & 0 \end{bmatrix}.$$

Find $\alpha \in S_1$ so that

$$|P_3 \circ V_2 \circ V_1(w_\alpha)| = \sup_{\lambda \in B_1} |P_3 \circ V_2 \circ V_1(w_\lambda)|.$$

If $T_{V_3}(w) = w_\alpha$,

$$\begin{aligned} |P_3 \circ V_2 \circ V_1 \circ V_3(w)| &\geq \sup_{\lambda \in B_1} |P_3 \circ V_2 \circ V_1(w_\lambda)| \\ &\geq |P_3 \circ V_2 \circ V_1(w_{t/v})| = \left(\sqrt{\frac{1}{1+\mu}}\right)^p \left(\sqrt{\frac{\mu}{1+\mu}}\right)^q \\ &= \sup_{z \in S} |z_2^p z_{n+1}^q| = \sup_{z \in \mathrm{cl}\,(K_\mu)} |z_2^p z_{n+1}^q| \\ &= \sup_{z \in \mathrm{cl}\,(K_\mu)} |P_3 \circ V_2 \circ V_1 \circ V_3(z)|. \end{aligned}$$

Therefore, if $w \notin \mathrm{cl}\,(K_\mu)$, there is a polynomial with modulus larger at w than any point of $\mathrm{cl}\,(K_\mu)$. Hence $\mathrm{cl}\,(K_\mu)$ is polynomially convex.

Proof of Theorem 2.1. By Proposition 2.2, it is enough to show that the one point compactification of $\mathrm{cl}\,(K_\mu)$ is the maximal ideal space of H_μ and that all multiplicative linear functionals of H_μ are point evaluations.

LEMMA 2.5. *H_μ is the closure of the linear span of the monomials*

$$\left\{ w^a \mid a \in Z^{2n},\ \sum_{j=1}^n a_{n+j} < \mu \sum_{j=1}^n a_j \quad or \quad a = 0 \right\}.$$

Proof of Lemma 2.5. $w^a \in H_\mu$ if and only if

$$G(w^a) = \prod_{j=1}^n z_j^{a_j} \bar{z}_j^{a_{n+j}} \in E_\mu.$$

If $p = \sum_{j=1}^n a_j$ and $q = \sum_{j=1}^n a_{n+j}$,

$$G(w^a) \in H_{p,0} \cdot H_{0,q} \subseteq \sum_{j=0}^q H_{p-j,q-j} \subseteq E_\mu \quad \text{if } q < \mu p.$$

E_μ is the closed linear span of the $H_{p,q}$ spaces with $q < \mu p$ or $p = q = 0$ and the $H_{p,q}$ spaces are linear spans of the functions $\{z_1^p \bar{z}_2^q \circ V \mid V \in U(n)\}$. If V has entries $(\alpha_{j,k})$,

$$z_1^p \bar{z}_2^q \circ V = \left(\sum_{k=1}^n \alpha_{k,1} z_k\right)^p \left(\sum_{m=1}^n \overline{\alpha_{m,2}} \bar{z}_m\right)^q$$

which shows that $z_1^p \bar{z}_2^q \circ V$ is a sum of monomials of degree p in z and q in $\bar{z}$. This proves the lemma.

Let ϕ be a multiplicative linear functional on H_μ. By Lemma 2.5, ϕ is determined by the values it assigns to the monomials w^a where $\sum_{j=1}^n a_{n+j} < \mu \sum_{j=1}^n a_j$. For $j \le n$, let $z_j = \phi(w_j)$.

Suppose $z_j = 0$ for each $j \le n$. Choose $w^a \in H_\mu$ with $a \ne 0$. Since $\sum_{j=1}^n a_{n+j} < \sum_{j=1}^n a_j$, for some $m > 0$,

$$\frac{m \sum_{j=1}^n a_{n+j}}{m \sum_{j=1}^n a_j - 1} < \mu.$$

Thus, for some $k \le$ n, $w^{ma}/w_k \in H_\mu$ and $\phi(w^a)^m = \phi(w^{ma}) = \phi(w_k)\phi(w^{ma}/w_k) = 0$. It follows that ϕ is point evaluation at the point at infinity.

Suppose for some $k \le n$ $z_k \ne 0$. Choose m so that $1/m < \mu$. For $j \le n$, define

$$z_{n+j} = \frac{\phi(w_k^m w_{n+j})}{z_k^m}.$$

If $w^a \in H_\mu$, let $p = \sum_{j=1}^n a_j$ and $q = \sum_{j=1}^n a_{n+j}$. Then

$$\begin{aligned} \phi(w^a) &= \phi(w_k)^{-mq}\phi(w_k)^{mq}\phi(w^a) \\ &= z_k^{-mq}\phi\left(\prod_{j=1}^n w_j^{a_j}\right)\phi\left(\prod_{j=1}^n w_k^{ma_{n+j}} w_{n+j}^{a_{n+j}}\right) \\ &= z_k^{-mq} \prod_{j=1}^n z_j^{a_j} \prod_{j=1}^n [\phi(w_k^m w_{n+j})^{a_{n+j}}] \\ &= z^a. \end{aligned}$$

Therefore, ϕ is point evaluation at the point $z \in \mathbf{C}^{2n}$. It follows from Proposition 2.3 that $z \in \mathrm{cl}\,(K_\mu)$.

Since functions in H_μ separate points in K_μ, distinct point evaluations on the compactification of cl (K_μ) give distinct multiplicative linear functionals on H_μ. This proves Theorem 2.1.

3. Special cases

Nagel and Rudin show that there are four types of U-algebras which are self-adjoint (that is, $\bar{f} \in X$ whenever $f \in X$.):

(a) $\mathbf{C}^1 = H_{0,0}$;

(b) $A_k = \{f \in C(S_{2n-1}) \mid f(e^{2\pi i/k}z) = f(z)$ for each $z \in S_{2n-1}\}$ where $k \in Z_+$;

(c) $D_1 = \{f \in C(S_{2n-1}) \mid f(\alpha z) = f(z)$ for $z \in S_{2n-1}$ and $\alpha \in S_1\}$;

(d) if $n = 2$, $D_2 = \{f \in C(S_{2n-1}) \mid f(z) = f(w)$ for z, $w \in S_{2n-1}$ with $\langle z, w\rangle = 0\}$.

THEOREM 3.1. *The maximal ideal space*

(a) *For* $\mathbf{C}^1$ *is a one point space,*

(b) *For* A_k *is the lens space* [4, p. 88] *consisting of* S_{2n-1} *where* $z, w \in S_{2n-1}$ *are identified whenever* $e^{2\pi i/k}z = w$.

(c) *For* D_1 *is* $\mathbf{C}P(n-1)$ [4, p. 146].

(d) *For* D_2 *is* $\mathbf{R}P(2)$ [4, p. 146].

Proof of Theorem 3.1. Each algebra given is isomorphic to the set of continuous functions on the set claimed to be its maximal ideal space. Since each set is compact and Hausdorff and the continuous functions on each set separate points on the set, the theorem follows.

Nagel and Rudin also show that if a U-algebra is not self adjoint, then its algebra pattern, Y, either satisfies $q \leq p$ for all $(p, q) \in Y$ or $p \leq q$ for all $(p, q) \in Y$. Moreover, $\{(p, q) \in Y \mid p = q\}$ is $\{(0, 0)\}$, $\{(p, p) \mid p \in Z\}$, or in the case $n = 2$ only $\{(2p, 2p) \mid p \in Z\}$. If X is a U-algebra with algebra pattern Y such that $(p, q) \in Y$ only if $p \leq q$, then X is isomorphic to $\bar{X} = \{f \mid \bar{f} \in X\}$ and $\bar{X}$ has an algebra pattern $\bar{Y}$ with $(p, q) \in \bar{Y}$ only if $q \leq p$. Therefore, when finding the maximal ideal spaces for the remaining U-algebras, it will be enough to consider those whose algebra pattern satisfies $(p, q) \in Y$ only if $q \leq p$.

For $\mu \in [0, 1]$ and $c \in Z$ let

$$Y_{\mu,c,0} = \{(p, q) \mid p = q = 0 \quad \text{or} \quad q < \mu p \quad \text{and} \quad c \text{ divides } p - q\}.$$

If μ is rational with $\mu = s/r$ where s and r are relatively prime positive integers and d is a positive integer, let

$$Y_{\mu,c,d} = Y_{\mu,c,0} \cup \{(mdr, mds) \mid m \in Z_+\}.$$

Then $Y_{\mu,c,d}$ is an algebra pattern provided that

(a) $c = d$ when $\mu = 0$,

(b) $d = 0$ when μ is irrational,

(c) c divides $d(r - s)$,

(d) $d = 0, 1$, or (if $n = 2$) $d = 2$ when $\mu = 1$.

Let $E_{\mu,c,d}$ be the closed U-algebra with algebra pattern $Y_{\mu,c,d}$. Let $L_{\mu,c,d}$ be the maximal ideal space of $E_{\mu,c,d}$. If $\mu \in (0, 1)$, $E_{\mu,1,0} = E_\mu$ discussed in Section 2. Then

$$E_{0,1,1} = \{f \in C(S_{2n-1}) \mid f \text{ extends to be continuous on cl}\,(B_n) \text{ and holomorphic on } B_n\}$$

and

$$E_{1,1,0} = \{f \in C(S_{2n-1}) \mid f \text{ extends to be continuous on cl}\,(B_n) \text{ such that for each } z \in S_{2n-1}, f(\alpha z) \text{ is holomorphic in } \alpha \in B_1\}.$$

THEOREM 3.2. *$E_{0,1,1}$ and $E_{1,1,0}$ both have* cl (B_n) *for a maximal ideal space.*

The case of $E_{0,1,1}$ is a classical result in function theory while the case of $E_{1,1,0}$ is a lesser known result of K. Hoffman and I. Singer [1].

THEOREM 3.3. *The maximal ideal space of $E_{1,1,1}$ is* $L_{1,1,1} = \{(z, [w]) \in \mathrm{cl}\,(B_n) \times \mathbf{C}P(n-1) \mid w \in S_{2n-1}$ *and* $z = \lambda w$ *for some* $\lambda \in \mathbf{C}^1\}$.

Proof of Theorem 3.3. Each $f \in E_{1,1,1}$ can be written as a sum $g + h$ where $g \in D_1$ (as in Theorem 3.1) and $h \in E_{1,1,0}$ where h extends to cl (B_n) with $h(0) = 0$. Let ϕ be a multiplicative linear functional on $E_{1,1,1}$. Then ϕ restricts to be a multiplicative linear functional on both D_1 and $E_{1,1,0}$. By Theorems 3.1 and 3.2 there are $w_0 \in S_{2n-1}$ and $z_0 \in \mathrm{cl}\,(B_n)$ so that $\phi(g) = g(w_0)$ and $\phi(h) = h(z_0)$ for all $g \in D_1$ and $h \in E_{1,1,0}$. Therefore, for $f \in E_{1,1,1}$ with $f = g + h$, $\phi(f) = \phi(g) + \phi(h) = g(w_0) + h(z_0)$. Suppose $z_0 = tw$ for $t \in \mathbf{C}^1$ and $w \in S_{2n-1}$. For all $g \in D_1$ and $h \in E_{1,1,0}$,

$$g \cdot h \in E_{1,1,0} \quad \text{and} \quad (g \cdot h)(z_0) = g(w)h(z_0).$$

Thus $g(w)h(z_0) = (g \cdot h)(z_0) = \phi(gh) = \phi(g)\phi(h) = g(w_0)h(z_0)$. It follows that $[w] = [w_0]$ in $\mathbf{C}P(n-1)$ so $z_0 = \lambda w_0$ for some $\lambda \in \mathbf{C}^1$. This proves the theorem.

For $\mu \in (0, 1)$, $L_{\mu,1,0}$ is the one point compactification of cl (K_μ) which is the maximal ideal space for $E_{\mu,1,0}$ by Theorem 2.1. The maximal ideal space of $E_{\mu,1,1}$ for rational μ is similar to $L_{\mu,1,0}$ except that $L_{\mu,1,1}$ is a more complicated compactification of cl (K_μ). If r and s are relatively prime positive integers with $\mu = s/r$, let L be the compactification of cl (K_μ) created by attaching the set $U(n) \times [0, 1]$ in such a way that a neighborhood base of $(W, y) \in U(n) \times [0, 1]$ consists of the sets

$$\{\gamma_{r,s,x,V}(\zeta) \mid \|W - V\| < \varepsilon, |y - x| < \rightarrow, |\zeta| < \varepsilon\}$$
$$\cup \{(V, x) \in U(n) \times [0, 1] \mid \|V - W\| < \varepsilon, |y - x| < \varepsilon\}$$

for small $\varepsilon > 0$ where $\gamma_{r,s,x,V}$ is the same as in Lemma 2.4.

THEOREM 3.4. *If $\mu = s/r$, the maximal ideal space of $E_{\mu,1,1}(L_{\mu,1,1})$ is the set L where two points (W, y) and $(V, x) \in U(n) \times [0, 1]$ are identified whenever*

$$\begin{bmatrix} 0 & \sqrt{1-y^2} & 0 & \cdots & 0 \\ y & 0 & 0 & \cdots & 0 \end{bmatrix} W = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{1-x^2} & 0 & \cdots & 0 \\ x & 0 & 0 & \cdots & 0 \end{bmatrix} V,$$

where $\omega_1^r = \bar{\omega}_2^{-s}$.

Proof of Theorem 3.4. Proposition 2.2 shows that $E_{\mu,1,0}$ is isomorphic to the algebra of functions continuous on the one point compactification of cl (K_μ) and holomorphic on K_μ. Similarly, $E_{\mu,1,1}$ is isomorphic to the algebra of functions continuous on L and holomorphic on K_μ. Call this algebra $H_{\mu,1,1}$.

Then $H_{\mu,1,1}$ is the closed linear span of $\{w^a \mid \sum_{j=1}^n a_{n+j} \leq \mu \sum_{j=1}^n a_j\}$. As in Theorem 2.1, if ϕ is a homomorphism of $H_{\mu,1,1}$ into $\mathbf{C}^1$ such that $\phi(w_j) \neq 0$ for some $j \leq n$, then ϕ is point evaluation at some point of cl (K_μ). If $\phi(w_j) = 0$ for all $j \leq n$, then $\phi(w^a) = 0$ whenever $\sum_{j=1}^n a_{n+j} < \mu \sum_{j=1}^n a_j$. On the other hand, using the monomials w^a with $\sum_{j=1}^n a_{n+j} = \mu \sum_{j=1}^n a_j$, one gets a set of relations

$$\phi(w^a)\phi(w^b) = \phi(w^c)\phi(w^d) \quad \text{when } a + b = c + d.$$

Solving a set of difference equation arising from these relations yields a point $z \in \mathbf{C}^{2n}$ such that $\phi(w^a) = z^a$ when $\sum_{j=1}^n a_{n+j} = \mu \sum_{j=1}^n a_j$. It follows that such a homomorphism arises as

$$\phi(f) = \lim_{\zeta \to 0} f \circ \gamma_{r,s,x,V}(\zeta)$$

for proper choice of x and V and that the space of all such ϕ is the continuous image of $U(n) \times [0, 1]$. The functions in $H_{\mu,1,1}$ do not separate points of $U(n) \times [0, 1] \subseteq L$ so some of these points must be identified to give the maximal ideal space of $H_{\mu,1,1}$. This yields $L_{\mu,1,1}$.

Notice that the maps $\{T_V \mid V \in U(n)\}$ which map cl (K_μ) onto itself extend continuously to $L_{\mu,1,1}$ by mapping $(W, x) \in U(n) \times [0, 1]$ to $T_V((W, x)) = (WV, x)$.

4. Final reductions

In Section 3 the maximal ideal spaces were given for each closed self-adjoint U-algebra and for the algebras $E_{\mu,1,1}$ for $\mu \in [0, 1]$. In this section the maximal ideal space $L_{\mu,c,d}$ is found for each $E_{\mu,c,d}$ (Theorem 4.1) and then it is shown that each remaining U-algebra has a maximal ideal space equivalent to one of the $E_{\mu,c,d}$ algebras (Theorem 4.2).

THEOREM 4.1. *Let* $\mu \in (0, 1)$, $c, d \in Z_+$, $\alpha = e^{2\pi i/c}$, *and* $\beta = e^{2\pi i/d}$.

(a) $L_{0,c,c}$ *is* cl (B_n) *with* $z, w \in$ cl (B_n) *identified if* $z = \alpha w$.

(b) $L_{\mu,c,0}$ *is* $L_{\mu,1,0}$ *with* $z, w \in$ cl (K_μ) *identified whenever* $z = T_{\alpha I}(w)$.

(c) *If* μ *is rational,* $L_{\mu,c,d}$ *is* $L_{\mu,1,1}$ *with* $z, w \in$ cl (K_μ) *identified whenever* $z = T_{\alpha I}(w)$ *and* $z, w \in U(n) \times [0, 1]$ *identified whenever* $z = T_{\beta I}(w)$.

(d) $L_{1,c,1}$ *is* $L_{1,1,1}$ *with* $(z_1, [w_1])$ *and* $(z_2, [w_2])$ *identified if* $[w_1] = [w_2]$ *and* $0 \neq z_1 = \alpha z_2$.

(e) *If* $n = 2$, $L_{1,c,2}$ *is* $L_{1,c,1}$ *with* $(0, [w_1])$ *and* $(0, [w_2])$ *identified if* $\langle w_1, w_2 \rangle = 0$.

Proof of Theorem 4.1. Consider the space $E_{\mu,c,d}$. If μ is rational let $X = E_{\mu,1,1}$ and if μ is irrational let $X = E_{\mu,1,0}$. The theorem will follow when it is shown that each homomorphism of $E_{\mu,c,d}$ into $\mathbf{C}^1$ is the restriction of a homomorphism from X into $\mathbf{C}^1$ because then one only has to identify homorphisms on X which are equal when restricted to $E_{\mu,c,d}$. Since X^* (see Section 1) is dense in X, it is enough to show that each homomorphism on $E_{\mu,c,d}$ extends to be a bounded homomorphism on X^*.

Let ϕ be a homomorphism from $E_{\mu,c,d}$ into $\mathbf{C}^1$. If $g \in H_{1,0}$, $g^c \in H_{c,0} \subseteq E_{\mu,c,d}$.

Case 1. Suppose $\phi(g^c) \neq 0$ for some $g \in H_{1,0}$. Let x be a cth root of $\phi(g^c)$. If $f \in H_{p,q} \subseteq X$, let $r \equiv q - p \pmod{c}$. Then $g^r f \in E_{\mu,c,d}$. Define $\phi^*(f) = \phi(g^r f)/x^r$. Then ϕ^* is well defined since $r = s + mc$ implies

$$\frac{\phi(g^r f)}{x^r} = \frac{\phi(g^s g^{mc} f)}{x^s x^{mc}} = \frac{\phi(g^{mc})}{x^{mc}} \frac{\phi(g^s f)}{x^s} = \frac{\phi(g^s f)}{x^s}.$$

One may extend ϕ^* by linearity to X^*. Since

$$\phi^*\left(\sum f_j \cdot \sum h_k\right) = \sum\sum \phi^*(f_j h_k) = \sum\sum \frac{\phi(g^{r_j + s_k} f_j h_k)}{x^{r_j + s_k}} = \sum\sum \phi^*(f_j)\phi^*(h_k)$$
$$= \phi^*\left(\sum f_j\right)\phi^*\left(\sum h_k\right)$$

where the f_j's and h_k's lie in $H_{p,q}$ spaces, ϕ^* is multiplicative.

To show that ϕ^* is bounded on X^* let $f \in X^*$ with $\|f\| = 1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{cd}$ be the (cd)th roots of 1 and $V_k = \alpha_k I \in U(n)$. Then for all $m \in Z_+$,

$$\sum_{k=1}^{cd} f^m \circ V_k \in E_{\mu,c,d} \quad \text{and} \quad \left\| \sum_{k=1}^{cd} f^m \circ V_k \right\| \leq \sum_{k=1}^{cd} \left\| f^m \circ V_k \right\| = cd.$$

Thus,

$$\left| \sum_{k=1}^{cd} \phi^*(f \circ V_k)^m \right| \leq \left| \phi\left(\sum_{k=1}^{cd} f^m \circ V_k \right) \right| \leq cd.$$

But this can happen for all m only if $|\phi^*(f \circ V_k)| \leq 1$ for all k which proves that ϕ^* is bounded. ϕ^* extends ϕ to X as desired.

Case 2. Suppose $\phi(g^c) = 0$ for all $g \in H_{1,0}$. Then $\phi(g) = 0$ for all $g \in H_{p,q} \subseteq E_{\mu,c,0}$ with $p > 0$. For $p > 0$ and $f \in H_{p,q} \subseteq E_{\mu,1,0}$ let $\phi^*(f) = 0$. If $d = 0$, this extends ϕ to ϕ^* on X. If $d > 0$ and $\mu = s/r$ where r and s are relatively prime, for all $g \in H_{r,s}$, $g^d \in E_{\mu,c,d}$. If $\phi(g^d) = 0$ for all $g \in H_{r,s}$, then ϕ is the trivial homomorphism and easily extends to X. If $g \in H_{r,s}$ with $\phi(g^d) \neq 0$, let x be a dth root of $\phi(g^d)$. Then ϕ^* extends to $H_{mr,ms}$ for $m \in Z_+$ as in Case 1 and ϕ^* extends by linearity to a multiplicative linear functional on X in the same way as in Case 1. This completes the proof.

THEOREM 4.2. *Let X be a closed U-algebra with algebra pattern Y containing* (0, 0). *Let*

$$\sup\{q/p \mid (p, q) \in Y, p \neq 0\} \leq 1.$$

Let $c = \text{g.c.d. } \{p - q \mid (p, q) \in Y\}$. *If* $\{(p, q) \in Y \mid q = \mu p\}$ *is empty, let* $d = 0$. *Otherwise let* $d = \text{g.c.d. } \{m \mid (mr, ms) \in Y\}$ *where* r *and* s *are relatively prime with* $\mu = s/r$. *Then the maximal ideal space of* X *is* $L_{\mu,c,d}$.

The idea of the proof will be similar to that of Theorem 4.1 except that less is known about Y, making the proof more technical.

LEMMA 4.3. (a) *For some* $N_0 \in Z$, $(mc, 0) \in Y$ *for all* $m \geq N_0$.

(b) *If* $\mu > 0$, *for each* $q \in Z$ *there is an* $N_q \in Z$ *such that* $(mc + q - j, q - j) \in Y$ *whenever* $m \geq N_q$ *and* $0 \leq j \leq q$.

(c) *If* $(p, q) \in Z^2$ *with* $q/p < \mu$ *such that* c *divides* $p - q$, *there exists* m, $k \in Z_+$ *with* $k \geq N_0$ *such that* $(mp - kc, mq) \in Y$.

Proof of Lemma 4.3. It follows from Proposition 1.1 (b)–(d) that g.c.d. $\{m \mid (m, 0) \in Y\} = c$. Part (a) follows.

If $\mu > 0$, there is a $(p, q) \in Y$ with $q > 0$. Proposition 1.1 (b)–(d) then shows that there are $(p, q) \in Y$ with arbitrarily large q. Since $(mc, 0) \in Y$ for all large m, (b) follows from Proposition 1.1 (f).

If $(p, q) \in Z^2$ where $q/p < \mu$ and c divides $p - q$, choose $(a, b) \in Y$ with $b/a > q/p$. Choose m^* such that $m^*(bp - qa) > 2N_0c$ and let $m = m^*b$. By Proposition 1.1 (b)(d)(f), for all $j \leq mq$,

$$(m^*qa + N_0c - j, m^*qb - j) = (m^*bp + N_0c + m^*(qa - bp) - j,\ m^*bq - j) \in Y.$$

So $(mp - kc - j,\ mq - j) \in Y$ where $kc = m^*(bp - qa) - N_0c \geq N_0c$. This gives (c).

LEMMA 4.4. *Let* ϕ *be a homomorphism from* X *to* $\mathbf{C}^1$. *Then* ϕ *is the restriction of a unique homomorphism* $\phi^*: E^*_{\mu,c,d} \to \mathbf{C}^1$.

Proof of Lemma 4.4. Let N_q be defined as in Lemma 4.3.

Case 1. Suppose that for some $m \geq N_0$ there is a $g \in H_{mc,0}$ such that $\phi(g) \neq 0$. Let $(p, q) \in Y_{\mu,c,d}$ and select $a \in Z_+$ such that $p + amc \geq N_q + q$. If $f \in H_{p,q}$, Proposition 1.1 (a) shows that $g^a f \in H_{amc,0} \cdot H_{p,q} \subseteq \sum_{j=0}^{q} H_{p+amc-j,q-j}$ which is contained in X by Lemma 4.3 (b). Define $\phi^*(f) = \phi(g^a f)/\phi(g)^a$. As in Theorem 4.1, ϕ^* extends by linearity to a homomorphism of $E^*_{\mu,c,d}$. Since $\phi^*(f) = \phi(g^a f)/\phi(g)^a$ must hold for all ϕ^* that extend ϕ, ϕ^* is uniquely determined.

Case 2. Suppose that for all $m \geq N_0$ and $g \in H_{mc,0}$, $\phi(g) = 0$. If $(p, q) \in Y_{\mu,c,d}$, $q/p < \mu$, and $f \in H_{p,q}$, define $\phi^*(f) = 0$. If $d = 0$, this defines ϕ^* on each $H_{p,q} \subseteq E^*_{\mu,c,d}$. If $\mu = 1$, then $d = 0$, 1, or 2. In each of these cases $(p, p) \in Y$ if and only if $(p, p) \in Y_{\mu,c,d}$ so ϕ is already defined on $H_{p,p}$ so set $\phi^* = \phi$ on $H_{p,p}$. If $\mu < 1$ and $d \neq 0$, $\mu = s/r$. The set $W = \{m \in Z \mid (mdr, mds) \in Y\}$ is closed under addition, and g.c.d. $W = 1$. Thus, there is an $N \in Z_+$ such that for all $m \geq N$, $(mdr, mds) \in Y$. Let $(a, b) \in Y_{\mu,c,d}$ with $b/a = \mu$. Then $m \geq Nrd/a$

implies $(ma, mb) \in Y$. Proposition 1.1 (b)(d)(e) shows that for all $m \geq 3Nrd/a$, $(H_{a,b})^m \subseteq X$. If $f \in H_{a,b}$ and $m \geq 3Nrd/a$, define $\phi^*(f) = 0$ if $\phi(f^m) = 0$ and

$$\phi^*(f) = \phi(f^{m+1})/\phi(f^m) \quad \text{otherwise,}$$

Extend ϕ^* by linearity. Again, ϕ^* becomes a multiplicative linear functional on $E^*_{\mu,c,d}$. The way ϕ^* was defined on $H_{a,b}$ with $b/a = \mu$ was clearly forced. If $H_{a,b} \in E_{\mu,c,0}$, then by Lemma 4.3 (c) there exist $m, k \in Z_+$ such that $(kc, 0)$ and $(ma - kc, mb) \in Y$. Proposition 1.1 (a)(b)(d) shows

$$(H_{a,b})^m \subseteq H_{kc,0} \cdot H_{ma-kc,mb}$$

so ϕ^* must be zero on $H_{a,b}$ since ϕ is zero on $H_{kc,0}$. Thus ϕ^* is the unique extension of ϕ.

Proof of Theorem 4.2. Lemma 4.4 shows that every homomorphism ϕ on X has a unique extension, ϕ^*, to $E^*_{\mu,c,d}$ so it is enough to prove that this extension is bounded. If $\mu < 1$, it follows from the proofs of Theorems 2.1, 3.2, and 3.4 that ϕ^* is bounded if and only if ϕ^* is bounded on $H_{p,q}$ for each $(p, q) \in Y_{\mu,c,d}$. But Proposition 1.1 (b)(d)(f) and Lemma 4.3 (b) imply that if $f \in H_{p,q} \subseteq E_{\mu,c,d}$, then $f^m \in X$ for some $m \in Z_+$. Thus $|\phi^*(f)| = |\phi(f^m)|^{1/m} \leq \|f\|$.

If $\mu = 1$, define I to be the linear direct sum of the $H_{p,q}$ spaces in $E_{1,c,d}$ with $q < p$ and define A to be the sum of the $H_{p,p}$ spaces in $E_{1,c,d}$. If $f \in I$, then f is contained in the span of finitely many $H_{p,q}$ spaces with $q \leq k$, c dividing $p - q$, and $q/p \leq \mu' < 1$. Then for $m \in Z_+$, f^m is contained in the span of $H_{p,q}$ spaces with $q \leq mk$, $m \leq p - q$, c dividing $p - q$, and $q/p \leq \mu'$. It follows from Lemma 4.3 and Proposition 1.1 (f), that for some $m \in Z_+$, $f^m \in X$. Thus, $|\phi^*(f)| = |\phi(f^m)|^{1/m} \leq \|f\|$, so ϕ^* is bounded on I. One has ϕ^* bounded on A since $A \subseteq X$. It then follows from Theorem 3.1 (a)(c)(d) and Theorem 3.2 that ϕ^* is bounded on $E^*_{1,c,d}$ as in the proof of Theorem 3.3.

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