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EXPONENTIAL SUMS OF SUM-OF-DIGIT FUNCTIONS

BY

ALAN H. STEIN

1. Let $\phi(x) = \Sigma_{n<x} q^{\beta(n)}$, where $q > 0$ and $\beta(n)$ represents the sum of the digits of n written to a given base b. If $q = b = 2$, then $\phi(x)$ represents the number of odd integers in the first x rows of Pascal's triangle. This case has been studied by Harborth [4] and Stolarsky [9], while the author has previously extended many of their results for arbitrary $q > 0$ [7].

We shall show that many of the same properties hold for arbitrary base b. Define

$$B = \phi(b), \quad \theta = \log B/\log b, \quad \psi(x) = \phi(x)/x^{\theta}. \tag{1.1}$$

We shall establish that $\phi(x)$ is on the order of x^{θ}, develop an exact formula for $\phi(x)$ and extend $\psi(x)$ to a continuous function on $\mathbf{R}^+$. In the course of doing so, we will also examine interesting properties of the function

$$h(t) = \frac{\log \dfrac{1-q^t}{1-q}}{\log t}.$$

2. We begin by developing several formulas for $\phi(x)$, each of which also yields a formula for $\psi(x)$.

Formula 1. $\phi(bx) = B\phi(x)$.

Proof. Write $\phi(bx) = \Sigma_{n<x}\Sigma_{m<b} q^{\beta(bn+m)}$. Since $\beta(bn+m) = \beta(n) + \beta(m)$ if $m < b$, we can factor out $q^{\beta(n)}$ to get

$$\phi(bx) = \sum_{n<x} q^{\beta(n)} \sum_{m<b} q^{\beta(m)} = \sum_{n<x} q^{\beta(n)}\phi(b) \tag{2.1}$$

and Formula 1 follows immediately.

Corresponding to Formula 1 we have:

Formula 1′. $\psi(bx) = \psi(x)$.

(Note. We frequently take advantage of $B = b^{\theta}$.)

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Formula 2. $\phi(ab^n + x) = \phi(ab^n) + q^{\beta(a)}\phi(x)$ if $x \leqq b^n$.

Proof. We have

$$\phi(ab^n + x) = \phi(ab^n) + \Sigma_{m<x} q^{\beta(ab^n+m)}.$$

Again, $\beta(ab^n + m) = \beta(a) + \beta(m)$, yielding

$$\phi(ab^n + x) = \phi(ab^n) + q^{\beta(a)}\Sigma_{m<x} q^{\beta(m)}$$

and Formula 2.

Formula 3. $\phi(b^n) = B^n$.

Proof. Use Formula 1 repeatedly along with the definition $B = \phi(b)$.

Formula 3′. $\psi(b^n) = 1$.

Formula 4.

$$\phi(ab^n) = \left(\frac{1-q^a}{1-q}\right)B^n \quad \text{if } a \leqq b.$$

Proof. By Formula 1,

$$\begin{aligned}\phi(ab^n) &= \phi(a)B^n \\ &= (1 + q + \ldots + q^{a-1})B^n \\ &= \left(\frac{1-q^a}{1-q}\right)B^n.\end{aligned}$$

Formula 5.

$$\phi(ab^n + x) = \frac{1-q^a}{1-q}B^n + q^a\phi(x) \quad \text{if } a < b, x \leqq b.$$

Proof. Combine Formulae 2, 4 and the fact that $\beta(a) = a$ if $a < b$.

If $b = q = 2$, then a must be 1 and we get $\phi(2^n + x) = 3^n + 2\phi(x)$, which was used effectively by both Harborth and Stolarsky.

We will use Formula 5 to establish and exact formula for $\phi(x)$.

Formula 6. Let $x = \Sigma_{i \leqq s} a_i b^{e_i}$ with $e_i > e_{i+1}$ and $a_i < b$. Then

$$\phi(x) = \sum_{i \leqq s} \left(q^{\Sigma_{j<i} a_j}\right) \cdot \frac{1-q^{a_i}}{1-q} \cdot B^{e_i}.$$

Proof. We use induction on s. For $s = 1$, Formula 6 reduces to Formula 4. So assume the exact formula holds for $s - 1$. Then, by Formula 5,

$$\phi(x) = \frac{1-q^{a_1}}{1-q}B^{e_1} + q^{a_1}\phi\left(\sum_{i=2}^{s} a_i q^i\right). \tag{2.2}$$

Using the induction, we have

$$(2.3) \qquad \phi(x) = \frac{1-q^{a_1}}{1-q} B^{e_1} + q^{a_1} \sum_{i=2}^{s} \left(q^{\sum_{j=2}^{i-1} a_j}\right)\left(\frac{1-q^{a_i}}{1-q}\right) B^{e_i},$$

which simplifies to the exact formula.

Note that the exact formula gives a second immediate proof of Formula 1. Recall that Formula 1′ shows that multiplying by b has no effect on ψ, suggesting that ψ may have easily obtained bounds. Indeed, if we let $b^n \leqq x \leqq b^{n+1}$, then $B^n \leqq x^\theta \leqq B^{n+1}$ and $B^n = \phi(b^n) \leqq \phi(x) \leqq \phi(b^{n+1)} = B^{n+1}$, so

$$\frac{1}{B} = \frac{B^n}{B^{n+1}} \leqq \frac{\phi(x)}{x^\theta} \leqq \frac{B^{n+1}}{B^n} = B,$$

yielding the following two formulas.

Formula 7.

$$\left(\frac{1}{B}\right) x^\theta \leqq \phi(x) \leqq B x^\theta.$$

Formula 7′.

$$\frac{1}{B} \leqq \psi(x) \leqq B.$$

We shall further show that $\phi(x) \leqq x^\theta$ if $q > 1$ and $\phi(x) \geqq x^\theta$ if $q < 1$. (Since $\phi(b^n) = B^n = (b^n)^\theta$, equality will hold infinitely often in either case.) Either sharpening involving Formula 7 involves the function

$$h(t) = \frac{\log \dfrac{1-q^t}{1-q}}{\log t}$$

for the following reason.

Suppose we want to show that, for $q > 1$, $\phi(x) \leqq x^\theta$. Then, in particular, we must have $\phi(a) \leqq a^\theta$ for $a < b$ and so we need

$$(2.4) \qquad \frac{1-q^a}{1-q} = \phi(a) \leqq a^\theta = a^{(\log B)/(\log b)} = B^{(\log a)/(\log b)}.$$

Since

$$B = \phi(b) = 1 + q + \ldots + q^{b-1} = \frac{1-q^b}{1-q},$$

we need

$$(2.5)\qquad \frac{1-q^a}{1-q} \leqq \left(\frac{1-q^b}{1-q}\right)^{(\log a)/(\log b)},$$

which will be true iff

$$(2.6)\qquad \left(\frac{1-q^a}{1-q}\right)^{1/(\log a)} \leqq \left(\frac{1-q^b}{1-q}\right)^{1/(\log b)},$$

which is true precisely when $h(a) \leqq h(b)$. Thus we are led to study the monotonicity of $h(t)$.

3. In order to analyze the behavior of $h(t)$, we shall make use of the following theorem [5, Theorem 148]:

If f, g, and f'/g' are positive increasing functions, then f/g either increases for all x in question, or decreases for all such x, or decreases to a minimum and then increases. In particular, if $f(0) = g(0) = 0$, the f/g increases for $x > 0$.

In analyzing the proof of Theorem 148, we observe the following. Consider the statement:

(3.1) If g is (positive, negative) and (increasing, decreasing), f'/g' is (increasing, decreasing) for $(t > t_0, t < t_0)$ and $f(t_o) = g(t_0) = 0$, then f/g is (increasing, decreasing) for $(t > t_0\, t < t_0)$.

In the if part of the statement choose one of the alternatives in each set of parentheses. If the first alternative is used an even number of times, then we can conclude that f/g is increasing in the relevant interval; otherwise we can conclude f/g is decreasing.

Now consider

$$h(t) = \frac{\log\dfrac{1-q^t}{1-q}}{\log t}, \quad t > 0.$$

We may write

$$h(t) = \frac{f(t)}{g(t)}, \quad \text{where } f(t) = \log\frac{1-q^t}{1-q} \quad \text{and} \quad g(t) = \log t.$$

We first note that g is positive for $t > 1$ and negative for $t < 1$, g is increasing for $t > 0$, and $f(1) = g(1) = 0$. We are left with analyzing

$$(3.2)\qquad \frac{f'(t)}{g'(t)} = \frac{(q^t\log q)/(q^t - 1)}{1/t} = \frac{q^t\log q^t}{q^t - 1}.$$

Using the fact that $u \log u/(u-1)$ increases for all positive $u \neq 1$, we see that f'/g' decreases if $g < 1$ and increases if $g > 1$.

We can now apply (3.1). If $q < 1$ and $t < 1$, then g is negative and increasing, while f'/g' is decreasing, so f/g is decreasing. If $t > 1$, then g is positive and increasing, while f'/g' is decreasing, so f/g is decreasing here as well. Similarly, we see that if $q > 1$, then f/g is increasing for all $t > 0$, $t \neq 1$.

We can simplify the result by observing that $h(t) \to (q \log q)/(q-1)$ as $t \to 1$ and redefining it there to remove the discontinuity and get

THEOREM 1. *If*

$$h(t) = \begin{cases} \dfrac{\log \dfrac{1-q^t}{1-q}}{\log t} & \text{if } t > 0, t \neq 1 \\[2ex] \dfrac{q \log q}{q-1} & \text{if } t = 1 \end{cases}$$

then h is increasing if $q > 1$ and decreasing if $q < 1$.

It is interesting to note that if one tries to analyze the monotonicity of h directly from its derivative, one is led to the expression

$$f(t) = \log tq^t \log q^t - \left(q^t - 1\right)\log \frac{q^t - 1}{q - 1},$$

which is difficult to analyze directly. However, from Theorem 1 we immediately obtain

$$f(t) \begin{cases} > 0 & \text{if } (q-1)(t-1) > 0 \\ < 0 & \text{if } (q-1)(t-1) < 0. \end{cases} \tag{3.3}$$

We may also choose to treat h as a function of two variables and, observing that both

$$\frac{\log \dfrac{1-q^t}{1-q}}{\log t} \quad \text{and} \quad \frac{q \log q}{q-1}$$

approach 1 as $q \to 1$, define

$$h(t,q) = \begin{cases} \dfrac{\log \dfrac{1-q^t}{1-q}}{\log t} & \text{if } q \neq 1, t \neq 1 \\[2ex] \dfrac{q \log q}{q-1} & \text{if } q \neq 1, t = 1 \\[1ex] 1 & \text{if } q = 1 \end{cases} \tag{3.4}$$

giving a function which is defined and continuous in the interior of the first quadrant.

4. We are now prepared to prove the following formulas.

Formula 8. $\phi(x) \leqq x^\theta$ if $q > 1$.

Formula 8′. $\psi(x) \leqq 1$ if $q > 1$.

Formula 9. $\phi(x) \geqq x^\theta$ if $q < 1$.

Formula 9′. $\psi(x) \geqq 1$ if $q < 1$.

We will prove Formula 8, omitting the essentially identical proof of Formula 9.

At the end of §2 we saw that, if $q > 1$ and $a < b$, then we could show $\phi(a) \leqq a^\theta$ if we could show $h(a) \leqq h(b)$. Having done that with Theorem 1, and recalling that $\phi(b) = B = b^\theta$, we proceed by induction. Recall again that we are looking only at the case $q > 1$. So suppose $\phi(x) \leqq x^\theta$ if $x \leqq b^n$ and assume $b^n \leqq x \leqq b^{n+1}$. Then we can write $x = ab^n + y$ for $a < b$ and $y \leqq b^n$. Using Formula 5, we write

$$\psi(x) = \frac{\phi(ab^n + y)}{(ab^n + y)^\theta} = \frac{\left(\frac{1 - q^a}{1 - q}\right) B^n + q^a \phi(y)}{(ab^n + y)^\theta}. \tag{4.1}$$

Using the induction step $\phi(y) \leqq y^\theta$ and then factoring and cancelling $B^n = (b^n)^\theta$ yields

$$\psi(x) \leqq \frac{\left(\frac{1 - q^a}{1 - q}\right) + q^a \left(\frac{y}{b^n}\right)^\theta}{\left(a + \frac{y}{b^n}\right)^\theta}. \tag{4.2}$$

Since $0 \leqq y/b^n \leqq 1$ and we need to show that the numerator is no greater than the denominator, we will consider the function

$$f(t) = \frac{1 - q^a}{1 - q} + q^a t^\theta - (a + t)^\theta, \tag{4.3}$$

and show that $f(t) \leqq 0$ on $[0, 1]$.

First observe that

$$f(0) = \frac{1 - q^a}{1 - q} - a^\theta = \phi(a) - a^\theta \leqq 0$$

since $a < b$ while

$$f(1) = \frac{1 - q^a}{1 - q} + q^a - (a + 1)^\theta = \phi(a + 1) - (a + 1)^\theta \leqq 0$$

since $a + 1 \leqq b$. Now look at $f'(t) = \theta q^a t^{\theta - 1} - \theta(a + t)^{\theta - 1}$. This starts out negative, then becomes positive and stays positive, showing that any maximum of f on an interval must occur at an endpoint. Thus

$$f(t) \leqq \max(f(0), f(1)) \leqq 0,$$

completing the proof of Formula 9′ and thus of Formula 9 as well.

If we combine Formulas 1 and 2 to write

$$\phi(ab^n + x) = B^n\phi(a) + q^{\beta(a)}\phi(x) \leqq B^n\phi(a) + q^{\beta(a)}x^\theta \quad \text{for } x \leqq b^n,$$

we can use the same technique to show:

Formula 10. $\psi(ab^n + x) \leqq \max\{\psi(a), \psi(a + 1)\}$ if $x \leqq b^n, q > 1$

Formula 11. $\psi(ab^n + x) \geqq \min\{\psi(a), \psi(a + 1)\}$ if $x < b^n, q < 1$.

One way of looking at this (for $q > 1$) is that if we start with an integer x and add more digits to the end of it, the value of ψ must stay below the maximum of $\psi(x)$ and $\psi(x + 1)$. Recalling Formula 1′, which means that adding 0's to the end of an integer does not change the value of ψ at all, leads us to the recognition that the value of $\psi(x)$ depends on the sequence of digits in x more than the size of x. We therefore extend the definition of ψ as follows.

(4.4) If $b^n x \in \mathbf{Z}^+$, then $\psi(x) = \psi(b^n x)$.

This extends the domain of ψ to all positive reals with finite representation to base b. We shall soon see that the domain of ψ can be extended to $\mathbf{R}^+$ in a natural way that turns ψ into a continuous function.

Now that we've shown that adding digits to an integer x cannot increase $\psi(x)$ very much (if $q > 1$), we will show how $\psi(x)$ can always be made smaller (for $q > 1$).

Formula 12. $\min(\psi(bx + r), \psi(bx - r)) < \psi(x)$ if $x \in \mathbf{Z}^+, r < b, q > 1$.

Formula 13. $\max(\psi(bx + r), \psi(bx - r)) > \psi(x)$ if $x \in \mathbf{Z}^+, r < b, q < 1$.

We first need to show:

Formula 14. If $q > 1, r < b$, then $\phi(bx + r) + \phi(bx - r) \leqq 2B\phi(x)$.

Formula 15. If $q < 1, r < b$, then $\phi(bx + r) + \phi(bx - r) \geqq 2B\phi(x)$.

To prove Formula 14, recall that

$$\phi(bx + r) = B\phi(x) + q^{\beta(x)}\phi(r) \tag{4.5}$$

while

$$\phi(bx - r) = \phi(bx) - \sum_{i=1}^{r} q^{\beta(bx-i)}. \tag{4.6}$$

Since $\beta(bx - i) \geqq \beta(x) + b - i - 1$, $q^{\beta(bx-i)} \geqq q^{\beta(x)}q^{b-i-1}$, so

$$\phi(bx - r) \leqq B\phi(x) - q^{\beta(x)} \sum_{i=1}^{r} q^{b-i-1}. \tag{4.7}$$

Since $r < b$ and $q > 1$, it is clear that

$$\begin{aligned}\phi(r) = q^0 + q^1 + \cdots + q^{r-1} &\leqq q^{b-r-1} + q^{b-r} + \cdots + q^{b-2} \\ &= \sum_{i=1}^{r} q^{b-i-1}\end{aligned} \tag{4.8}$$

and thus adding (4.5) and (4.7) yields Formula 14. If $q < 1$, then the inequalities in (4.7) and (4.8) reverse, giving Formula 15.

In order to prove Formula 12, assume that $q > 1$ and Formula 12 doesn't work. Then $\psi(bx + r) \geqq \psi(x)$ and $\psi(bx - r) \geqq \psi(x)$, so we have

$$\frac{\phi(bx + r)}{(bx + r)^{\theta}} \geqq \frac{\phi(x)}{x}$$

and thus

$$\phi(bx + r) \geqq B\phi(x)\left(1 + \frac{r}{bx}\right)^{\theta}. \tag{4.9}$$

Similarly,

$$\phi(bx - r) \geqq B\phi(x)\left(1 - \frac{r}{bx}\right)^{\theta}. \tag{4.10}$$

Adding (4.9) and (4.10) gives

$$\phi(bx + r) + \phi(bx - r) \geqq \left\{\left(1 + \frac{r}{bx}\right)^{\theta} + \left(1 - \frac{r}{bx}\right)^{\theta}\right\} B\phi(x). \tag{4.11}$$

By Formula 14, this can only hold if

$$\left(1 + \frac{r}{bx}\right)^{\theta} + \left(1 - \frac{r}{bx}\right)^{\theta} \leqq 2,$$

which cannot be true since $\theta > 1$ if $q > 1$, completing the proof of Formula 12.

If $q < 1$, then the sense of each inequality reverses and we get a contradiction again since $\theta < 1$.

In the binary case with $b = q = 2$, Formula 12 reduces to

$$\min\{\psi(2x+1), \psi(2x-1)\} < \psi(x)$$

which led Harborth [4] to conjecture that by starting with $n_0 = 1$ and then choosing $n_{r+1} = 2n_r \pm 1$, whichever minimized $\psi(2n_r \pm 1)$, he would get a decreasing sequence $\{\psi(n_r)\}$ converging to $\liminf \psi(x)$. If $b > 2$ then the corresponding sequence would be obtained by choosing $1 < n_0 < b$ so as to minimize $\psi(n_0)$ and then take $n_{r+1} = bn_r \pm s$ with $s < b$ chosen to minimize $\psi(bn_r \pm s)$. We conjecture that this will work in general and will later show what we believe that sequence represents in terms of the graph of ψ. First, however, we must establish that we can extend ψ to a continuous function on R^+.

5. In this section we will show that if two integers have similar representations to base b, then they must yield similar values for ψ.

First assume that we start with an integer $x \geqq b^N$ and obtain y from x by repeatedly multiplying by b and adding integers smaller than b. To this end, consider $b^{n+1} > u \geqq b^n, 0 \leqq r < b$. Then

$$\begin{aligned}\psi(bu+r) &= \frac{\phi(bu+r)}{(bu+r)^\theta} \\ &\geqq \frac{\phi(bu)}{(bu)^\theta\left(1+\frac{r}{bu}\right)^\theta} \\ &\geqq \frac{\psi(bu)}{\left(1+\frac{1}{b^n}\right)^\theta} \\ &= \frac{\psi(u)}{\left(1+\frac{1}{b^n}\right)^\theta}\end{aligned} \tag{5.1}$$

Thus

$$\psi(y) \geqq \frac{\psi(x)}{\prod_{n \geqq N}\left(1+\frac{1}{b^n}\right)^\theta}. \tag{5.2}$$

Since $\prod_{n \geqq N}(1 + 1/b^n) \to 1$ as $N \to \infty$, we can assume that $\psi(y)$ can't be much smaller than $\psi(x)$ by making x large enough.

On the other hand,

$$\psi(bu + r) \leqq \frac{\phi(bu + r)}{(bu)^\theta} = \psi(u) + \frac{q^{\beta(u)}\phi(r)}{(bu)^\theta}. \tag{5.3}$$

Since $\phi(r) \leqq \phi(b) = b^\theta$ and $\beta(u) \leqq (n + 1)(b - 1)$,

$$\psi(bu + r) \leqq \psi(u) + \frac{q^{(n+1)(b-1)}}{(b^n)^\theta}. \tag{5.4}$$

Recall that $(b^n)^\theta = B^n$ and $B = 1 + q + \cdots + q^{b-1} \geq q^{b-1}(1 + 1/q)$, so

$$\frac{q^{(n+1)(b-1)}}{(b^n)^\theta} \leqq \frac{q^{(n+1)(b-1)}}{(q^{b-1})^n\left(1 + \frac{1}{q}\right)^n} = \frac{q^{b-1}}{\left(1 + \frac{1}{q}\right)^n}$$

and thus

$$\psi(bu + r) \leqq \psi(u) + q^{b-1}\Big/\left(1 + \frac{1}{q}\right)^n \tag{5.5}$$

and

$$\psi(y) \leqq \psi(x) + q^{b-1} \sum_{n \geqq N} \frac{1}{\left(1 + \frac{1}{q}\right)^n}. \tag{5.6}$$

Since $\sum_{n \geqq N}(1 + 1/q)^N \to \infty$ as $N \to \infty$, $\psi(y)$ can't get much bigger than $\psi(x)$. This completes our proof that $\psi(y)$ can be made arbitrarily close to $\psi(x)$ by making x large enough and gives us almost everything we need to extend ψ to a continuous function on $\mathbf{R}^+$. Note that it suffices to be able to extend ψ to a continuous function on $[1, \infty)$, since we can then define $\psi(x)$ on $(0, 1)$ by finding an n such that $b^n x \geqq 1$ and letting $\psi(x) = \psi(b^n x)$. We actually need only show that, for x, $y \geqq 1$ with finite representations to base b, $\psi(x)$ and $\psi(y)$ can be made arbitrarily close by requiring that x and y be close enough together. We will be able to do that with a few simple observations once we demonstrate that for integer x large enough, $\psi(x + 1)$ and $\psi(x)$ will be arbitrarily close together.

We write

$$\psi(x + 1) - \psi(x) = \phi(x)\left\{\frac{1}{(x + 1)^\theta} - \frac{1}{x^\theta}\right\} + \frac{q^{\beta(x)}}{(x + 1)^\theta}. \tag{5.7}$$

Since

$$\frac{1}{(x+1)^\theta} - \frac{1}{x^\theta} = O\left(\frac{1}{x^{\theta+1}}\right),$$

the first term on the right of (5.7) is $O(1/x)$. Also,

$$\beta(x) \leqq (b-1)\frac{\log x}{\log b} + 1,$$

so

$$\frac{q^{\beta(x)}}{(x+1)^\theta} = O\left(1/x^{\theta-(b-1)\log q/\log b}\right). \tag{5.8}$$

Since

$$\theta = \frac{\log B}{\log b} = \frac{\log(1+q+\cdots+q^{b-1})}{\log b},$$

$$\theta - \frac{(b-1)\log q}{\log b} = \frac{\log\left(1 + \dfrac{1+q+\cdots+q^{b-2}}{q^{b-1}}\right)}{\log b} > 0$$

and thus $q^{\beta(x)}/(x+1)^\theta \to 0$ as $x \to \infty$. Since both terms on the right side of (5.7) approach 0 when x gets large, $\psi(x)$ and $\psi(x+1)$ must be arbitrarily close for large integral x.

Now consider the following process, which will be referred to as Process P. Start with an integer u. Take either u or $u+1$ and then keep multiplying by b and adding non-negative integers smaller than b. Finally, put a radix point somewhere. Our discussion above guarantees that, for large enough u, if x and y are both obtained from u by Process P, then $\psi(x)$ and $\psi(y)$ will arbitrarily close.

In order to show that ψ can be extended to a continuous function on $\mathbf{R}^+$, it remains only to show that if two numbers x, $y \geqq 1$ are close together, then each can be obtained via Process P with the same large u. To see that this is true, suppose $x \leqq y \leqq x + b^{-n}$. Then either x and y agree to the first n digits after the radix point and u can be obtained by using those digits without the radix point or else there is some $m \geqq n$ such that $x + b^{-m}$ and y agree to the first m digits after the radix point and we can use the appropriate digits of x to generate u.

We now complete our definition of ψ on $\mathbf{R}^+$ as follows:

(a) If x is a positive integer, then $\psi(x) = \phi(x)/x^\theta$.

(b) If $x > 0$ has a finite representation to base b, then $\psi(x) = \psi(b^n x)$, where n is chosen so that $b^n x$ is an integer.

(c) If $x > 0$ does not have a finite representation to base b, then choose a sequence $\{x_n\}$ of positive reals with finite representations and let $\psi(x) = \lim_{n \to \infty} \psi(x_n)$.

It is clear from the definition and the discussion preceding it that ψ is continuous on $\mathbf{R}^+$ and that the relevant formulas developed for integral x still hold.

Since $\psi(x) = \psi(bx)$ for all $x \in \mathbf{R}^+$, we must have:

Formula 16.

$$\inf_{x \in [1, b]} \psi(x) = \liminf_{n \to \infty} \psi(n).$$

Formula 17.

$$\sup_{x \in [1, b]} \psi(x) = \limsup_{n \to \infty} \psi(n).$$

Furthermore, since ψ is continuous on $[1, b]$, there must exist x^*, X^* such that

$$\psi(x^*) = \inf \psi(x) \quad \text{and} \quad \psi(X^*) = \sup \psi(X).$$

Consider the special case $b = q = 2$ which has been closely examined. Now consider the sequence $\{x_r\}$ where x_r is obtained by rounding x^* off to r places and dropping the radix point. We conjecture that this sequence is the same as the sequence $\{n_r\}$ proposed by Harborth.

6. The estimates that enabled us to extend the domain of ψ can also be used to obtain error estimates for approximations to $\inf \psi(x)$ and $\sup \psi(x)$.

First consider the case where $q > 1$. Here we try to estimate $\inf \psi(x)$, since $\sup \psi(x) = 1$ trivially. Choose x_n by taking the first n digits of x^* and eliminating the radix point. Then

$$\lim_{n \to \infty} \psi(x_n) = \psi(x^*) = \inf \psi(x)$$

and we can use the estimate (5.2) to get

$$\psi(x^*) \geqq \frac{\psi(x_N)}{\prod_{n \geqq N} \left(1 + \frac{1}{b^n}\right)^\theta} \tag{6.1}$$

and thus

$$\psi(x^*) \geqq \psi(x_N) - \psi(x_N) \left\{1 - \frac{1}{\prod_{n \geqq N} \left(1 + \frac{1}{b^n}\right)^\theta}\right\}. \tag{6.2}$$

Since $\psi(x_N) \leqq 1$, the error in using $\psi(x_N)$ to estimate $\inf \psi(x)$ is bounded by

$$E = 1 - \frac{1}{\prod_{n \geqq N} \left(1 + \frac{1}{b^n}\right)^{\theta}}. \tag{6.3}$$

Since

$$\prod_{n \geqq N} \left(1 + \frac{1}{b^n}\right) \leqq e^{1/b^{N-1}},$$

we have

$$E \leqq 1 - e^{-\theta/b^{N-1}} \leqq \frac{\theta}{b^{N-1}}. \tag{6.4}$$

On the other hand, if we consider the case where $q < 1$ and try to estimate $\sup \psi(x)$ by defining x_r by truncating X^*, we can use (5.6) to show that the error is bounded by

$$E \leqq q^{b-1} \sum_{n \geqq N} \frac{1}{\left(1 + \frac{1}{q}\right)^n}. \tag{6.5}$$

Since

$$\sum_{n \geqq N} \frac{1}{\left(1 + \frac{1}{q}\right)^N} \leqq \frac{1}{\left(1 + \frac{1}{q}\right)^{N-1} \log\left(1 + \frac{1}{q}\right)},$$

we have

$$E \leqq \frac{q^{b-1}}{\left(1 + \frac{1}{q}\right)^{N-1} \log\left(1 + \frac{1}{q}\right)}. \tag{6.6}$$

Since $q < 1$ and $1 + 1/q \geqq 2$, we can get a bound

$$E \leqq \frac{1}{2^{N-1} \log 2} \tag{6.7}$$

which is valid for all b and all $q < 1$.

7. We can consider $\inf \psi(x)$ and $\sup \psi(x)$ as functions of q as well as of b. We shall prove the following formulas for fixed b.

Formula 18.

$$\lim_{q\to 0^+} \sup \psi(x) = 1.$$

Formula 19.

$$\lim_{q\to\infty} \inf \psi(x) = 0.$$

To prove Formula 18, recall that if $q < 1$, then

$$1 \leqq \psi(x) \leqq B = 1 + q + \cdots + q^{b-1} \to 1 \quad \text{as } q \to 0.$$

To prove Formula 19, observe that

$$\psi(b+1) = \frac{\phi(b) + q^{\beta(b)}}{(b+1)^{\theta}} = \frac{B+q}{B^{\log(b+1)/\log b}}.$$

As $q \to \infty$, $B \sim q^{b-1}$, so

$$\psi(b+1) \sim \frac{q^{b-1}}{q^{(b-1)\log(b+1)/\log b}} \to 0 \quad \text{as } q \to \infty.$$

We can also show:

Formula 20.

$$\lim_{q\to 1} \sup \psi(x) = 1.$$

Formula 21.

$$\lim_{q\to 1} \inf \psi(x) = 1.$$

To prove Formula 20, we may assume that $q < 1$ (since $\sup \psi(x) = 1$ if $q > 1$) and use (6.7) to show that

$$1 \leqq \sup \psi(x) \leqq \max_{N \leqq b^N} \psi(n) + \frac{1}{2^{N-1}\log 2}. \tag{7.1}$$

First fix N and observe that for fixed n, $\phi(n) \to n$ as $q \to 1$, so $\psi(n) \to 1$ and

$$1 \leqq \sup \psi(x) \leqq 1 + \frac{1}{2^{N-1}\log 2}. \tag{7.2}$$

Letting $N \to \infty$ yields Formula 20.

We prove Formula 21 in a similar manner, using (6.4) to obtain

$$\min_{n \leqq b^N} \psi(n) - \frac{\theta}{b^{N-1}} \leqq \inf \psi(x) \leqq 1. \tag{7.3}$$

As $q \to 1$, $\theta \to 1$, so

$$1 - \frac{1}{b^{N-1}} \leqq \inf \psi(x) \leqq 1 \tag{7.4}$$

and we obtain Formula 21 by letting $N \to \infty$.

If $q > 1$, we already know that $0 < 1/B \leqq \psi(x) \leqq 1$, so that $0 < \psi(x) \leqq 1$ with $\psi(x) = 1$ occurring whenever $x = b^n$. The results above show that $\psi(x)$ can take on values arbitrarily close to 0.

If $q < 1$, we know that $1 \leqq \psi(x) < B$ with $\psi(x) = 1$ whenever $x = b^n$. Since B can be made arbitrarily large by taking b large and q close to 1, we have the question of where $\psi(x)$ can take on arbitrarily large values. Unfortunately, the last formulas do little to help resolve that question.

8. We conclude by discussing some open questions.

Does the sequence $\psi(n_r)$ described by Harborth for the binary case actually converge to $\inf \psi(x)$?

Can we obtain n_r by rounding x^* as described in Section 5?

Clearly x^* (or X^* where appropriate) does not have a finite representation to base b. However, is it rational or irrational?

We have shown that ψ is continuous on $\mathbf{R}^+$. Is it differentiable?

Since $\psi(bx) = \psi(x)$, it suffices to study ψ on the interval $[1, b]$. If $q > 1$, then $\psi(1) = \psi(b) = 1$, but $\psi(x) < 1$ if $1 < x < b$. Computer aided calculations indicate that as x increases from 1 to b, $\psi(x)$ tends to decrease monotonically until x is fairly close to x^*, then oscillates quite a bit before monotonically increasing. Is this actually the case and, if so, how close to x^* do the oscillations occur?

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University of Connecticut at Waterbury
Waterbury, Connecticut