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A NOTE ON THE VOLUME OF A RANDOM POLYTOPE IN A TETRAHEDRON

BY

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1. Introduction

The following problem in integral geometry was presented by Klee [5] in 1969:

From a simplex of unit volume in Euclidean d-space $d + 1$ points are chosen independently and uniformly at random. What is the expected volume $\mathscr{V}(d)$ of their convex hull?

Let us consider the more general question of determining the expected volume $\mathscr{V}(d, n)$ of the convex hull of $n \geqq d + 1$ points chosen independently and uniformly from the given d-simplex. If $d = 1$, almost trivial calculations yield

$$\mathscr{V}(1, n) = 1 - \frac{2}{n+1}.$$

In the case $d = 2$, the expected volume of the convex hull of n points chosen independently and uniformly from an arbitary convex polygon was derived in [1]; especially,

$$\mathscr{V}(2, n) = 1 - \frac{2}{n+1} \sum_{k=1}^{n} \frac{1}{k}.$$

For $d = 3$, it is only known (cf. [2]) that

$$\mathscr{V}(3,5) = \frac{5}{2} \mathscr{V}(3,4).$$

In the following we give a representation of $\mathscr{V}(3, n)$ by a threefold definite integral, whence we deduce that

$$1 - \mathscr{V}(3, n) \sim \frac{3}{4} \frac{\log^2 n}{n}$$

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as n tends to infinity. (Compare this result to $1 - \mathscr{V}(1, n) \sim 2/n$ and to $1 - \mathscr{V}(2, n) \sim 2(\log n)/n$.)

A final section contains a remark on a recent paper by Hall [4] concerning the probability that a random triangle in a ball is acute.

2. An integral representation of $\mathscr{V}(3, n)$

In the following we use some arguments due to Efron [3] which are included here for the sake of completeness.

The convex hull of $n \geqq 4$ points chosen at random from the given tetrahedron Δ is a polyhedron. Denote by V_n, E_n and F_n the expected number of its vertices, its edges and its facets. As all facets are triangles with probability one, it follows that $E_n = \frac{3}{2}F_n$; further, Euler's theorem $V_n - E_n + F_n = 2$ implies that $V_n = \frac{1}{2}F_n + 2$. If $n + 1$ points are chosen independently and uniformly at random, each of them is a vertex with probability $1 - \mathscr{V}(3, n)$, whence $V_{n+1} = (n + 1)(1 - \mathscr{V}(n, 3))$, and thus,

$$\mathscr{V}(3, n) = 1 - \frac{1}{n+1}V_{n+1} = 1 - \frac{2}{n+1} - \frac{1}{2(n+1)}F_{n+1}.$$

To derive F_{n+1}, note that the convex hull of three points P_1, P_2 and P_3 is a facet if all the other $n - 2$ points lie on one and the same side of the plane $\varepsilon(P_1, P_2, P_3)$ spanned by P_1, P_2 and P_3. This event occurs with probability

$$\int_\Delta \int_\Delta \int_\Delta \left[\tilde{\mathscr{V}}^{n-2} + (1 - \tilde{\mathscr{V}})^{n-2}\right] dP_1\, dP_2\, dP_3,$$

where $\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(P_1, P_2, P_3)$ denotes the smaller of the two parts of Δ cut off by $\varepsilon(P_1, P_2, P_3)$. As there are $\binom{n+1}{3}$ possibilities of choosing three points out of $n + 1$, it follows that F_{n+1} is $\binom{n+1}{3}$ times this integral, whence

$$\mathscr{V}(3, n) = 1 - \frac{2}{n+1} - \frac{(n-1)n}{12} \int_\Delta \int_\Delta \int_\Delta \left[\tilde{\mathscr{V}}^{n-2} + (1 - \tilde{\mathscr{V}})^{n-2}\right] dP_1\, dP_2\, dP_3.$$

As each of the points P_1, P_2 and P_3 is determined by three coordinates, $\mathscr{V}(3, n)$ is expressed by a ninefold integral; but as the integrand depends only on the plane $\varepsilon(P_1, P_2, P_3)$ (not on the points themselves) and as the plane is determined for example merely by its distance from some origin and two angles, the integral may be transformed by means of a classical formula in integral geometry due to Blaschke (cf. Santaló [6, p. 201 and p. 204]) into a threefold one:

$$\mathscr{V}(3, n) = 1 - \frac{2}{n+1} - \frac{(n-1)n}{6} \int_0^{p(\phi, \theta)} \int_0^{2\pi} \int_0^{\pi} \left[\tilde{\mathscr{V}}^{n-2} + (1 - \tilde{\mathscr{V}})^{n-2} \right] \times \tilde{s}^3 \tilde{t} \sin \theta \, d\theta \, d\phi \, dp,$$

where $p(\phi, \theta)$ is the support function of Δ with respect to a fixed origin, $\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(p, \phi, \theta)$ is—as above—the volume of the smaller of the two parts of Δ cut off by the plane $\varepsilon = \varepsilon(p, \phi, \theta)$ orthogonal to the direction (ϕ, θ) and with distance p from the origin, $\tilde{s} = \tilde{s}(p, \phi, \theta)$ is the area of $\varepsilon \cap \Delta$, and $\tilde{t} = \tilde{t}(p, \phi, \theta)$ is the expected area of the convex hull of three random points in $\varepsilon \cap \Delta$. Thus, the problem of determining $\mathscr{V}(3, n)$ is reduced to the problem of determining $\tilde{t}$; i.e., the three-dimensional problem is reduced to a two-dimensional one which has been solved in [1]. (Note that $\varepsilon \cap \Delta$ is either a triangle or a quadrangle. In the first case, $\tilde{t} = \frac{1}{12}\tilde{s}$ as $\mathscr{V}(2, 3) = \frac{1}{12}$ (cf. the introduction); in the second case, the situation is more complicated as all quadrangles $\varepsilon \cap \Delta$ cannot be transformed into one another by affine transformations; i.e., the ratio of $\tilde{t}$ and $\tilde{s}$ depends on p, ϕ and θ.)

3. The asymptotic behaviour of $\mathscr{V}(3, n)$

THEOREM. *Denote by $\mathscr{V}(3, n)$ the expected volume of the convex hull of $n \geqq 4$ points chosen independently and uniformly from a tetrahedron of unit volume. Then*

$$1 - \mathscr{V}(3, n) \sim \frac{3}{4} \frac{\log^2 n}{n}$$

as n tends to infinity.

Proof. It is relatively easy to see from the following considerations that the asympotic behaviour of

$$\int_0^{p(\phi, \theta)} \int_0^{2\pi} \int_0^{\pi} \left[\tilde{\mathscr{V}}^{n-2} + (1 - \tilde{\mathscr{V}})^{n-2} \right] \tilde{s}^3 \tilde{t} \sin \theta \, d\theta \, d\phi \, dp$$

is determined merely by those hyperplanes ε whose intersections $\varepsilon \cap \Delta$ are triangles, in which case the corresponding integral may be replaced by

$$\frac{1}{12}\int\int\int\left[\tilde{\mathscr{V}}^{n-2} + (1-\tilde{\mathscr{V}})^{n-2}\right]\tilde{s}^4 \sin\theta \, d\theta \, d\phi \, dp.$$

Further, $\mathscr{V}(3, \mathrm{n})$ does not depend on the shape of the tetrahedron (ratios of volumes are invariant under nonsingular affine transformations); let us thus consider the tetrahedron with vertices

$$V_0(0,0,0),\ V_1\left(\sqrt[3]{6},0,0\right),\ V_2\left(0,\sqrt[3]{6},0\right) \quad \text{and} \quad V_3\left(0,0,\sqrt[3]{6}\right).$$

To calculate the integral corresponding to all hyperplanes intersecting the edges V_0V_1, V_0V_2 and V_0V_3, we choose V_0 as the origin and put

$$(1) \qquad (a, b, c) := (p\cos\phi\sin\phi,\ p\sin\phi\sin\theta,\ p\cos\theta).$$

This yields

$$\frac{1}{192}\int\int\int\left[\left(\frac{(a^2+b^2+c^2)^3}{6abc}\right)^{n-2} + \left(1-\frac{(a^2+b^2+c^2)^3}{6abc}\right)^{n-2}\right] \times \frac{(a^2+b^2+c^2)^9}{(abc)^4}\, da\, db\, dc.$$

By the further substitution

$$(2) \qquad (u, v, w) := \left(\frac{a^2+b^2+c^2}{a}, \frac{a^2+b^2+c^2}{b}, \frac{a^2+b^2+c^2}{c}\right)$$

we obtain

$$\frac{1}{192}\int_0^{\sqrt[3]{6}}\int_0^{\sqrt[3]{6}}\int_0^{\sqrt[3]{6}}\left[\left(\frac{uvw}{6}\right)^{n-2} + \left(1-\frac{uvw}{6}\right)^{n-2}\right]u^2v^2w^2\, du\, dv\, dw$$
$$= \frac{9}{8}\int_0^1\int_0^1\int_0^1\left[(xyz)^{n-2} + (1-xyz)^{n-2}\right]x^2y^2z^2\, dx\, dy\, dz.$$

The integral corresponding to all hyperplanes intersecting the edges V_0V_1, V_1V_2 and V_1V_3 is identical to the integral corresponding to all hyperplanes intersecting the edges V_0V_2, V_1V_2 and V_2V_3 as well as to the integral corresponding to all hyperplanes intersecting the edges V_0V_3, V_1V_3 and V_2V_3. We therefore consider only the last one. We choose V_3 as the origin; transformation (1) yields

$$\frac{1}{192}\iiint\left[\left(\frac{(a^2+b^2+c^2)^3}{6(a+c)(b+c)c}\right)^{n-2}+\left(1-\frac{(a^2+b^2+c^2)^3}{6(a+c)(b+c)c}\right)^{n-2}\right]$$
$$\times\frac{(a^2+b^2+c^2)^9}{(a+c)^4(b+c)^4c^4}\,da\,db\,dc,$$

and by

$$(2')\qquad (u,v,w):=\left(\frac{a^2+b^2+c^2}{a+c},\frac{a^2+b^2+c^2}{b+c},\frac{a^2+b^2+c^2}{c}\right)$$

we obtain

$$\frac{1}{192}\int_0^{\sqrt[3]{6}}\int_0^{\sqrt[3]{6}}\int_0^{\sqrt[3]{6}}\left[\left(\frac{uvw}{6}\right)^{n-2}+\left(1-\frac{uvw}{6}\right)^{n-2}\right]u^2v^2w^2\,du\,dv\,dw$$
$$=\frac{9}{8}\int_0^1\int_0^1\int_0^1\left[(xyz)^{n-2}+(1-xyz)^{n-2}\right]x^2y^2z^2\,dx\,dy\,dz.$$

It is easy to derive

$$\int_0^1\int_0^1\int_0^1\left[(xyz)^{n-2}+(1-xyz)^{n-2}\right]x^2y^2z^2\,dx\,dy\,dz$$
$$=\frac{1}{\binom{n+1}{3}}\left[\frac{1}{3}\sum_{k=1}^{n+1}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}-\frac{1}{2}\sum_{k=1}^{n+1}\frac{1}{k}+\frac{1}{3}\right.$$
$$\left.-\frac{1}{2(n+1)}+\frac{1}{3(n+1)^2}\right]$$

by means of

$$\int_0^1\frac{1-(1-x)^n}{x}\,dx=\sum_{k=1}^{n}\frac{1}{k}.$$

As n tends to infinity, it follows from

$$\sum_{k=1}^{n+1}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sim\frac{1}{2}\log^2 n$$

that

$$1-\mathscr{V}(3,n)\sim\frac{3}{4}\frac{\log^2 n}{n}.$$

4. A remark on a paper by G.R. Hall on acute triangles in a ball

Recently, Hall [4] derived an integral representation for the probability p_d that three random points, chosen independently and uniformly from the d-dimensional unit ball, form an acute triangle. In dimensions 2 and 3, he deduced that $p_2 = 4/\pi^2 - 1/8 \cong 0{,}2803$ and $p_3 = 33/70 \cong 0{,}4714$.

As the values of p_d in a sense illuminate the geometry of the d-dimensional unit ball, these values are of some interest also for $d > 3$. It can be shown by cumbersome, but essentially standard methods that

$$p_{2m+1} = m\binom{2m}{m}^2 2^{2m} \sum_{k=0}^{m} \binom{2k}{k} \frac{3m+k+1}{(m+k)(4m+2k+1)} \frac{1}{\binom{2m+k}{m}\binom{4m+2k}{2m+k}} - \frac{1}{2} - 2^{2m-1} \frac{\binom{2m}{m}\binom{4m}{2m}}{\binom{4m}{m}\binom{6m+1}{2m}},$$

$$p_{2m+2} = \frac{2^{4m}}{\binom{2m}{m}} \left[\sum_{k=0}^{m} \frac{2^{2k}}{\binom{2k}{k}} \frac{3m+k+3}{2k+1} \frac{1}{\binom{2m+k}{m}\binom{2m+k+2}{2}} + \frac{1}{(2m+1)^2\binom{2m}{m}} \right] \frac{1}{\pi^2} + \frac{1}{4} - \frac{3}{2^{2m+4}} \frac{\binom{4m+4}{m+1}}{\binom{2m+2}{m+1}}.$$

Note that p_{2m+1} is a rational number, while $p_{2m+2} = q_{2m+2}/\pi^2 - r_{2m+2}$ with rational numbers q_{2m+2} and r_{2m+2}. For example,

$$p_4 = \frac{256}{45}\frac{1}{\pi^2} + \frac{1}{32} \cong 0{,}6077,$$

$$p_5 = \frac{1\,415}{2\,002} \cong 0{,}7068,$$

$$p_6 = \frac{2\,048}{315}\frac{1}{\pi^2} + \frac{31}{256} \cong 0{,}7798,$$

$$p_7 = \frac{231\,161}{277\,134} \cong 0{,}8341,$$

$$p_8 = \frac{4\,194\,304}{606\,375}\frac{1}{\pi^2} + \frac{89}{512} \cong 0{,}8747,$$

$$p_9 = \frac{9\,615\,369}{10\,623\,470} \cong 0{,}9051.$$

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