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# NOETHERIAN $\mathbf{Z}_p$ $[[T]]$-MODULES, ADJOINTS, AND IWASAWA THEORY

BY

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## Introduction

Let $p$ be a rational prime and set $\Lambda = \mathbf{Z}_p\ [[T]]$. We define a functor $\alpha$ on the category of noetherian torsion $\Lambda$-modules. This map associates to each module $X$ its adjoint $\alpha(X)$ which has no non-trivial finite $\Lambda$-submodules. There exist $\Lambda$-module homomorphisms $x \to \alpha(x)$ and $\alpha(x) \to x$, each having finite kernel and cokernel. These maps are *not* canonical but there is a canonical homomorphism from $x$ to $\alpha(\alpha(x))$, again having finite kernel and cokernel.

Let $k_n$ be the $n$th layer of the basic $\mathbf{Z}_p$-extension $k_\infty$ of a number field $k$. Fix disjoint finite sets $S$ and $R$ of places of $k$ with $S$ containing all the nonarchimedean places and $R$ containing no primes above $p$. By $A_n$ we signify the $p$-part of the $R$-generalized $S$-class group of $k_n$. Set

$$H = \mathrm{Hom}_{\mathbf{Z}_p}\Big(\varinjlim A_n, \mathbf{Q}_p/\mathbf{Z}_p\Big)$$

where the limit is with respect to the natural maps induced by extension of $S$-ideals. $H$ has a natural structure as $\mathbf{Z}_p[G(k_\infty/k)]$-module and we identify $\mathbf{Z}_p[G(k_\infty/k)]$-modules and $\Lambda$-modules by requiring that $(T+1)$ act as a topological generator $\gamma$ of $G(k_\infty/k)$. Thus $H$ is a $\Lambda$-module. We show that $H$ may be interpreted as an adjoint. In particular, it is noetherian.

Let $A = \varinjlim A_n$ where the limit is with respect to the norm maps. We show that $A$ is pseudo-isomorphic to $H$ and interpret $A$ as a Galois group. Our study of $H$ and $A$ depends on class field theory and the results may be thought of as analogs of the Artin isomorphism for $A_n$.

The exposition of this paper is based on Iwasawa's presentation in a course at Princeton during Spring, 1980. It seemed useful for us to present it here since the results are discussed only briefly in the extant literature. Our new contribution is to work with $R$-generalized $S$-class groups. Iwasawa only considered the usual class groups.

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## 1. Noetherian $\Lambda$-modules

Let $\Lambda = \mathbf{Z}_p[[T]]$ be the ring of formal power series with coefficients in $\mathbf{Z}_p$, endowed with the $(p, T)$-adic topology. Let $P$ denote the set of all height 1 primes of $\Lambda$. A height 1 prime of $\Lambda$ is generated by the rational prime $p$ or by a unique irreducible, distinguished polynomial (i.e., an irreducible polynomial $T^n + a_1 T^{n-1} + \cdots + a_n$ with $a_i \in p\mathbf{Z}_p$, $i = 1, \ldots, n$). If $\mathfrak{p} \in P$, we define the localization

$$\Lambda_{\mathfrak{p}} = \{\alpha/\beta \mid \alpha, \beta \in \Lambda, \beta \notin \mathfrak{p}\}.$$

Let $X$ be a noetherian $\Lambda$-module. If $\mathfrak{p} \in P$, set

$$X_{\mathfrak{p}} = X \otimes \Lambda_{\mathfrak{p}} = \{x \otimes 1/\beta \mid x \in X, \beta \in \Lambda, \beta \notin \mathfrak{p}\}.$$

Note that $x \otimes 1/\beta$ is the zero element of $X$ precisely when $\alpha x = 0$ for some $\alpha \in \Lambda$, $\alpha \notin \mathfrak{p}$. It is a standard algebraic exercise [1] using this observation to prove:

PROPOSITION (1.1). (i) *If $X$ is* **not** *torsion, then $X_{\mathfrak{p}} \neq 0$ $\forall \mathfrak{p} \in P$.*
(ii) *If $X$ is torsion, then $X_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p} \in P$.*
(iii) *$X$ is finite if and only if $X_{\mathfrak{p}} = 0$ $\forall \mathfrak{p} \in P$.*

If $\phi$: $X \to Y$ is a homomorphism of noetherian $\Lambda$-modules, we define the localization

$$\phi_{\mathfrak{p}}: X_{\mathfrak{p}} \to Y_{\mathfrak{p}}, \qquad x \otimes 1/\beta \mapsto \phi(x) \otimes 1/\beta.$$

The operation of localization is exact [1]. The homomorphism $\phi$ is said to be a pseudo-isomorphism if $\phi_{\mathfrak{p}}$ is an isomorphism $\forall \mathfrak{p} \in P$. By (1.1) (iii) and the exactness of localization, this is equivalent with $\phi$ having finite kernel and cokernel. We write $X \sim Y$ if there exists a pseudo-isomorphism $X \to Y$. In general $X \sim Y$ does *not* imply $Y \sim X$. For example, we have $(p, T) \sim \Lambda$ (by the inclusion map) but $\Lambda \nsim (p, T)$ since $\Lambda$ is regular of dimension 2.

PROPOSITION (1.2) [3]. *Pseudo-isomorphism is an equivalence relation on the category of noetherian, torsion $\Lambda$-modules.*

Noetherian $\Lambda$-modules are characterized up to pseudo-isomorphism by the following well-known structure theorem:

THEOREM (1.3) [3]. *If $X$ is a noetherian, torsion $\Lambda$-module, there exists a unique $\Lambda$-module of the form*

$$E(X) = \Lambda^{e_0} \oplus \prod_{i=1}^{s} \Lambda/\mathfrak{p}_i^{e_i}$$

*($\mathfrak{p}_i \in P$, $s$, $e_0$ non-negative integers, $e_1, \dots, e_s$ positive integers) with $X \sim E(X)$. $X$ is torsion precisely when $e_0 = 0$.*

Modules of the form $E(X)$ are called elementary. In the notation of (1.3), the divisor of $X$ (in the divisor group of $\Lambda$) is $\operatorname{div} X = \sum_{i=1}^{s} e_i \mathfrak{p}_i$.

## 2. Adjoints

For the remainder of this paper we assume that the noetherian $\Lambda$-module $X$ is torsion. Therefore, by (1.1) (ii), $\prod_{\mathfrak{p} \in P} X_{\mathfrak{p}}$ is a finite product. In fact, $X_{\mathfrak{p}} \neq 0$ precisely for those $\mathfrak{p}$ which divide $\operatorname{div} X$. We have a natural $\Lambda$-module homomorphism

(2.1) $$\psi_X : X \to \prod_{\mathfrak{p} \in P} X_{\mathfrak{p}}, \qquad x \mapsto \{x \otimes 1\}_{\mathfrak{p} \in P}.$$

THEOREM (2.2). Ker $\psi_X$ *is the maximal finite $\Lambda$-submodule of $X$.*

*Proof.* Let $Y$ be a $\Lambda$-submodule of $X$ and $\phi$: $Y \to X$ the natural inclusion. The commutativity of

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle \psi_Y} & & \downarrow{\scriptstyle \psi_X} \\ \prod_{\mathfrak{p} \in P} Y_p & \xrightarrow[\prod \phi_{\mathfrak{p}}]{} & \prod_{\mathfrak{p} \in P} X_{\mathfrak{p}} \end{array}$$

yields the following strıng of equivalences and hence our assertion:

$$\begin{aligned} Y \subseteq \ker \psi_X &\Leftrightarrow \psi_X|_Y \text{ is the zero map} \\ &\Leftrightarrow \psi_X \circ \phi \text{ is the the zero map} \\ &\Leftrightarrow \prod_{\mathfrak{p} \in P} \phi_{\mathfrak{p}} \circ \psi_Y \text{ is the zero map} \\ &\Leftrightarrow \psi_Y \text{ is the zero map (by the exactness of localization)} \\ &\Leftrightarrow Y_{\mathfrak{p}} = 0\ \forall \mathfrak{p} \in P \\ &\Leftrightarrow Y \text{ is finite (by (2.1)(iii)).} \end{aligned}$$

■

COROLLARY (2.3). *If $E$ is elementary, then $\psi_E$ is injective.*

The cokernel of $\psi_X$ is also of interest. Its Pontryagin dual, denoted $\alpha(X)$, is called the adjoint of $X$:

$$\alpha(X) = \mathrm{Hom}_{\mathbf{Z}_p}\left(\mathrm{coker}\,\psi_X, \mathbf{Q}_p/\mathbf{Z}_p\right).$$

Recalling (1.1)(iii), the following is immediate from the definition of $\alpha(X)$.:

PROPOSITION (2.4). *If $X$ is finite, $\alpha(X) = 0$.*

If $X$ and $Y$ are both noetherian torsion $\Lambda$-modules, so is $X \oplus Y$ and $\alpha(X \oplus Y) \simeq \alpha(x) \oplus \alpha(Y)$.

If $f$: $X \to Y$ is a homomorphism of $\Lambda$-modules, we define the adjoint map $\alpha(f)$: $\alpha(Y) \to \alpha(X)$ by

$$\begin{aligned}\alpha(f)(G)\left(\{x \otimes 1/\beta\}_{\mathfrak{p}\in P} \mathrm{mod}\,\mathrm{Im}(\psi_X)\right) \\ = G\left(\{f(x) \otimes 1/\beta\}_{\mathfrak{p}\in P} \mathrm{mod}\,\mathrm{Im}(\psi_Y)\right) \forall G \in \alpha(Y).\end{aligned}$$

PROPOSITION (2.5). *Let $1 \to Z \to X \to Y \to 1$ be an exact sequence of noetherian, torsion $\Lambda$-modules. Then $1 \to \alpha(Y) \to \alpha(X) \to \alpha(Z)$ is exact and $\#(\mathrm{coker}(\alpha(X) \to \alpha(Z))) < \infty$.*

*Proof.* Apply the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 1 \to & Z \to & X \to & Y \to 1 \\ & \downarrow \psi_Z & \downarrow \psi_X & \downarrow \psi_Y \\ 1 \to & \prod_{\mathfrak{p}\in P} Z_{\mathfrak{p}} \to & \prod_{\mathfrak{p}\in P} X_{\mathfrak{p}} \to & \prod_{\mathfrak{p}\in P} Y_{\mathfrak{p}} \to 1 \end{array}$$

to obtain the exactness of

$$\ker \psi_Y \to \mathrm{coker}\,\psi_Z \to \mathrm{coker}\,\psi_X \to \mathrm{coker}\,\psi_Y \to 1.$$

Taking Pontryagin duals yields our assertion since $\ker \psi_Y$ is finite by (2.2). ∎

To see the explicit structure of $\alpha(X)$, we introduce the notion of $X$-admissible sequence. A sequence $\{\sigma_n\}_{n\geq 0}$ of elements of $\Lambda$ will be called *$X$-admissible* provided that $\sigma_0 \in (p, T)\Lambda$, $\sigma_n \neq 0\ \forall n \gg 0$, $\sigma_{n+1} \in \sigma_n(p, T)\Lambda\ \forall n \geq 0$, and the $\sigma_n$ are all disjoint from $\mathrm{div}\, X$. Given $X$, there always exists such a sequence. Theorem (2.7) will give us a useful realization of $\alpha(x)$ in terms of an $X$-admissible sequence. First we need:

LEMMA (2.6). *Suppose $\{\sigma_n\}_{n\geq 0}$ is X-admissible. Then the map*

$$\phi_X\colon X\otimes_\Lambda\Big\{\bigcup_{n\geq 0}1/\sigma_n\Lambda\Big\}\to\prod_{\mathfrak{p}/\operatorname{div}X}X_{\mathfrak{p}},$$
$$x\otimes 1/\sigma_n\to\{x\otimes 1/\sigma_n\}\text{ embedded diagonally}$$

*is an isomorphism of $\Lambda$-modules. Moreover, the diagram*

$$\begin{tikzcd}
X \arrow[rr, "\psi_X"] \arrow[d, "x \mapsto (x\otimes 1)"'] & & \prod_{\mathfrak{p}\in P} X_{\mathfrak{p}} \\
X\otimes_\Lambda \Lambda \arrow[rr, "(x\otimes\lambda)\to(x\otimes\lambda)"'] & & X\otimes_\Lambda\{\bigcup_{n\geq 0}1/\sigma_n\Lambda\} \arrow[u, "\phi_X"']
\end{tikzcd}$$

is commutative.

*Proof.* Commutativity is immediate from the definitions of the maps involved.

(i) *Injectivity of $\phi_X$.*

$$\begin{aligned}\phi_X(x\otimes 1/\sigma_n)=0 &\Rightarrow x\otimes 1/\sigma_n=0\text{ in }X_{\mathfrak{p}}\ \forall\mathfrak{p}\in P\\ &\Rightarrow x\otimes 1=0\text{ in }X_{\mathfrak{p}}\ \forall\mathfrak{p}\in P\\ &\Rightarrow x\in\ker\psi_X.\end{aligned}$$

Therefore, since $\ker\psi_X$ is finite by (2.2), if $x\otimes 1/\sigma_n\in\ker\phi_X$, we may choose a positive integer $a$ with $(p,T)^a\Lambda=0$. By the definition of an $X$-admissible sequence, $(\sigma_{n+a}/\sigma_n)\in(p,T)^a\Lambda$ so

$$(\sigma_{n+a}/\sigma_n)x=0.$$

Consequently,

$$(x\otimes 1/\sigma_n)=((\sigma_{n+a}/\sigma_n)x\otimes 1/\sigma_{n+a})=0\text{ in }X\otimes_\Lambda\Big\{\bigcup_{n\geq 0}1/\sigma_n\Lambda\Big\}.$$

(ii) *Surjectivity of $\phi_X$.* It suffices to show that if $x\in X$, $\mathfrak{q}\in P$, $\eta\in\Lambda$, and $\eta\notin\mathfrak{q}$, then

$$\underset{\substack{\uparrow\\ \mathfrak{q}\text{-th place}}}{(0,\ldots,0,x\otimes 1/\eta,0,\ldots,0)}\in\operatorname{Im}\phi_X.$$

Since $\mathfrak{q}\in P$, $\mathfrak{q}=f(T)\Lambda$ where $f(T)=p$ or $f(T)$ is irreducible distinguished.

Choose a non-zero $\lambda \in \Lambda$ with $\lambda x = 0$. We can do this because $X$ is a noetherian torsion $\Lambda$-module. We write $\lambda = f^b\lambda'$ where $\lambda' \in \Lambda$ is prime to $f$. Set $Y = \lambda' X/\lambda'^2\eta X$; $Y$ has been chosen so that $Y_{\mathfrak{p}} = 0\ \forall_{\mathfrak{p}} \in P$.

$$\begin{cases} \quad \mathfrak{p} \neq \mathfrak{q} \Rightarrow f \notin \mathfrak{p} \Rightarrow 1/f^b \in \Lambda_{\mathfrak{p}} \Rightarrow \lambda' X_{\mathfrak{p}} = x \otimes_\Lambda \lambda'\Lambda_{\mathfrak{p}} \\ \quad\quad = \lambda\left(x \otimes 1/f^b\Lambda_{\mathfrak{p}}\right) \subseteq \lambda x \otimes_\Lambda \Lambda_{\mathfrak{p}} = 0 \Rightarrow Y_p = 0, \\ f \nmid \lambda' \Rightarrow \lambda' \notin \mathfrak{q} \Rightarrow 1/\lambda' \cdot 1/\eta \in \Lambda_{\mathfrak{q}} \Rightarrow \lambda'^2\eta X_{\mathfrak{q}} = \lambda' X_{\mathfrak{q}} \Rightarrow Y_{\mathfrak{q}} = 0. \end{cases}$$

Therefore $Y$ is finite by (1.1)(iii) so we may choose a positive integer $c$ with $(p, T)^c\Lambda Y = 0$. This implies that there exists $y \in Y$ with $\sigma_c\lambda' x = \lambda'^2\eta y$.

$$\lambda' y \otimes 1/\sigma_c = \begin{cases} \lambda'(y \otimes 1/\sigma_c) \in \lambda' X_{\mathfrak{p}} = 0 & \text{in } X_{\mathfrak{p}} \quad \text{if} \quad \mathfrak{p} \neq \mathfrak{q} \\ \lambda'^2\eta y \otimes 1/\eta\sigma_c\lambda' = \sigma_c\lambda' \otimes 1/\eta\sigma_c\lambda' = x \otimes 1/\eta & \text{in } X_{\mathfrak{q}} \end{cases}$$

Therefore

$$\phi(\lambda' y \otimes 1/\sigma_c) = (0,\ldots,0,\underset{\substack{\uparrow \\ \mathfrak{q}\text{-th place}}}{x \otimes 1/\eta},0\ldots0) \qquad \blacksquare$$

THEOREM (2.7). *Let $\{\sigma_n\}_{n\geq 0}$ be $X$-admissible. Then*

$$\begin{aligned} \alpha(X) &\simeq \operatorname{Hom}_{\mathbf{Z}_p}\left(X \otimes_\Lambda \left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Big/\Lambda, \mathbf{Q}_p/\mathbf{Z}_p\right) \\ &\simeq \operatorname{Hom}_{\mathbf{Z}_p}\left(\varinjlim X/\sigma_n X, \mathbf{Q}_p/\mathbf{Z}_p\right) \\ &\simeq \varprojlim \operatorname{Hom}_{\mathbf{Z}_p}\left(X/\sigma_n X, \mathbf{Q}_p/\mathbf{Z}_p\right) \end{aligned}$$

*where the direct limit is with respect to the maps*

$$\begin{aligned} &X/\sigma_n X \quad \to X/\sigma_m X, \\ &x \bmod \sigma_n X \mapsto (\sigma_m/\sigma_n)x \quad \bmod \sigma_m X, \quad m \geq n \geq 0 \end{aligned}$$

and the inverse limit is with respect to the induced maps.

*Proof.* From the exactness of

$$X \otimes_\Lambda \Lambda \to X \otimes_\Lambda \left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right) \to X \otimes_\Lambda \left(\left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Big/\Lambda\right) \to 1$$

and the commutativity of the diagram of (2.6), we see that the isomorphism $\phi_X$ induces $X \otimes_\Lambda ((\bigcup_{n\geq 0} 1/\sigma_n\Lambda)/\Lambda) \simeq \operatorname{coker} \psi_X$. Therefore

$$\alpha(X) \simeq \operatorname{Hom}_{\mathbf{Z}_p}\left( X \otimes_\Lambda \left( \left( \bigcup_{n\geq 0} 1/\sigma_n\Lambda \right) \Big/ \Lambda \right), \mathbf{Q}_p/\mathbf{Z}_p \right).$$

Observe that $(\bigcup_{n\geq 0} 1/\sigma_n\Lambda)/\Lambda = \bigcup_{n\geq 0}(1/\sigma_n\Lambda)/\Lambda = \varinjlim (1/\sigma_n\Lambda)/\Lambda$. But the natural diagram

$$\begin{array}{ccc} (1/\sigma_n\Lambda)/\Lambda & \hookrightarrow & (1/\sigma_m\Lambda)/\Lambda \\ \downarrow\wr & & \downarrow\wr \\ \Lambda/\sigma_n\Lambda & \longrightarrow & \Lambda/\sigma_m\Lambda \\ \lambda \bmod \sigma_n\Lambda & \mapsto & (\sigma_m/\sigma_n)\lambda \bmod \sigma_m\Lambda \qquad (m \geq n) \end{array}$$

is commutative, whence $(\bigcup_{n\geq 0} 1/\sigma_n\Lambda)\Lambda = \varinjlim \Lambda/\sigma_n\Lambda$ and

$$\begin{aligned} \left( \left( \bigcup_{n\geq 0} 1/\sigma_n\Lambda \right) \Big/ \Lambda \right) \otimes_\Lambda X &\simeq \left( \varinjlim \Lambda/\sigma_n\Lambda \right) \otimes_\Lambda X \\ &\simeq \varinjlim (\Lambda/\sigma_n\Lambda \otimes X) \\ &\simeq \varinjlim (X/\sigma_n X). \end{aligned}$$

Hence

$$\alpha(x) \simeq \operatorname{Hom}_{\mathbf{Z}_p}\left( \varinjlim X/\sigma_n X, \mathbf{Q}_p/\mathbf{Z}_p \right) \simeq \varprojlim \operatorname{Hom}_{\mathbf{Z}_p}(X/\sigma_n X, \mathbf{Q}_p/\mathbf{Z}_p) \quad \blacksquare$$

COROLLARY (2.8).

$$\alpha(X) \simeq \operatorname{Hom}_\Lambda\left( X, \operatorname{Hom}\left( \left( \bigcup_{n\geq 0} 1/\sigma_n\Lambda \right) \Big/ \Lambda, \mathbf{Q}_p/\mathbf{Z}_p \right) \right) \simeq \operatorname{Hom}_\Lambda(X, \mathbf{Q}_p[[T]])$$

*Proof.* The first isomorphism follows from the isomorphism

$$\alpha(X) \simeq \operatorname{Hom}_{\mathbf{Z}_p}\left( X \otimes_\Lambda \left( \bigcup_{n\geq 0} 1/\sigma_n\Lambda \right) \Big/ \Lambda, \mathbf{Q}_p/\mathbf{Z}_p \right)$$

of (2.7) by basic commutative algebra. We now calculate

$$\operatorname{Hom}_{\mathbf{Z}_p}\left( \left( \left( \bigcup_{n\geq 0} 1/\sigma_n\Lambda \right), \Lambda \right), \mathbf{Q}_p/\mathbf{Z}_p \right)$$

by embedding $\Lambda$ in $\mathbf{Q}_p((T)) = \{\sum_{i=1}^{N} a_i T^i | N \in \mathbf{Z}, a_i \in \mathbf{Q}_p\}$. Using this embedding we have

$$\begin{aligned}\bigcup_{n\geq 0}(1/\sigma_n\Lambda) &= \bigcup_{n\geq 0}(1/T^n)\Lambda \\ &= \left\{\sum_{i=1}^{N} a_i T^i | N \in \mathbf{Z}, a_i \in \mathbf{Z}_p\right\} \\ &=_{\text{def}} \mathbf{Z}_p((T))\end{aligned}$$

and

$$\left(\bigcup_{n\geq 0}(1/\sigma_n\Lambda)\right)\Big/\Lambda = \mathbf{Z}_p((T))/\Lambda.$$

Let $w \in \mathbf{Q}_p[[T]]$, $\xi \in \mathbf{Z}_p((T))$. Then $w\xi \in \mathbf{Q}_p((T))$. Let $\operatorname{res}(w\xi)$ be the coefficient of $T^{-1}$ in the power series $w\xi$. Set $\phi_w(\xi) = \operatorname{res}(w\xi) \bmod \mathbf{Z}_p \in \mathbf{Q}_p/\mathbf{Z}_p$. Then $\phi_w : \mathbf{Z}_p((T)) \to \mathbf{Q}_p/\mathbf{Z}_p$ is a $\mathbf{Z}_p$-homomorphism and $\Lambda \subset \ker \phi_w$. Therefore $\phi_w$ induces a $\mathbf{Z}_p$-homomorphism

$$\bar{\phi}_w : \mathbf{Z}_p((T))/\Lambda = \left(\left(\textstyle\bigcup_{n\geq 0}(1/\sigma_n)\Lambda\right)/\Lambda\right) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

We thus have a $\Lambda$-homomorphism

$$\mathbf{Q}_p[[T]] \to \operatorname{Hom}_{\mathbf{Z}_p}\left(\left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Big/\Lambda, \mathbf{Q}_p/\mathbf{Z}_p\right), \qquad w \mapsto \bar{\phi}_w$$

and this induces a $\Lambda$-isomorphism

$$\mathbf{Q}_p[[T]]/\Lambda \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{Z}_p}\left(\left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Big/\Lambda, \mathbf{Q}_p/\mathbf{Z}_p\right).$$

Therefore $\alpha(x) \simeq \operatorname{Hom}_\Lambda(X, \mathbf{Q}_p[[T]]/\Lambda)$. ■

Using Corollary (2.8) it is not difficult to prove that elementary torsion $\Lambda$-modules are self-adjoint.

THEOREM (2.9). *If $E$ is an elementary torsion $\Lambda$-module, then $E \simeq \alpha(E)$.*

*Proof.* Since the adjoint respects direct sums, it suffices to verify that $\Lambda/\mathfrak{p}^e \simeq \alpha(\Lambda/\mathfrak{p}^e)$ $\forall \mathfrak{p} \in P$ and for all positive integers $e$.

*Case* (i) $\mathfrak{p} \neq T\Lambda$. Let $\sigma_n = T^n$. Then $\{\sigma_n\}$ is $(\Lambda/\mathfrak{p}^e)$-admissible. According to (2.8),

$$\alpha(\Lambda/\mathfrak{p}^e) \simeq \operatorname{Hom}_\Lambda\left(\Lambda/\mathfrak{p}^e, \operatorname{Hom}_{\mathbf{Z}_p}\left(\left(\left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Big/\Lambda\right), \mathbf{Q}_p/\mathbf{Z}_p\right)\right).$$

On the other hand, since $\not{p} \neq T\Lambda$ we have a bilinear pairing

$$\Lambda/\not{p}^e \times \Lambda/\not{p}^e \to \mathbf{Q}_p[[T]]/\Lambda,$$
$$(\lambda \bmod \not{p}^e, \lambda' \bmod \not{p}^e) \mapsto \frac{\lambda\lambda'}{f^e} \bmod \Lambda$$

where $\not{p} = f\Lambda$ and $f = p$ or $f$ is a distinguished irreducible polynomial not equal to $T$. This pairing induces an isomorphism

$$\Lambda/\not{p}^e \simeq \mathrm{Hom}_\Lambda(\Lambda/\not{p}^e, \mathbf{Q}_p[[T]]/\Lambda).$$

*Case* (ii). $\not{p} = T\Lambda$. There exists a (unique) topological automorphism of $\Lambda$ mapping $T$ to $T + \not{p}$ and fixing $\mathbf{Z}_p$. This allows us to reduce to case (i). ■

COROLLARY (2.10). *Let $X$ be a noetherian, torsion $\Lambda$-module. Then:*
(i) *$\alpha(X) \sim X$. In particular $\alpha(X)$ is a noetherian, torsion $\Lambda$-module.*
(ii) *The $\Lambda$-module $\alpha(X)$ has no non-trivial finite $\Lambda$-submodules.*

*Proof.* By (1.2) and (1.3), there is an exact sequence

$$0 \to A \to E(X) \to X \to B \to 0$$

where $A$ and $B$ are finite and $E(X)$ is elementary. Since $E(X)$ has no finite $\Lambda$-submodules, $A = 0$ and

$$0 \to E(X) \to X \to B \to 0$$

is exact. Taking adjoints and recalling (2.4) and (2.5), we obtain an *injective* pseudo-isomorphism $\alpha(X) \hookrightarrow \alpha(E)$. Therefore $\alpha(X) \sim \alpha(E) \simeq E \sim X$ and $\mathrm{Ker}\,\psi_{\alpha(X)} \hookrightarrow \mathrm{Ker}\,\psi_{\alpha(E)} = 0$. Recalling (2.2), we are done. ■

Corollary (2.10)(ii) asserts that $X$ and $\alpha(X)$ are pseudo-isomorphic but does *not* give a canonical pseudo-isomorphism. On the other hand, Iwasawa has remarked that using (2.7) one may prove that there is a canonical pseudo-isomorphism from $X$ to $\alpha(\alpha(x))$. We now explain this. Applying (2.8) twice we obtain an isomorphism

$$h : \alpha(\alpha(X)) \to \mathrm{Hom}_\Lambda\left(\mathrm{Hom}_\Lambda\left(X, \mathrm{Hom}_{\mathbf{Z}_p}\left(\left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Big/\Lambda, \mathbf{Q}_p/\mathbf{Z}_p\right)\right),\right.$$
$$\left.\mathrm{Hom}_{\mathbf{Z}_p}\left(\left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Lambda, \mathbf{Q}_p/\mathbf{Z}_p\right)\right)$$

Which depends only upon choice of an $X$-admissible sequence. But we have a natural map

$$g: X \to \operatorname{Hom}_\Lambda\left(\operatorname{Hom}_\Lambda\left(X, \operatorname{Hom}_{\mathbf{Z}_p}\left(\left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Big/\Lambda, \mathbf{Q}_p/\mathbf{Z}_p\right)\right),\right.$$
$$\left.\operatorname{Hom}_{\mathbf{Z}_p}\left(\left(\bigcup_{n\geq 0} 1/\sigma_n\Lambda\right)\Big/\Lambda, \mathbf{Q}_p/\mathbf{Z}_p\right)\right),$$
$$x \mapsto \rho_x$$

where $\rho_x(j) = j(x)\ \forall j \in \operatorname{Hom}_\Lambda(X, \operatorname{Hom}((\cup 1/\sigma_n\Lambda), \mathbf{Q}_p/\mathbf{Z}_p))$. The map

$$h^{-1} \circ g: X \to \alpha(\alpha(X))$$

is a pseudo-isomorphism and one may check that it is independent of choice of $X$-admissible sequence.

## 3. Iwasawa theory and Galois groups

Let $\mathbf{Q}_\infty$ denote the unique $\mathbf{Z}_p$-extension of $\mathbf{Q}$ and set $k_\infty = k\mathbf{Q}_\infty$. Define $k_n$ to be the unique intermediate field $k \subseteq k_n \subseteq k_\infty$ satisfying $[k_n : k] = p^n$. Set $\Gamma = G(k_\infty/k)$ and fix a topological generator $\gamma$ of $\Gamma$. We will identify $\mathbf{Z}_p[\Gamma]$-modules and $\Lambda$-modules by requiring that $\gamma$ act as $T + 1$.

PROPOSITION (3.1). *A prime of $k$ is ramified in $k_\infty$ precisely when it lies above the rational prime $p$. The decomposition group in $k_\infty$ of each finite prime $\mathfrak{p}$ of $k$ is equal to $G(k_\infty/k_{n_\mathfrak{p}})$ for some integer $n_\mathfrak{p}$, $0 \leq n_\mathfrak{p} \leq \infty$.*

The proof of the first claim (which is well known) depends on the multiplicative ramification index. The second does not appear to have been published but Iwasawa included it in his 1971 course [2]. Its proof is similar to that of the first but depends on analysis of residue degrees rather than ramification indices.

Fix disjoint finite sets $S$ and $R$ of places of $k$ with $S$ containing all the nonarchimedean places and $R$ containing no primes above $p$. By (3.1), there are only finitely many primes of $k_\infty$ lying above $p$ or $S$.

Let $s$ denote the number of primes of $k_\infty$ lying above $p$ or primes of $S$ and let $n(S) = \max\{n \mid \mathfrak{p} \in S \text{ or } \mathfrak{p}/p\}$—with $n_\mathfrak{p}$ as in (3.1). (Note that when $S = \{\mathfrak{p} \mid \mathfrak{p}/p \text{ or } \mathfrak{p}/\infty\}$, $s$ is Iwasawa's "$s$" and $n(S)$ is his "$n_0$" (see [3]).)

Let $M_\infty$ denote the maximal abelian $p$-extension of $k_\infty$ which is unramified outside $R$. Iwasawa [3] has shown that $G(M_\infty/k_\infty)$ is a noetherian $\mathbf{Z}_p[\Gamma]$-module where the action of $\Gamma$ on $G(M_\infty/k_\infty)$ is by inner automorphisms.

Let $L_n$ denote the maximal abelian $p$-extension of $k_n$ which is unramified outside $R$ and in which every prime above $S$ is completely decomposed, and let $N_n$ be the maximal abelian extension of $k_n$ contained in $L_\infty$. Then

$$M_\infty \supseteq L_\infty \supseteq N_n \supseteq L_n \text{ for all } n \geq n(S)$$

and

$$L_\infty = \bigcup_{n \geq 0} N_n = \bigcup_{n \geq 0} L_n.$$

Consider the following result.

THEOREM (3.2). *$G(L_\infty/k_\infty)$ is a noetherian, torsion $\mathbf{Z}_p[\Gamma]$-module.*

Since $G(L_\infty/k_\infty)$ is isomorphic to a quotient of $G(M_\infty/k_\infty)$, it is a noetherian $\mathbf{Z}_p[\Gamma]$-module. Therefore standard techniques in Iwasawa theory [3] show that it suffices to prove that the essential rank of

$$G(L_\infty/k_\infty)/(\gamma^{p^n} - 1)G(L_\infty/k_\infty)$$

is bounded as $n \to \infty$. (The essential rank of a $\mathbf{Z}_p$-module $X$ is $\dim_{\mathbf{Q}_p} X \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.) But

$$(\gamma^{p^n} - 1)G(L_\infty/k_\infty) = G(L_\infty/N_n) \; \forall n \geq n(s)$$

so the following lemma suffices to complete the proof of (3.2).

LEMMA (3.3). *The essential rank of $G(N_n/k_\infty)$ is at most $s$.*

*Proof.* Fix $n \geq n(S)$. Denote the finite primes of $N_n$ lying above $S$ or $p$ by $\mathfrak{q}_1, \dots \mathfrak{q}_s$. The canonical map

$$G(N_n/k_n) \to G(N_n/k_n)/G(N_n/k_\infty) \to G(k_\infty/k_n),$$
$$\sigma \to \sigma|_{k_\infty}$$

maps the inertia group $T_i(n)$ of $\mathfrak{q}_i$ in $k_n$ onto the inertia group of $\mathfrak{q}_i|_{k_\infty}$ in $k_n$ which is $G(k_\infty/k_n)$. However, since we have chosen $n \geq n(S)$, the extension $N_n/k_\infty$ is unramified at all places not lying above $R$ and hence $T_i(n) \cap G(N_n/k_\infty) = 1$. It follows that we have a direct product decomposition

$$G(N_n/k_n) = T_i(n) \times G(N_n/k_\infty). \tag{3.4}$$

We let $T(n) = T_1(n) \dots T_s(n)$ (semidirect product). Recalling the definition of $L_n$, we see that $T(n) = G(N_n/L_n)$.

By class field theory (see (4.1)), $G(L_n/k_n)$ is finite. Consequently,

$$\begin{aligned}\text{ess rank } G(N_n/k_n) &= \text{ess rank } G(L_n/k_n)\\ &= \text{ess rank } T_1(n)\dots T_s(n)\\ &\le \sum_{i=1}^{s} \text{ess rank } T_i(n)\\ &= \sum_{i=1}^{s} \text{ess rank } \mathbf{Z}_p\\ &= s. \qquad \blacksquare\end{aligned}$$

We next use the field $N_n$ to show that $G(L_\infty/k_\infty L_n)$ is the image of $G(L_\infty/k_\infty L_{n(S)})$ under the natural map

$$\nu_{n(S),n} = \left(\gamma^{p^n} - 1\right)/\left(\gamma^{p^{n(S)}} - 1\right).$$

THEOREM (3.5). *For $n \ge n(S)$,*

$$G(L_\infty/k_\infty L_n) = \left(\gamma^{p^n} - 1\right)/\left(\gamma^{p^{n(S)}} - 1\right)G\left(L_\infty/k_\infty L_{n(S)}\right).$$

*Proof.* Fix $n \ge n(S)$. Denote the primes of $L_\infty$ lying above $S$ or $p$ by $\mathfrak{p}_1,\dots,\mathfrak{p}_s$. Let $T_i$ denote the inertia group of $\mathfrak{p}_i$ in $k_{n(S)}$. Reasoning as in the proof of (3.3) we find

$$(3.6)\qquad T_i \cap G(L_\infty/k_\infty) = 1, \quad T_i G(L_\infty/k_\infty) = G\left(L_\infty/k_{n(S)}\right).$$

Hence there is a canonical isomorphism

$$(3.7)\quad T_i \xrightarrow{\sim} G\left(L_\infty/k_{n(S)}\right)/G(L_\infty/k_\infty) \simeq G\left(k_\infty/k_{n(S)}\right), \quad i = 1,\dots,s.$$

Let $g_j \in T_j$ denote the inverse image of $\gamma' = \gamma^{p^{n(S)}}$ under (3.6), $j = 1,\dots,s$. By ((3.6) with $i = 1$), there exists $x_j \in G(L_\infty/k_\infty)$ and $h_j \in T_1$ such that $g_j = x_j h_j$. We observe that $h_j$ is mapped to $\gamma'$ under ((3.7) with $i = 1$). Therefore, $h_j = g_1$. It follows that the inertia group of $\mathfrak{p}_j$ in $k_n$ is topologically generated by

$$\begin{aligned} g_j^{p^{n-n(S)}} &= \left(x_j g_1\right)^{p^{n-n(S)}} = x_j^{(1+g_1+\dots+g_1^{(p^{n-n(S)}-1)})} g_1^{p^{n-n(S)}}\\ &= x_j^{(1+\gamma'+\dots+\gamma'^{(p^{n-n(S)}-1)})} g_1^{p^{n-n(S)}}\\ &= \left(1+\gamma'+\dots\gamma'^{(p^{n-n(S)}-1)}\right)x_j g_1^{p^{n-n(S)}}\\ &= \nu_{n(S),n} x_j g_1^{p^{n-n(S)}}.\end{aligned}$$

Let $T^n(j)$ denote the inertia group of $\mathfrak{p}_j|_{N_n}$ in $k_n$. Since the restriction map from $G(L_\infty/k_n)$ to $G(N_n/k_n)$ maps inertia groups of $\mathfrak{p}_j$ surjectively, this implies that $T^n(j)$ is topologically generated by $(\nu_{n(S),n}x_j|_{N_n} g_1^{p^{n-n(S)}}$. Note that $(\nu_{n(S),n}x_j)|_{N_n} \in G(N_n/k_\infty)$, $g_1^{p^{n-n(S)}} \in T_1(n)$, and we have the *direct* product decomposition (3.4):

$$G(N_n/k_n) = T_1(n) \times G(N_n/k_\infty).$$

Let

$$T(n) = T_1(n)\dots T_s(n) = T_1(n) \times R,$$
$$R = T(n) \cap G(N_n/k_\infty) = G(N_n/k_\infty L_n).$$

From the above analysis of $T_j(n)$ we see that $R$ is the closed subgroup of $G(N_n/k_\infty)$ generated by $(\nu_{n(S),n}x_j)|_{N_n}$, $j = 1,\dots, s$ and is congruent to the closed subgroup of $G(L_\infty/k_\infty)/(\gamma^{p^n} - 1)G(L_\infty/k_\infty)$ generated by

$$(\nu_{n(S),n}x_j)|_{N_n} \operatorname{mod}(\gamma^{p^n} - 1)G(L_\infty/k_\infty), \quad j = 1,\dots, s.$$

(For the second equality we use the fact that restriction to $N_n$ maps $G(L_\infty/k_\infty)$ *canonically* to $G(N_n/k_\infty) = G(L_\infty/k_\infty)/(\gamma^{p^n} - 1)G(L_\infty/k_\infty)$). Therefore

(3.8) $G(L_\infty/k_\infty L_n)$ is the closed subgroup of $G(L_\infty/k_\infty)$ generated by $\nu_{n(S),n}x_j$, $j = 1,\dots, s$, and $(\gamma^{p^n} - 1)G(L_\infty/k_\infty)$.

The $x_j$ were chosen independently of $n$ and (3.8) holds for *all* $n \geq n(S)$. In particular

(3.9) $G(L_\infty/k_\infty L_{n(S)})$ is the closed subgroup of $G(L_\infty/k_\infty)$ generated by $x_j$, $j = 1,\dots s$, and $(\gamma^{p^{n(S)}} - 1)G(L_\infty/k_\infty)$.

Combining (3.8) and (3.9) we obtain the desired equality:

$$G(L_\infty/k_\infty L_{n(S)}) = \nu_{n(S),n}G(L_\infty/k_\infty L_n). \qquad \blacksquare$$

Since $k_n = k_\infty \cap L_n$, Galois theory yields the following corollary.

COROLLARY (3.10). *Given $n \geq n(S)$, we have a canonical isomorphism*:

$$G(L_\infty/k_\infty)/\nu_{n(S),n}G(L_\infty/k_\infty L_{n(S)}) \xrightarrow{\sim} G(k_\infty L_n/k_\infty) \xrightarrow{\sim} G(L_n/k_n).$$

## 4. R-generalized $S$-class groups and $\Lambda$-modules

Let $\theta_n = \{\alpha \mid \alpha \in k_n,\ |\alpha|_\mathfrak{p} \leq 1\ \forall \notin S\}$, the ring of $S$-integers of $k_n$. Let $I_n$ be the invertible $\theta_n$-submodules of $k_n$ which are prime to $R$ (the "$S$-ideals" prime to $R$) and let $P_n = \{\alpha\theta_n \mid \alpha \equiv 1 \bmod \mathfrak{p}\ \forall \mathfrak{p} \in R\}$. The $R$-generalized

$S$-class group is $I_n/P_n$ and we write $A_n$ for the Sylow $p$-subgroup of $I_n/P_n$. At each finite level, the Artin map of class field theory gives us a classical isomorphism

$$A_n \xrightarrow{\sim} G(L_n/k_n), \qquad \mathfrak{a} \bmod P_n \mapsto \langle \mathfrak{a}, L_n/k_n \rangle. \tag{4.1}$$

Combining this with (3.10) we obtain:

PROPOSITION (4.2). *Given $n \geq n(S)$, we have a canonical isomorphism*

$$\begin{gathered} A_n \xrightarrow{\sim} G(L_\infty/k_\infty)/\nu_{n(S),n} G(L_\infty/k_\infty L_{n(S)}), \\ \mathfrak{a} \bmod P_n \mapsto \sigma \bmod \nu_{n(S),n} G(L_\infty/k_\infty L_{n(S)}) \end{gathered}$$

*where* $\langle \mathfrak{a}, L_n/k_n \rangle = \sigma|_{k_n}$.

The isomorphisms of (4.2) are useful for studying $\varinjlim A_n$, where the limit is with respect to the maps induced by extension of ideals, because they piece together nicely. More precisely:

PROPOSITION (4.3). *Given $m \geq n \geq n(S)$, we have the commutative diagram*

$$\begin{array}{ccc} A_n & \xrightarrow{\sim} & G(L_\infty/k_\infty)/\nu_{n(S),n} G(L_\infty/k_\infty L_{n(S)}) \\ \downarrow & & \downarrow \\ A_m & \xrightarrow{\sim} & G(L_\infty/k_\infty)/\nu_{n(S),m} G(L_\infty/k_\infty L_{n(S)}) \end{array}$$

*where the isomorphisms are as in* (4.2), *the map on the left is the natural map induced by extension of S-ideals, and the righthand map is*

$$\sigma \bmod \nu_{n(S),n} G(L_\infty/k_\infty L_{n(S)}) \mapsto \nu_{n,m}\sigma \bmod \nu_{n(S),m} G(L_\infty/k_\infty L_{n(S)})$$

*Proof.* Argue as in [3, Theorem 7].

THEOREM (4.4).

$$\begin{aligned} \varinjlim_{n \geq 0} A_n &\simeq \varinjlim_{n \geq n(S)} A_n \\ &\simeq \varinjlim_{n \geq n(S)} G(L_\infty/k_\infty)/\nu_{n(S),n} G(L_\infty/k_\infty L_{n(S)}) \\ &\simeq \varinjlim_{n \geq n(S)} G(L_\infty/k_\infty L_{n(S)})/\nu_{n(S),n} G(L_\infty/k_\infty L_{n(S)}) \end{aligned}$$

(*as* $\mathbf{Z}_p[\Gamma]$-*modules*) *where the limits are with respect to the maps of* (4.3).

*Proof.* The first isomorphism is clear by the definition of direct limits and the second is immediate from Proposition (4.3). The last isomorphism follows from the exact sequence

$$\begin{aligned}1 &\to G(L_\infty/k_\infty L_{n(S)})/G(L_\infty/k_\infty L_{n(S)})\\ &\to G(L_\infty/k_\infty)/\nu_{n(S),n}G(L_\infty/k_\infty L_{n(S)})\\ &\to G(L_\infty/k_\infty)/G(L_\infty/k_\infty L_{n(S)}) \to 1\end{aligned}$$

by taking direct limits with respect to the map of (4.3). This is because $G(L_\infty/k_\infty)/G(L_\infty/k_\infty L_{n(S)})$ is finite so

$$\varinjlim_{n \geq n(S)} \left(G(L_\infty/k_\infty)/G(L_\infty/k_\infty L_{n(S)})\right)$$

is trivial. ■

We identify $\mathbf{Z}_p[\Gamma]$-modules and $\Lambda$-modules by requiring that $\gamma$ act as $T+1$. Note that this identifies $\nu_{n(S),n(S)+n}$ with

$$\begin{aligned}&\left((1+T)^{p^{n+n(S)}}-1\right)/\left((1+T)^{p^{n(S)}}-1\right)\\ &\quad= \prod_{i=n(S)+1}^{n(S)+n}\left((1+T)^{p^i}-1\right)/\left((1+T)^{p^{i-1}}-1\right)\\ &\quad= \prod_{i=n(S)+1}^{n(S)+n}\left(\sum_{j=0}^{p-1}(1+T)^{jp^{n-1}}\right) \in \Lambda.\end{aligned}$$

THEOREM (4.5). *As $\Lambda$-modules,*

$$\begin{aligned}H &=_{\text{def}} \operatorname{Hom}_{\mathbf{Z}_p}\left(\varinjlim A_n, \mathbf{Q}_p/\mathbf{Z}_p\right)\\ &\simeq \varprojlim_{n\geq 0} \operatorname{Hom}_{\mathbf{Z}_p} G(L_\infty/k_\infty L_{n(S)})\\ &\qquad /\nu_{n(S),n+n(S)}G(L_\infty/k_\infty L_{n(S)}), \mathbf{Q}_p/\mathbf{Z}_p)\\ &\simeq \alpha\left(G(L_\infty/k_\infty L_{n(S)})\right)\\ &\sim G(L_\infty/k_\infty L_{n(S)})\\ &\sim G(L_\infty/k_\infty).\end{aligned}$$

*Proof.* The first isomorphism follows directly from (4.4) and the second from (2.7) since $\{\nu_{n(S),n(S)+n}\}_{n\geq 0}$ is $G(L_\infty/k_\infty L_{n(S)})$-admissible. The first pseudo-isomorphism follows from (2.10)(i) and the second is obvious from (2.4) and (2.5). ■

Theorem (4.5) may be regarded as the result for $k_\infty$ corresponding to the isomorphism $A_n \simeq G(L_n/k_n)$ for finite $n$. However, one has a second analog based on the following lemma.

LEMMA (4.6). *Given $m \geq n \geq n(S)$, we have the commutative diagram*

$$\begin{array}{ccc} A_m & \xrightarrow{\sim} & G(L_\infty/k_\infty)/\nu_{n(S),m}G(L_\infty/k_\infty L_{n(S)}) \\ \text{norm}\downarrow & & \downarrow \\ A_n & \xrightarrow{\sim} & G(L_\infty/k_\infty)/\nu_{n(S),n}G(L_\infty/k_\infty L_{n(S)}) \end{array}$$

*where the isomorphisms are as in* (4.2) *and the map on the right is the canonical surjection.*

*Proof.* Argue as in [3, Theorem 7]. ■

The commutativity of (4.6) may be extended to the larger commutative diagram

$$\begin{array}{ccccc} A_m & \xrightarrow{\sim} & G(L_\infty/k_\infty)/\nu_{n(S),m}G(L_\infty/k_\infty L_{n(S)}) & \xrightarrow[\sim]{\text{Galois theory}} & G(L_m/k_m) \\ \downarrow & & \downarrow & & \downarrow \text{restriction} \\ A_n & \xrightarrow{\sim} & G(L_\infty/k_\infty)/\nu_{n(S),n}G(L_\infty/k_\infty L_{n(S)}) & \xrightarrow[\sim]{\text{Galois theory}} & G(L_n/k_n). \end{array}$$

The diagram induces an isomorphism of indirect limits:

$$\varprojlim_{n \geq n(S)} A_n \xrightarrow{\sim} \varprojlim_{n \geq n(S)} G(L_n/k_n)$$

where the limit of class groups is with respect to norm maps and the limit of Galois groups is with respect to restriction maps. On the other hand, $\bigcup_{n \geq n(S)} L_n = L_\infty$ so

$$\varprojlim_{n \geq n(S)} G(L_n/k_n) \xrightarrow{\sim} G(L_\infty/k_\infty).$$

We therefore have proved:

THEOREM (4.7). *We have a canonical $\Lambda$-module isomorphism*

$$A =_{\text{def}} \varprojlim A_n \xrightarrow{\sim} G(L_\infty/k_\infty)$$

*where the inverse limit is with respect to norm maps.*

Recalling (4.5) we have;

COROLLARY (4.8). *H and A are pseudo-isomorphic $\Lambda$-modules.*

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