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# A CLASS OF EXTREME $L_p$ CONTRACTIONS, $p \neq 1, 2, \infty$, AND REAL $2 \times 2$ EXTREME MATRICES

BY

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## 1. Introduction

For any two Banach spaces $\mathbf{E}, \mathbf{F}$, denote by $\mathscr{L}(\mathbf{E}, \mathbf{F})$ the Banach space of all bounded linear operators from $\mathbf{E}$ to $\mathbf{F}$. The scalar field may be the reals or the complex numbers. An operator $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ is a *contraction* if it is in the closed unit ball $\mathscr{U}(\mathbf{E}, \mathbf{F})$ of $\mathscr{L}(\mathbf{E}, \mathbf{F})$; it is an *extreme* contraction if it is an extreme point of $\mathscr{U}(\mathbf{E}, \mathbf{F})$. The set $\mathscr{E}(\mathbf{E}, \mathbf{F})$ of extreme contractions in $\mathscr{U}(\mathbf{E}, \mathbf{F})$ has been identified in the cases where $\mathbf{E}$ and $\mathbf{F}$ are both (a) Hilbert spaces [9], (b) $L_\infty$ spaces over $\sigma$-finite measure spaces [4], [8], [23] or (c) $L_1$ spaces [8, Theorem 2]. For related results, see [1], [5], [6], [7], [15], [19], [20], [23], [24], [25]. In case (a), $\mathscr{E}(\mathbf{E}, \mathbf{F})$ consists of isometries and coisometries (adjoints of isometries from the dual $\mathbf{F}'$ to the dual $\mathbf{E}'$). In the case where $\mathbf{E}$ and $\mathbf{F}$ are both $L_p$ space, $p \neq 1, 2, \infty$, it follows from strict convexity of the unit balls of $\mathbf{F}$ and $\mathbf{E}'$ that isometries and coisometries are still extreme, but the complete description of $\mathscr{E}(\mathbf{E}, \mathbf{F})$ is yet unresolved. In this article, starting with a simple inequality (Lemma 2.1), we establish (in Theorem 2.8) a sufficient condition for extremeness of an operator $T$ from an $L_p$ space $\mathbf{E}$ to an $L_q$ space $\mathbf{F}$ in terms of the isometric vectors of $T$ and of $T^*$, illustrated with examples. We then use this and the apparatus developed in [10] to generalize (in Theorem 3.8) a recent result of R. Grząślewicz [6] which characterizes, in the case $1 < p = q < \infty$, $p \neq 2$, those extreme contractions belonging to a subset of $\mathscr{L}(\mathbf{E}, \mathbf{F})$ consisting of, in our terminology, *semidisjunctive* operators. ([6] considers only $l_p$ spaces.) We observe with interest that in cases (b) and (c), the extreme contractions are semidisjunctive. We prove (in Theorem 4.4) that operators in the weak* closed convex hull of semidisjunctive extreme contractions have contractive linear moduli. Using results on real $2 \times 2$ extreme matrices derived in Section 5, we extend the characterization in Theorem 3.8 to a larger class of contractions (Theorem 6.4). Finally we construct extreme contractions not of one of the two main types established in Proposition 2.6 and Theorem 2.8.

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## 2. A sufficient condition for extreme contractions

LEMMA 2.1. *Let $u$ and $v \neq 0$ be scalars. If $2 < q < \infty$, then*

$$|u + v|^q + |u - v|^q > 2|u|^q + q|u|^{q-2}|v|^2. \tag{2.1}$$

*If $0 < q < 2$ and $u \neq 0$, the reverse inequality holds.*

*Proof.* By homogeneity, we need only consider the case $|u| = 1$.

$$\text{L.H.S.} = (1 + c + t)^{q/2} + (1 + c - t)^{q/2} \equiv f(t),$$

where $t = 2\,\mathrm{Re}\, u\bar{v}$ and $c = |v|^2$. Hence $|t| \leq 2\sqrt{c}$.
If $2 < q < \infty$, $sf'(s) \geq 0$ for $|s| < 2\sqrt{c}$, and so

$$\text{L.H.S.} \geq f(0) = 2(1 + c)^{q/2} = 2 + qc(1 + \theta c)^{q/2-1} > 2 + qc = \text{R.H.S.}$$

for some real number $0 < \theta < 1$, by the mean value theorem.
If $0 < q < 2$, the inequalities are reversed.

Let $1 \leq p, q \leq \infty$ be fixed and let $\mathbf{E} \equiv L_p(X, \mathscr{F}, \mu)$ and $\mathbf{F} \equiv L_q(Y, \mathscr{G}, \nu)$ be the usual Lebesgue spaces over arbitrary measure spaces. For each $A \in \mathscr{F}$, define $\mathbf{E}_A$ to be the subspace of all $f \in \mathbf{E}$ with support $\operatorname{supp} f \equiv \{f \neq 0\} \subset A$. Define $\mathbf{F}_B$ for $B \in \mathscr{G}$ similarly. $\mathbf{E}_A$ is identified with $L_p(A, \mathscr{F} \cap A, \mu|\mathscr{F} \cap A)$. For $p < \infty$, write $(\mathbf{E}_A)' = (\mathbf{E}')_A$ as $\mathbf{E}'_A$. The norms in $\mathbf{E}, \mathbf{F}$, etc., will all be denoted by $\|\cdot\|$, as no confusion seems likely.

THEOREM 2.2. *Let $2 < p \leq \infty$ and $2 \leq q < \infty$, and $O \neq T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ be such that $\|Tf\| = \|T\| \cdot \|f\|$ for some $0 \neq f \in \mathbf{E}$. Then, with $A = \operatorname{supp} f$ and $B = \operatorname{supp} Tf$,* (i) *$T\mathbf{E}_{A^c} \subset \mathbf{F}_{B^c}$ if $p \leq q$, and* (ii) *$T\mathbf{E}_{A^c} = \{0\}$ if $p > q$.*

*Proof.* Let $0 \neq g \in \mathbf{E}_{A^c}$, if that exists, and $\lambda$ be any positive number. Then

$$q\lambda^2 h \leq |Tf + \lambda Tg|^q + |Tf - \lambda Tg|^q - 2|Tf|^q, \tag{2.2}$$

where $h = |Tf|^{q-2}|Tg|^2$ if $2 < q < \infty$, by Lemma 2.1, and $h = |Tg|^2$ if $q = 2$, by the parallelogram law (with equality holding). Integrating (2.2) we get

$$q\lambda^2 \int h\, d\nu \leq 2\|T\|^q(\|f \pm \lambda g\|^q - \|f\|^q). \tag{2.3}$$

When $2 < p < \infty$, the right-hand side of (2.3) is, as $\lambda \to 0$, of the order of

$$(\|f\|^p + \lambda^p\|g\|^p)^{q/p} - \|f\|^q = O(\lambda^p),$$

and when $p = \infty$, it is 0 for $\lambda = \|f\|/\|g\|$. Hence in either case, $h = 0$. Further, when $p > q > 2$, comparing $\|Tf + \lambda Tg\|$ and $\|f + \lambda g\|$ for small enough $\lambda > 0$, we infer that $Tg = 0$. The conclusions follow.

COROLLARY 2.3. *If* $2 < p = q < \infty$, *and* $T$: $\mathbf{E} \to \mathbf{F}$ *is an isometry, then* $Tf.Tg = 0$ a.e. *iff* $f.g = 0$ a.e.

*Remark* 2.4. (i) In the case of a matrix operator on real $l_p$, ($2 < p = q < \infty$), a special case of Theorem 2.2, namely when $f$ is a coordinate vector, was proved by Hennefeld [7, Lemma 2.2][2] using an inequality resembling (2.1). Grząślewicz [5, Lemma 1] essentially reproved this by a different method, which can be adapted to the complex case.

(ii) In Theorem 2.2 (i), in general we may not have $T\mathbf{E}_A \subset \mathbf{F}_B$. See (2.12).

(iii) Lemma 2.1 remains valid if $u$ and $v$ are vectors in a Hilbert space, with $|u|$ and $|v|$ taken as Hilbert space norms, and $u\bar{v}$ in the proof as an inner product $\langle u, v\rangle$. Similarly Theorem 2.2 and Corollary 2.3 are still valid if $\mathbf{E}$ is a Bochner $L_p$ space of Banach-valued functions and $\mathbf{F}$ a Bochner $L_q$ space of Hilbert-valued ones.

(iv) Corollary 2.3 was proved by Lamperti more generally for $0 < p = q < \infty$, $p \neq 2$ [18, Theorem 3.1], using a case of Lemma 3.2. See Remark 3.3.

Denote by $\mathscr{C}(l_p)$ the set of compact operators on real or complex sequence space $l_p$. Thanks to Theorem 2.2, the following proposition, which Hennefeld [7, Theorem 2.4] proved for real $l_p$, is true also for complex $l_p$, by the same method used in [7]. (The case $p = 1$ can be treated directly.)

PROPOSITION 2.5. *For* $1 \leq p < \infty$ *and* $p \neq 2$, *the unit ball of* $\mathscr{C}(l_p)$ *is the norm closed convex hull of its extreme points.*

For a contraction $T: \mathbf{E} \to \mathbf{F}$, define

$$\mathscr{N}(T) = \{f \in \mathbf{E} : \|Tf\| = \|f\|\}, \tag{2.4}$$

and

$$\overline{\operatorname{span}}\,\mathscr{N}(T) = \text{weakly closed } (= \text{norm closed}) \text{ linear span of } \mathscr{N}(T). \tag{2.5}$$

PROPOSITION 2.6. *Let* $1 \leq p \leq \infty$ *and* $1 < q < \infty$. *Let* $T \in \mathscr{U}(\mathbf{E}, \mathbf{F})$.

(a) *If* $R \in \mathscr{L}(\mathbf{E}, \mathbf{F})$, $T \pm R \in \mathscr{U}(\mathbf{E}, \mathbf{F})$ *and* $0 \neq f \in \mathscr{N}(T)$, *then* $Rf = 0$.

(b) *If* $\overline{\operatorname{span}}\,\mathscr{N}(T) = \mathbf{E}$, *then* $T$ *is extreme.*

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[2] I am grateful to Dr. M. Feder for drawing my attention to this work.

*Proof.* Part (a) follows from strict convexity of $\mathbf{F}$, and (b) follows from (a).

For any $A \in \mathscr{F}$, and any $\mathscr{F}$-measurable function $f$, define

$$f_A = \begin{cases} f \text{ on } A \\ 0 \text{ on } A^c, \end{cases} \tag{2.6}$$

and similarly $g_B$, for any $B \in \mathscr{G}$ and any $\mathscr{G}$-measurable function $g$. For any operator $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$, and any pair $\mathbf{E}_A$ and $\mathbf{F}_B$, define $T_{BA} \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ by

$$T_{BA} f = (Tf_A)_B \quad \text{for all } f \in \mathbf{E}. \tag{2.7}$$

$T_{BA}$ is also regarded as an operator in $\mathscr{L}(\mathbf{E}_A, \mathbf{F}_B)$. This does not affect its norm. $(T_{BA})^* = (T^*)_{AB}$, which we shall write simply as $T^*_{AB}$.

An arbitrary measure space $(X, \mathscr{F}, \mu)$ can be reconstituted as a direct union of finite ones, without altering the $L_p$ spaces over it, for all $1 \leq p < \infty$. Indeed, let $\{A_i\}$ be a maximal family of mutually disjoint (modulo null sets) subsets of $X$ of positive finite measures. Let $(X', \mathscr{F}', \mu')$ be the disjoint direct union of all

$$(A_i, \mathscr{F} \cap A_i, \mu|\mathscr{F} \cap A_i).$$

Then $L_p(X, \mathscr{F}, \mu)$ can be identified with $L_p(X', \mathscr{F}', \mu')$ isometrically and lattice isomorphically. (Cf. [17, §15, corollary to Theorem 3]). We shall assume in the sequel that such reconstitution has been made for the underlying measure spaces of $\mathbf{E}$ and $\mathbf{F}$. This done, $(X, \mathscr{F}, \mu)$ retains most of the nice properties of the $\sigma$-finite case. Mostly, every subfamily of $\mathscr{F}$ has a supremum. This will be used in Theorem 2.8 and Section 3. From this also, each *projection band* of $\mathbf{E}$, i.e., each closed linear subspace $\mathbf{E}_1$ of $\mathbf{E}$ such that $f \in \mathbf{E}_1$ implies $\mathbf{E}_{\operatorname{supp} f} \subset \mathbf{E}_1$, is of the form $\mathbf{E}_A$ for some $A \in \mathscr{F}$, and, of course, conversely, so that some of the concepts used here, e.g., $f_A$, $\mathbf{E}_A$, $T_{BA}$, can be expressed in the lattice theoretic language in terms of projection bands.

By the following result, we need only consider extreme contractions in the case $p > 2$.

LEMMA 2.7. *If $1 < p$, $q < \infty$, then $T \in \mathscr{E}(\mathbf{E}, \mathbf{F})$ if and only if $T^* \in \mathscr{E}(\mathbf{F}', \mathbf{E}')$.*

*Proof.* This follows from the reflexivity of both $\mathbf{E}$ and $\mathbf{F}$.

We now present our first main result. An analogous one can be formulated for the $q = 2$.

THEOREM 2.8. *Let* $2 < p \leq q < \infty$. *Suppose that* $T \in \mathscr{U}(\mathbf{E}, \mathbf{F})$ *is such that for some* $A \in \mathscr{F}$, (*which may be* $\varnothing$),
(i) $\overline{\operatorname{span}}\{f_A \colon f \in \mathscr{N}(T)\} = \mathbf{E}_A$,
*and*
(ii) $\overline{\operatorname{span}}\{g_{B^c} \colon g \in \mathscr{N}(T^*)\} = F'_{B^c}$,
*where* $B = \sup\{\operatorname{supp} Tf \colon f \in \mathscr{N}(T) \cap \mathbf{E}_A\}$.
*Then* $T$ *is extreme, and* $T_{BA^c} = O$.

*Proof.* Let $R \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ be such that $T \pm R \in \mathscr{U}(\mathbf{E}, \mathbf{F})$. If $f \in \mathbf{E}_A \cap \mathscr{N}(T)$, then by Proposition 2.6(a), $Rf = 0$ and so $f \in \mathscr{N}(T \pm R)$. By Theorem 2.2, we must have

$$T_{\operatorname{supp} Tf . A^c} = R_{\operatorname{supp} Tf . A^c} = O.$$

Consequently by definition of $B$,

$$T_{BA^c} = R_{BA^c} = O. \tag{2.8}$$

If now $f \in \mathscr{N}(T)$, then $Rf = 0$. So $0 = R_{BX} f = R_{BA} f_A$, in view of (2.8). By (i), we conclude that

$$R_{BA} = O. \tag{2.9}$$

Similarly if $g \in \mathscr{N}(T^*)$, then $R^* g = 0$. So $R^*_{XB^c} g_{B^c} = 0$, because of (2.8) and (2.9). By (ii), $R^*_{XB^c} = 0$ and so

$$R_{B^c X} = O. \tag{2.10}$$

Summing up (2.8)–(2.10), $R = O$. Thus $T$ is extreme and by (2.8), $T_{BA^c} = O$.

*Remark* 2.9. It can be shown that Proposition 2.6 and Theorem 2.8 remain valid if $\mathbf{E}$ and $\mathbf{F}$ are Bochner $L_p$ and $L_q$ spaces of Hilbert-valued functions.

For any scalar $a$, define $a^{p-1} = \operatorname{sgn} a . |a|^{p-1}$ if $p > 1$. This will be used on $L_p$ vectors. By the following lemma, the conditions in Theorem 2.8 can be expressed solely in terms of $\mathscr{N}(T)$.

LEMMA 2.10. *Let* $1 < p$, $q < \infty$ *and* $T \in \mathscr{U}(\mathbf{E}, \mathbf{F})$. *Then* $0 \neq f \in \mathscr{N}(T)$ *if and only if*

$$T^*\left(\overline{Tf}\right)^{q-1} = \|f\|^{q-p} \bar{f}^{p-1}, \tag{2.11}$$

*in which case* $(\overline{Tf})^{q-1} \in \mathscr{N}(T^*)$.

*Proof.* (2.11) implies $f \in \mathscr{N}(T)$ by duality action on $f$. The converse follows from the fact that $\|f\|^{q-p} \bar{f}^{p-1}$ is the unique element of $\mathbf{E}'$ of norm

$\|f\|^{q-1}$, which value it assumes on $f/\|f\|$, while $T^*(\overline{Tf})^{q-1}$ has the same property if $f \in \mathcal{N}(T)$, in which case $\|(\overline{Tf})^{q-1}\| = \|Tf\|^{q-1} = \|f\|^{q-1}$, so that $(\overline{Tf})^{q-1} \in \mathcal{N}(T^*)$ also.

In the following, let $l_p^n$ denote the $l_p$ space on $n$ unit masses. The following example, in particular (2.12), seems to be hitherto unknown.

*Example* 2.11. Let $2 < p = q < \infty$. Let $(a, b) > (0,0)$ be a unit $l_p^2$ vector and

$$\rho = \begin{pmatrix} a^{p-1} & b^{p-1} \\ -tb & ta \end{pmatrix},$$

where $t > 0$, an $l_p^2$ contraction isometric on $(a, b)$ and in some other direction. (See Theorem 5.1(d)(iii) for the existence of $\rho$. It has precisely two isometric directions, both real [14].) Let $r, s > 0$ be such that $r^{p'} + s^{p'} = 1$, where $p' = p/(p-1)$. Then

$$\sigma = \begin{pmatrix} a^{p-1} & b^{p-1} & 0 \\ -rtb & rta & s \end{pmatrix}$$

is an extreme contraction from $l_p^3$ to $l_p^2$. Indeed,

$$\begin{aligned} \|\sigma(x, y, z)\|^p &= |a^{p-1}x + b^{p-1}y|^p + |-rtbx + rtay + sz|^p \\ &\le |a^{p-1}x + b^{p-1}y|^p + (|-tbx + tay|^p + |z|^p)(r^{p'} + s^{p'})^{p-1} \\ &= \|\rho(x, y)\|^p + |z|^p \\ &\le (|x|^p + |y|^p) + |z|^p \\ &= \|(x, y, z)\|^p, \end{aligned}$$

where the first inequality (Hölder's) becomes an equality if and only if

$$z = (s/r)^{p'-1}t(-bx + ay),$$

and so does the second if and only if $(x, y) \in \mathcal{N}(\rho)$. It follows that $\sigma$ is a contraction isometric only along $(a, b, 0)$ and some $(x', y', z')$, with $(x', y') \in \mathcal{N}(\rho)$ not in the direction of $(a, b)$. By Lemma 2.10 and Theorem 2.8, $\sigma$ is extreme. (Alternatively, $\sigma^*$ satisfies the condition of Proposition 2.6(b).) Similarly the matrix

$$\tau = \begin{pmatrix} ua^{p-1} & ub^{p-1} & 0 \\ va^{p-1} & vb^{p-1} & 0 \\ -rtb & rta & s \end{pmatrix}, \quad (u, v) > (0,0),\ u^p + v^p = 1,$$

is an extreme contraction on $l_p^3$, with $\mathcal{N}(\tau) = \mathcal{N}(\sigma)$.

Concrete examples for $\sigma$ are obtained by taking

$$\rho = 2^{1/p-1}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

a contraction isometric in the directions of $(1, \pm 1)$ only. (See Remark 3.3 and use Lemma 2.10). Hence $\sigma$ is extreme and isometric in exactly two directions, those of $(1, 1, 0)$ and some $(1, -1, z')$. Take $u = v = 2^{-1/p}$. Then

$$\tau = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -c & c & d \end{pmatrix}, \quad c, d > 0, 2c^{p'} + d^{p'} = 1, \tag{2.12}$$

is extreme and isometric only in the directions of $(1, 1, 0)$ and $(-c, c, d)^{p'-1}$.

Theorem 3.8 contains other examples.

## 3. Semidisjunctive extreme contractions

In what follows, let $1 < p = q < \infty$. Let $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$. $T$ is said to be *disjunctive* (or *Lamperti*, see [10]) if it maps functions with disjoint supports to functions with disjoint supports. $T$ is *codisjunctive* if $T^*$ is disjunctive. If $T$ is such that $T\mathbf{E}_A \subset \mathbf{F}_B$ and $T\mathbf{E}_{A^c} \subset \mathbf{F}_{B^c}$, then it is a *direct sum* of $U \equiv T_{BA}$ and $V \equiv T_{B^cA^c}$, written $T = U \oplus V$, subordinate to the band decompositions $\mathbf{E} = \mathbf{E}_A \oplus \mathbf{E}_{A^c}$ and $\mathbf{F} = \mathbf{F}_B \oplus \mathbf{F}_{B^c}$. (We allow in degenerate cases $\mathbf{E}_A = \mathbf{E}$ or $\{0\}$, and likewise $\mathbf{F}_B = \mathbf{F}$ or $\{0\}$.) If $T = U \oplus V$, with $U$ disjunctive and $V$ codisjunctive, then $T$ is *semidisjunctive*. The class of such operators will be denoted by $\mathscr{S}(\mathbf{E}, \mathbf{F})$. It contains isometries and coisometries, as these are disjunctive and codisjunctive respectively [10]. $T$ is a *hemiisometry* if it is isometric on some $\mathbf{E}_A$ and annihilates $\mathbf{E}_{A^c}$. $T$ is *coextensive* if $\sup\{\operatorname{supp} Tf\colon f \in \mathbf{E}\} = Y$. This is equivalent to $T^*\mathbf{F}'_B \neq \{0\}$ if $\mathbf{F}'_B \neq \{0\}$. Dually, $T$ is *extensive* if $T\mathbf{E}_A \neq \{0\}$ for all $\mathbf{E}_A \neq \{0\}$. For $u \in \mathbf{E}'$ and $g \in \mathbf{F}$, the tensor product $T \equiv g \otimes u \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ is defined by $Tf = \langle f, u\rangle g$, $(f \in \mathbf{E})$. So

$$\|g \otimes u\| = \|g\| \cdot \|u\|. \tag{3.1}$$

The structure of a disjunctive $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ is described in [10, Theorems 4.1 and 4.2], when $\mathbf{E} = \mathbf{F}$ and the measure space is $\sigma$-finite. It is still valid in general after the reconstitution of the measure spaces has been made. Thus,

$$Tf(y) = h(y)\Phi f(y) \quad \text{for all } f \in \mathbf{E}, \tag{3.2}$$

where $h$ is a measurable function on $Y$ and $\Phi$ is a linear operator on measurable functions induced by a Boolean $\sigma$-homomorphism, denoted also by $\Phi$, from $(X, \mathscr{F}, \mu)$ to $(Y, \mathscr{G}, \nu)$, (see [10, Definition 4.1]), having formal properties of composition operators and such that $\Phi 1_A = 1_{\Phi A}$. Furthermore, there is a bounded, non-negative measurable function $D(T)$ on $X$, such that

$$(3.3) \qquad \|Tf\|^p = \int |h|^p \Phi |f|^p \, d\nu = \int D(T)|f|^p \, d\mu \quad \text{for all } f \in \mathbf{E}.$$

Denote a multiplication operator by the measurable function that induces it. By (3.2), if $\alpha$ is a bounded measurable function, then

$$(3.4) \qquad T \circ \alpha = \Phi\alpha \circ T.$$

It also follows from (3.2) that for each $A \in \mathscr{F}$,

$$(3.5) \qquad T = T_{(\Phi A)A} \oplus T_{(\Phi A)^c A^c}.$$

Define $\delta(T) = D(T)^{1/p}$. The next lemma is essentially contained in [10, Theorem 4.3].

LEMMA 3.1. *If $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ is disjunctive, then*

$$(3.6) \qquad T = S \circ \delta(T)$$

*for a disjunctive $S \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ with $\delta(S) = 1_{\operatorname{supp} \delta(T)}$ and the same associated $\sigma$-homomorphism.*

*Proof.* Define $S$ by $T \circ (\delta(T)^{-1})_{\operatorname{supp} \delta(T)}$. The results follow from (3.3).

The following lemma occurs in [6]. (See [14] for a sharper inequality.)

LEMMA 3.2. *Let $a, b > 0$ and $x, y$ be scalars. If $0 < p \leq 2$, then*

$$(3.7) \qquad |ax - a^{1-p/2}b^{p/2}y|^p + |bx + a^{p/2}b^{1-p/2}y|^p \leq (a^p + b^p)(|x|^p + |y|^p).$$

*If $p \geq 2$, the reverse inequality holds.*

*Remark* 3.3. The case $a = b = 1$ is a pair of the classical Clarkson inequalities [3, Theorem 2]. In this case, if $p \neq 2$, then equality holds in (3.7) if and only if $xy = 0$. These follow from Lemma 2.1. See [18, Lemma 2.1] and [21, Lemma 15.14] for other proofs.

LEMMA 3.4. *$T = U \oplus V$ is a contraction if and only if so are $U$ and $V$.*

*Proof.* Suppose $U = T_{BA}$ and $V = T_{B^c A^c}$. For any $f \in \mathbf{E}$, $\|f\|^p = \|f_A\|^p + \|f_{A^c}\|^p$ and $\|Tf\|^p = \|Uf_A\|^p + \|Vf_{A^c}\|^p$. The conclusion follows.

COROLLARY 3.5. *If $T = U \oplus V$ is an extreme contraction, then so are $U$ and $V$.*

LEMMA 3.6. *Let $1 < p \leq 2$ and let $T \in \mathcal{U}(\mathbf{E}, \mathbf{F})$ be disjunctive but not codisjunctive, and annihilate some $\mathbf{E}_{A^c} \neq \{0\}$. Then $T$ is not extreme.*

*Proof.* Since $T^*$ is not disjunctive, there exist non-zero $v_1, v_2 \in \mathbf{F}'$ with disjoint supports $B, B'$ respectively such that $A' \equiv \operatorname{supp} T^*v_1 \cap \operatorname{supp} T^*v_2 \neq \phi$. Consideration of duality action shows that $A' \subset A$. By (3.5) and Corollary 3.5 we need only consider the case $A' = A$. Hence $T = T_{BA} + T_{B^cA}$. There are isometries $U\colon \mathbf{E}_A \to \mathbf{F}_B$ and $V\colon \mathbf{E}_A \to \mathbf{F}_{B^c}$ and strictly positive, measurable functions $\xi, \zeta$ on $A$ such that

$$T_{BA} = U \circ \xi \quad \text{and} \quad T_{B^cA} = V \circ \zeta. \tag{3.8}$$

These follow from (3.3), Lemma 3.1 and the fact that $T^*v_1 = \xi U^*v_1$ and $T^*v_2 = \zeta V^*v_2$ have support $A$. By (3.3) again,

$$\xi^p + \zeta^p = D(T) \leq 1. \tag{3.9}$$

Let $C = A^c$. Take $e \in \mathbf{E}_A$ and $u \in \mathbf{E}'_C$ each of norm 1 and define $W = e \otimes u$. Let

$$R = \left(U \circ \xi^{1-p/2}\zeta^{p/2} - V \circ \xi^{p/2}\zeta^{1-p/2}\right) \circ W.$$

Then $R$ is of rank 1 and annihilates $\mathbf{E}_A$. For all $f \in \mathbf{E}$,

$$\begin{aligned} \|(T \pm R)f\|^p &= \int \left\{|\xi f_A \pm \xi^{1-p/2}\zeta^{p/2}Wf_C|^p + |\zeta f_A \mp \xi^{p/2}\zeta^{1-p/2}Wf_C|^p\right\} d\mu \\ &\leq \int (\xi^p + \zeta^p)(|f_A|^p + |Wf_C|^p)\, d\mu \\ &\leq \|f_A\|^p + \|f_C\|^p = \|f\|^p, \end{aligned}$$

by (3.8), Lemma 3.2, (3.9) and (3.1). Thus $T \pm R \in \mathcal{U}(\mathbf{E}, \mathbf{F})$ and $T$ is not extreme.

*Remark* 3.7. Lemma 3.6 may not be true without disjunctiveness. Let

$$T = 2^{1/p-1}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

$T^*$ is an extreme contraction by Lemma 3.2 and Proposition 2.6(b), and so is $T$ by Lemma 2.7.

THEOREM 3.8. *Let* **E** *and* **F** *be* $L_p$ *spaces. Consider* $\mathscr{S}(\mathbf{E},\mathbf{F}) \cap \mathscr{E}(\mathbf{E},\mathbf{F})$.

(a) *If* $2 < p$, *it consists of isometries and direct sums of a coextensive hemiisometry and a coisometry.*

(b) *If* $1 < p < 2$, *it consists of coisometries and direct sums of an isometry and the adjoint of a coextensive hemiisometry.*

(*The direct sums in* (a) *and* (b) *may degenerate to one of the component types.*)

*Proof.* By duality, we need only prove (a). The sufficiency part follows from Proposition 2.6(b) and Theorem 2.8. Conversely let $T \in \mathscr{S}(\mathbf{E},\mathbf{F}) \cap \mathscr{E}(\mathbf{E},\mathbf{F})$.

Case (a)(i). $T$ disjunctive. $T$ must be a hemiisometry. For otherwise $\delta \equiv \delta(T) \le 1$ and $\{0 < \delta < 1\} \ne \phi$. $T = S \circ \delta$, with $S$ as in Lemma 3.1. Hence

$$O \ne R \equiv S \circ (1 - \delta) \in \mathscr{L}(\mathbf{E},\mathbf{F}) \quad \text{and} \quad T \pm R \in \mathscr{U}(\mathbf{E},\mathbf{F}),$$

a contradiction. If $T$ is neither isometric nor coextensive, then for some $\mathbf{E}_A \ne \{0\}$ and some $\mathbf{F}_B \ne \{0\}$, $T = T_{B^cA^c} \oplus O_{BA}$. In view of (3.1), $O_{BA}$ is not extreme in $\mathscr{U}(\mathbf{E}_A, \mathbf{F}_B) \ne \{O\}$. Neither is $T$, by Corollary 3.5. Thus $T$ is either an isometry or a coextensive hemiisometry.

Case (a)(ii). $T = U \oplus V$, with $V = T_{B^cA^c}$ codisjunctive but not disjunctive, and $U = T_{BA}$ disjunctive. Readjusting the decomposition $\mathbf{F} = \mathbf{F}_B \oplus \mathbf{F}_{B^c}$ if necessary, we can assume $U$ coextensive (relative to $\mathbf{F}_B$, which may be $\{0\}$). By Corollary 3.5, $U$ and $V$ are extreme. As in the first part of (a)(i), $U$ and $V^*$ are hemiisometries, since the proof there is valid for $1 < p < \infty$. By Lemma 3.6, $V^*$ has to be an isometry.

*Remark* 3.9. $\tau$ in (2.12) is an extreme $l_p^3$ contraction, not semidisjunctive and not isometric in three directions, thus not among those extreme contractions considered in Proposition 2.6(b) and Theorem 3.8. See also Section 7.

## 4. Linear moduli and weak* closed convex hull of $\mathscr{S}\mathscr{E}$

As **F** is reflexive, $\mathscr{L}(\mathbf{E},\mathbf{F}) = \mathscr{L}(\mathbf{E},\mathbf{F}'') = \mathbf{G}'$, where **G** is the $\pi$-norm completion of the algebraic tensor product $\mathbf{E} \otimes \mathbf{F}'$ [26, Lemma 4.1.2]. So the $\mathbf{E} \otimes \mathbf{F}'$-topology of $\mathscr{L}(\mathbf{E},\mathbf{F})$ is its weak* topology. By the Banach-Alaoglu and the Krein-Milman theorems, $\mathscr{U}(\mathbf{E},\mathbf{F}) = \overline{\text{conv}}^{w^*}\mathscr{E}(\mathbf{E},\mathbf{F})$, the weak* closed convex hull of $\mathscr{E}(\mathbf{E},\mathbf{F})$. This fact and a result on $\overline{\text{conv}}^{w^*}\mathscr{S}\mathscr{E}$ (Theorem 4.4) imply that, in general,

$$\mathscr{S}\mathscr{E} \equiv \mathscr{S}(\mathbf{E},\mathbf{F}) \cap \mathscr{E}(\mathbf{E},\mathbf{F}) \ne \mathscr{E}(\mathbf{E},\mathbf{F}).$$

$S \in \mathscr{L}(\mathbf{E},\mathbf{F})$, necessarily positive, is said to (*absolutely*) *majorize* $T \in \mathscr{L}(\mathbf{E},\mathbf{F})$, or be an (*absolute*) *majorant* of $T$, if $|Tf| \le S|f|$, ($f \in \mathbf{E}$). Extending an idea in [2], we say that $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ has a (bounded) *linear modulus* $|T| \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ if $|T|$ is a least majorant of $T$, i.e., $|T|$ majorizes $T$, and all majorants of $T$ majorize $|T|$. Clearly $|T|$ is unique if it exists.

LEMMA 4.1. *Let $1 \leq p,\ q \leq \infty$ and let $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$. If for a positive constant $K$, any $f \in \mathbf{E}^+$ and any finite measurable partition $\mathscr{D} = \{A^1, A^2, \dots, A^n\}$ of $X$,*

$$\|f^*(\mathscr{D})\| \leq K\|f\|, \quad \textit{where}\ f^*(\mathscr{D}) = \sum_{i=1}^{n} |Tf_{A^i}|, \tag{4.1}$$

*then $T$ has a linear modulus $|T|$ of norm not greater than $K$. Furthermore, for $f \in \mathbf{E}^+$,*

$$|T|f = \sup\{f^*(\mathscr{D})\}, \tag{4.2}$$

*and*

$$|T|f = \sup\{|Tg| : |g| \leq f\}. \tag{4.3}$$

*Proof.* The construction is basically the same as in [2] for the case $p = q = 1$. Let $f \in \mathbf{E}^+$. For $q < \infty$, there exist successively finer $\mathscr{D}^1, \mathscr{D}^2, \dots$ such that

$$\|f^*(\mathscr{D}^n)\| \uparrow \sup \|f^*(\mathscr{D})\|$$

and $f^*(\mathscr{D}^n)$ increases to an $\mathbf{F}^+$ vector, designated $|T|f$, which majorizes all $f^*(\mathscr{D})$ and has norm $\leq K\|f\|$. This follows from the facts that $f^*(\mathscr{D})$ increases with refinement of $\mathscr{D}$, and $\|g\|$ is strictly increasing in $g \in \mathbf{F}^+$. For $q = \infty$, if $Y$ is a direct union of $\{Y^\alpha\}$, $\nu Y^\alpha < \infty$ for each $\alpha$, replace $\|f^*(\mathscr{D})\|$ by $\langle 1_{Y^\alpha}, f^*(\mathscr{D})\rangle$, etc., to get each $1_{Y^\alpha}|T|f$, whence $|T|f$. In either case, (4.2) follows. Linearity of $|T|$ on $\mathbf{E}^+$ is easy to establish for simple functions, and its general validity follows by approximation. Similarly for the majorant property. The supremum in (4.3) exists and is majorized by $|T|f$ since $|Tg| \leq |T|\,|g| \leq |T|f$ if $|g| \leq f$, and $\mathbf{F}$ is an order complete vector lattice. By the argument in [2], (4.3) follows. Hence $|T|$, linearly extended, is the least majorant of $T$.

COROLLARY 4.2. *Every $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ having a majorant $R$ has $|T|$ of norm not greater than $\|R\|$.*

*Proof.* We have $f^*(\mathscr{D}) \leq R|f|$ for all $f \in \mathbf{E}$. The conclusion follows.

*Remark* 4.3. (i) Lemma 4.1 remains valid if $0 < \min\{p, q\} < 1$, with "norm" defined in $\mathbf{E}$ as $(\int |f|^p\, d\mu)^{1/p}$ and similarly in $\mathbf{F}$. When $p \leq \min\{q, 1\}$

$\equiv r$, (4.1) always holds with $K = \|T\|$, as, by appropriate Minkowski inequalities,

$$\left\|\sum |Tf_{A^i}|\right\| \leq \left(\sum \|Tf_{A^i}\|^r\right)^{1/r} \leq \left(\sum \|Tf_{A^i}\|^p\right)^{1/p} \leq \|T\| \cdot \|f\|.$$

Hence $|T|$ exists of norm $\|T\|$. The same is true for $0 < p \leq q = \infty$, for the sup in (4.3) exists with norm not greater than $\|T\| \cdot \|f\|$ and, as shown in [2], majorizes all $f^*(\mathscr{D})$. For $1 = p \leq q < \infty$, $|T|^* = |T^*|$, as can be shown by the use of (4.2). See also the treatment given in [22], chapter IV, for the cases $p = 1$ or $q = \infty$.

(ii) If $1 < p = q < \infty$ and $T$ is a positive contraction, then $\mathscr{N}(T)$ is a closed vector sublattice [12] (cf. Proposition 5.6). Such a $T$ satisfying the conditions of Theorem 2.8 is thus a direct sum of a coextensive hemiisometry and a coisometry—a case of Theorem 3.8(a). This extends to a $T$ with contractive $|T|$, as the conditions imply $T = \xi|T| \circ \zeta$ for signum functions $\xi, \zeta$ [12].

THEOREM 4.4. *Let $1 < p = q < \infty$. Every $T \in \overline{\operatorname{conv}}^{w^*}\mathscr{S}\mathscr{E}$ has contractive $|T|$.*

*Proof.* As disjunctive and codisjunctive contractions have obvious contractive moduli, so do semidisjunctive ones. Each $S \in \operatorname{conv}\mathscr{S}\mathscr{E}$ has a contractive majorant and so contractive $|S|$, by Corollary 4.2. Let $T \in \overline{\operatorname{conv}}^{w^*}\mathscr{S}\mathscr{E}$. With the notation in Lemma 4.1, there exists $g \in (\mathbf{F}')^+$ of norm 1 such that

$$\|f^*(\mathscr{D})\| = \langle f^*(\mathscr{D}), g\rangle = \sum_{i=1}^{n} \langle Tf_{A^i}, \bar{\xi}_i g\rangle,$$

where $\xi_i = \operatorname{sgn} Tf_{A^i}$. Hence there exist $S_1, S_2, \ldots \in \operatorname{conv}\mathscr{S}\mathscr{E}$ such that

$$\begin{aligned} \|f^*(\mathscr{D})\| &= \sum_{i=1}^{n} \lim_{j\to\infty} \langle S_j f_{A^i}, \bar{\xi}_i g\rangle \\ &\leq \limsup_{j\to\infty} \sum_{i=1}^{n} \langle |S_j| f_{A^i}, g\rangle \\ &= \limsup_{j\to\infty} \langle |S_j| f, g\rangle \\ &\leq \|f\| \cdot \|g\| = \|f\|. \end{aligned}$$

By Lemma 4.1, contractive $|T|$ exists.

*Remark* 4.5. If $T \in \mathscr{E}(\mathbf{E}, \mathbf{F})$ is not in $\mathscr{S}\mathscr{E}$, it may not have a contractive $|T|$. Example 2.11 contains an extreme $l_p^2$ contraction $\rho$ with $|\rho| = 2^{1/p-1}(1,1) \otimes (1,1)$, of norm $2^{1/p}$, $p \geq 2$.

## 5. Real 2 × 2 extreme contractions

In this section, we prove some results about extreme contractions on $l_p^2$, $p \neq 1, 2, \infty$, that will be used in proving Theorem 6.4 and counter-examples in Section 7. Theorem 5.4 generalizes [5, Theorem] to both real and complex $l_p^2$. Note that even in the case of real $l_p^2$, which [5] treats, our method is different, and this case does not imply the complex $l_p^2$ case, and vice versa. The proofs are given for complex $l_p^2$, with the real case following by restricting the angle $\omega$ to 0 and $\pi$.

THEOREM 5.1. *Let $\tau$ be a linear operator on (real or complex) $l_p^2, 1 < p < \infty$, $p \neq 2$. If $\|\tau\| = 1$, in particular if $\tau$ is an extreme contraction, then it is of the form*

$$\tau_s = (u, v) \otimes (\bar{x}, \bar{y})^{p-1} + se^{i\phi}(-\bar{v}, \bar{u})^{p-1} \otimes (-y, x),\ s \geq 0, \tag{5.1}$$

*where $(x, y)$ and $(u, v)$ are unit vectors in $l_p^2$ and $\phi$ is an angle. Furthermore:*

(a) *$\tau_s$ is an extreme contraction for exactly one value $s^* \geq 0$ of $s$ for each set of parameters, and $s^*$ is upper semicontinuous jointly in the parameters $(x, y)$, $(u, v)$ and $e^{i\phi}$, and continuous in $e^{i\phi}$.*

(b) *$s^* = 0$ if and only if $xy = 0 \neq uv$ when $p > 2$ or $xy \neq 0 = uv$ when $1 < p < 2$, with*

$$\tau_0 = \begin{cases} g \otimes (1,0) \text{ or } g \otimes (0,1) & \text{when } p > 2, \\ (1,0) \otimes h \text{ or } (0,1) \otimes h & \text{when } 1 < p < 2, \end{cases} \tag{5.2}$$

*for $g$ a unit vector in $l_p^2$ and $h$ one in $(l_p^2)'$, both with non-zero coordinates.*

(c) *Let $\sigma = e^{i\phi}\operatorname{sgn}(xy\overline{uv})$. If* (i) *$|u| = |x|$, $\sigma = 1, 0$, or* (ii) *$|u| = |y|$, $\sigma = -1, 0$, then $s^* = 1$ and $\tau_1$ is*

$$\operatorname{diag}(\varepsilon_1, \varepsilon_2) \quad (\textit{for } \text{(i)})$$

*or*

$$\begin{pmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & 0 \end{pmatrix} \quad (\textit{for } \text{(ii)}),$$

*with $|\varepsilon_1| = |\varepsilon_2| = 1$; moreover, $\tau_{s^*}$ is an isometry only in these cases.*

(d) *In all other sub-cases of $\sigma = \pm 1, 0$, namely* (i) *$|u| \neq |x|$, $\sigma = 1$,* (ii) *$|u| \neq |y|$, $\sigma = -1$, and* (iii) *$xy \neq 0 = uv$ when $p > 2$, or $xy = 0 \neq uv$ when $1 < p < 2$, $\tau_{s^*}$ is isometric in two directions; moreover, in case* (i),

$$\min\left\{\left|\frac{y}{v}\right|^{p-1}\left|\frac{u}{x}\right|, \left|\frac{x}{u}\right|^{p-1}\left|\frac{v}{y}\right|\right\} < s^* < \left|\frac{xy}{uv}\right|^{p/2-1},$$

*and in case* (ii), *the same bounds hold for $s^*$ with $x$ and $y$ interchanged.*

*Proof.* If $\|\tau\| = 1$, which is the case if $\tau$ is an extreme contraction, $\tau$ maps some unit vector $(x, y)$ to another, $(u, v)$. $\tau - (u, v) \otimes (\bar{x}, \bar{y})^{p-1}$ annihilates $(x, y)$ and so is of rank 1 or 0. By Lemma 2.10, its dual annihilates $(\bar{u}, \bar{v})^{p-1}$ and is also of rank 1 or 0. It follows that $\tau$ is of the form (5.1). When $xyuv \neq 0$, up to isometric factors $\operatorname{diag}(\operatorname{sgn} u, \operatorname{sgn} v)$ and $\operatorname{diag}(\operatorname{sgn} \bar{x}, \operatorname{sgn} \bar{y})$, and with $e^{i\phi}$ changed to $e^{i\phi}\operatorname{sgn}(\overline{xyuv})$, (and when $xyuv = 0$, replace any 0 among $x, y, u, v$ by 1 in all these expressions), we can replace $x, y, u, v$ by $|x|$, etc. Up to $n = 0, 1$ or 2 isometric factors $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $\tau$ and with $e^{i\phi}$ changed to $(-1)^n e^{i\phi}$, we can assume $|x| \geq |y|$ and $|u| \geq |v|$. We now need only consider the case $x \geq y \geq 0$ and $u \geq v \geq 0$.

For each $r \geq 0$ and each angle $\omega$, $\tau_s$ maps $f(x, r, \omega) = (x, y) + re^{i\omega}(-y, x)^{p-1}$ to $f(u, rs, \phi + \omega)$. Define

$$\begin{aligned}(5.3) \quad F(x, r, \omega) &= \|f(x, r, \omega)\|^p \\ &= |x - re^{i\omega}y^{p-1}|^p + |y + re^{i\omega}x^{p-1}|^p \\ &= x^{-p}|xye^{-i\omega} + rx^p|^p + y^{-p}| - xye^{-i\omega} + ry^p|^p \\ &= x^{-p}G(rx^p + xy\cos\omega) + y^{-p}G(ry^p - xy\cos\omega),\end{aligned}$$

where $G(z) = Z^{p/2}$, $Z = z^2 + \lambda^2$ ($z$ real), and $\lambda = xy\sin\omega$. Hence

$$(5.4) \qquad F_r(x, r, \omega) = G'(rx^p + xy\cos\omega) + G'(ry^p - xy\cos\omega).$$

Now $G'(0) = 0$ and $G'(z) = pzZ^{(p-2)/2}$ is an odd function, positive and strictly increasing for $z > 0$. (To see this when $1 < p < 2$, rewrite $G'(z)$ as $p[Z^{p-1} - \lambda^2/Z^{2-p}]^{1/2}$.) Although derived for $y > 0$, (5.4) is clearly also true for $y = 0$. Since the two arguments for $G'$ in (5.4) add up to $r$, the numerically larger one is positive if $r > 0$. It follows that (5.4) is positive for $r > 0$, and $F(x, r, \omega)$ is a strictly increasing function of $r \geq 0$. (This also follows geometrically from $(-y, x)^{p-1}$ being tangent to the unit $l_p^2$ "sphere" at $(x, y)$, even in the complex case.) Hence

$$\|\tau_s f(x, r, \omega)\|^p = F(u, rs, \phi + \omega)$$

is a strictly increasing function of $s \geq 0$ for each $r > 0$ and each $\omega$. Therefore

$$(5.5) \qquad F(u, rs, \phi + \omega) = F(x, r, \omega), \quad \text{i.e., } \|\tau_s f\| = \|f\|,$$

for $s$ a unique $s' = s'(\phi, x, u, r, \omega) > 0$ for each $f = f(x, r, \omega)$ with $r > 0$, since

$$F(u, 0, \phi + \omega) = 1 = F(x, 0, \omega) < F(x, r, \omega).$$

Further, equality in (5.5) becomes $>$ (resp. $<$) if $s >$ (resp. $<$) $s'$. Evidently these hold also for $f = (-y, x)^{p-1}$. Except for multiples of $(x, y)$, each vector is a multiple of one of those considered. Hence $s'$ induces a function $s''$ on the unit $l_p^2$ vectors except those in the direction of $f = (x, y)$, for which (5.5) holds for all $s \geq 0$. By the continuity of $F$ and its strict monotonicity proved above, $s''$ is continuous jointly in the parameters $(x, y)$, $(u, v)$ and $e^{i\phi}$ and the variable $f$. Hence the infimum $s^*$ of $s''$ over all $f$, i.e., that of $s'$ over all $r > 0$ and $\omega$, is an upper semicontinuous function in the parameters. With $x$, $u$ fixed, $s^*$ is a function of $\phi$. Let

$$s^{**}(\phi) = \liminf s' \quad \text{as } re^{i\omega} \to 0.$$

Define

$$R^* = \left\{se^{i\phi} : 0 \leq s \leq s^*(\phi),\ \phi \text{ any angle}\right\},$$

and similarly $R^{**}$ with $s^*$ replaced by $s^{**}$. Clearly $R^* \subset R^{**}$, and $\tau_s$ in (5.1) is a contraction if and only if $se^{i\phi} \in R^*$. Evidently $R^*$ is convex and hence $s^*(\phi)$ is continuous in $e^{i\phi}$.

By continuity, $s^*$ is equal to either (I) some $s''(f)$ or (II) $s^{**} < \infty$. In case (I), $\tau_{s^*}$ is isometric in two directions, and is extreme by Proposition 2.6(b).

In case (II) if for some operator $\rho$, $\tau_{s^*} \pm \rho$ are contractions, then by strict convexity of the $l_p^2$ norm, they map $(x, y)$ to $(u, v)$, and are of the form (5.1), with possibly different values for $se^{i\phi}$. We shall show in individual cases that $R^{**}$ is strictly convex (which is trivial in real $l_p^2$), and so in case (II), $\tau_{s^*} = \tau_{s^{**}}$ is extreme. Indeed, $s^{**}(\phi)$ is characterized as the supremum of all $s \geq 0$ for which $\tau_s$ in (5.1) is contractive on all unit vectors close enough to $(x, y)$. It follows that $R^{**}$ is convex, and consequently the curve $\phi \mapsto s^{**}(\phi)e^{i\phi}$ is continuous (where $s^{**}(\phi) < \infty$) and forms the boundary of $R^{**}$.

If $y = v = 0$, then $\tau_s = \operatorname{diag}(1, se^{i\phi})$. Obviously $s'' = 1 = s^{**} = s^*$, and $\tau_1$, an isometry, is extreme. This partly proves (c).

When $y > 0$, Taylor expansion gives

$$\begin{aligned}(5.6)\qquad F(x, r, \omega) &= 1 + \tfrac{1}{2}p(xy)^{p-2}r^2\left[1 + (p-2)\cos^2\omega\right] \\ &\quad + \tfrac{1}{6}p(p-2)(xy)^{p-3}(x^p - y^p)r^3 \\ &\quad \times \cos\omega\left[3 + (p-4)\cos^2\omega\right] + \cdots\end{aligned}$$

and similarly for $F(u, rs, \phi + \omega)$, when $v > 0$.

For (b) and (d)(iii), we need only consider the case $p > 2$, by Lemmas 2.7 and 2.10. If $y = 0 < v$, then comparison of $F(1, r, \omega) = 1 + r^p$ and (5.6) for $F(u, rs, \phi + \omega)$ $(s > 0)$ as $r \to 0$ shows that $\tau_s$ is a contraction only if $s = 0$. Hence $s^*(\phi) = s^{**}(\phi) = 0$. Thus $R^{**} = \{0\}$, strictly convex. So $\tau_0$ is extreme. To complete the proof of (b), observe that in case (I), $s^* > 0$, and in all subcases of (II) except (b), $s^* = s^{**} > 0$ (see below). Reciprocally, if $v = 0 < y$, then $s^{**} = \infty$. This is case (I) and proves (d)(iii).

Now suppose $y, v > 0$. By (5.5), $rs' \to 0$ as $re^{i\omega} \to 0$. Hence by (5.5) and using (5.6) to compare $r^2$ terms, as $r \to 0$, $s'(\phi, x, u, r, \omega)$ tends to

$$s'(\phi, x, u, 0+, \omega) = \left[\left(\frac{xy}{uv}\right)^{p-2} H(\phi, \omega)\right]^{1/2} > 0, \tag{5.7}$$

where

(5.8)

$$H(\phi, \omega) = \frac{1 + (p-2)\cos^2\omega}{1 + (p-2)\cos^2(\phi + \omega)} = \frac{p + (p-2)\cos 2\omega}{p + (p-2)\cos 2(\phi + \omega)} > 0,$$

and

$$s^{**}(\phi) = s'(\phi, x, u, 0+, \omega') \tag{5.7$'$}$$

where $\omega'$ is an angle minimizing $H$. Now

$$H_\omega(\phi, \omega) = \frac{4(p-2)\sin\phi\{p\cos(\phi + 2\omega) + (p-2)\cos\phi\}}{[p + (p-2)\cos 2(\phi + \omega)]^2}.$$

Hence if $\sin\phi = 0$, $H(\phi) \equiv 1$; and if $\sin\phi \neq 0$,

$$p\cos(\phi + 2\omega') = -(p-2)\cos\phi \tag{5.9}$$

and $-(p-2)\sin\phi\sin(\phi + 2\omega') \geq 0$. By (5.8) and (5.9), $\sin(\phi + 2\omega') \neq 0$ and

(5.10)

$$\begin{aligned}\min_\omega H &= H(\phi, \omega')\\ &= \frac{p[\sin^2(\phi + 2\omega') + \cos^2(\phi + 2\omega')] + (p-2)[\cos\phi\cos(\phi + 2\omega') + \sin\phi\sin(\phi + 2\omega')]}{p[\sin^2(\phi + 2\omega') + \cos^2(\phi + 2\omega')] + (p-2)[\cos\phi\cos(\phi + 2\omega') - \sin\phi\sin(\phi + 2\omega')]}\\ &= \frac{\sin(\phi + 2\omega')}{\sin(\phi + 2\omega')} \cdot \frac{p\sin(\phi + 2\omega') + (p-2)\sin\phi}{p\sin(\phi + 2\omega') - (p-2)\sin\phi}\\ &= \frac{M(\phi) - |(p-2)\sin\phi|}{M(\phi) + |(p-2)\sin\phi|},\end{aligned}$$

where $M(\phi) = [4(p-1) + (p-2)^2\sin^2\phi]^{1/2}$. Hence, (even if $\sin\phi = 0$),

$$H^{1/2}(\phi, \omega') = \frac{2\sqrt{p-1}}{M(\phi) + |(p-2)\sin\phi|}. \tag{5.10$'$}$$

From (5.7), (5.7$'$) and (5.10$'$), it is routine to show that the curve $\phi \mapsto {}^{**}(\phi)e^{i\phi}$

has no linear part. Thus $R^{**}$ is strictly convex, and extremeness in case (II) is established. This completes the proof of (a).

To complete the proof of (c), it remains to consider the case $s = s^* = s^{**}$, $y, v > 0$ and $\phi = 0$ or $\pi$, since $H$ in (5.8) is constant ($= 1$) relative to $\omega$, as will follow if $\tau_{s^*}$ is an isometry, only for these angles. Now $s = [xy/(uv)]^{p/2-1}$, by (5.7). By (5.6), the Taylor expansions of $F(x, r, \omega')$ and $F(u, rs, \phi + \omega')$ in $r$ are equal up to the $r^2$ term, and so contraction requirement implies that the $r^3$ terms are equal. By (5.9), $\cos \omega' \neq 0$ and so

$$(xy)^{p-3}(x^p - y^p) = (uv)^{p-3}(u^p - v^p)s^3\varepsilon,$$

or

$$(uv)^{p/2}(x^p - y^p) = (xy)^{p/2}(u^p - v^p)\varepsilon,$$

where $\varepsilon = 1$ if $\phi = 0$, and $\varepsilon = -1$ if $\phi = \pi$. On squaring we have

$$u^p v^p(1 - 4x^p y^p) = x^p y^p(1 - 4u^p v^p),$$

and so $v^p(1 - v^p) = y^p(1 - y^p)$, or $(v^p - y^p)(1 - v^p - y^p) = 0$, whence $v = y$, and $s = 1$. Further, if $\phi = \pi$, then in addition $x = y$. This transforms to the case $\phi = 0$ by the method given at the beginning. For $\phi = 0$,

$$\tau_1 = (x, y) \otimes (x, y)^{p-1} + (-y, x)^{p-1} \otimes (-y, x) = \text{identity operator}.$$

This proves (c).

Now if $u \neq x$, $vy \neq 0$ and $e^{i\phi} = 1$, then we have just shown that this is case (I) and that $s^* < s^{**} = [xy/(uv)]^{p/2-1}$. Let $s$ be the minimum occurring in (d)(i). Then $\tau_s \geq O$, with exactly one matrix element equal to 0. With $f = (x, y) > (0, 0)$, $T = \tau_s$ satisfies (2.11). By [11, Theorem 4], $T$ is a contraction. By Proposition 5.1 below, $T$ has only one isometric direction. Hence $T \neq \tau_{s^*}$. So $s < s^*$. This proves (d)(i). (d)(ii) is proved similarly.

COROLLARY 5.2. *If $\tau_{s^{**}}$ is a contraction (with $s^{**}$ as defined in the proof), then it is extreme.*

*Remark* 5.3. (i) For $p > 2$, $\phi$ any angle and $(x, y) = (1, 0)$,

$$s^* = 0 \quad \text{when } (u, v) = \left((1 - v^p)^{1/p}, v\right) \textit{ with } 0 < v < 1,$$

but

$$s^* = 1 \quad \text{when } (u, v) = (1, 0).$$

So $s^*$ is not continuous in $(u, v)$.

(ii) For $1 < p < 2$, $\phi$ any angle, $(x, y) = (1,0)$ and $(u, v) > (0,0)$, $\min s^* = 2^{-(2-p)/p}$ [14]; cf. $s^* \geq (uv)^{1-p/2}$, a less sharp lower bound given by Lemma 3.2. Sup $s^*$ ($< 1$) is also found [14]. Compare with (c); again $s^*$ is not continuous in $(u, v)$. So by (i) and duality, $s^*$ is not continuous in either $(x, y)$ or $(u, v)$, if $p \neq 2$.

(iii) Theorem 5.1 is used to prove nonextremeness of some classes of contractions. Indeed, Lemma 3.6 can be proved by using (a) and (d)(iii) instead of Lemma 3.2. (a) and (d)(i) will be used to prove Theorem 6.2. (c) and (d)(ii) can be used to prove [13] that a contractive projection on $L_p$ ($1 < p < \infty$, $p \neq 2$) that is not the identity operator is not extreme.

(iv) (5.2) is the limit of $\tau_{s^*}$ in (d)(i) or (ii).

THEOREM 5.4. *Let $\tau$ be a real contraction on real or complex $l_p^2$ ($p \neq 1, 2, \infty$), with a real isometric unit vector. Then $\tau$ is extreme if and only if it is of the form* (5.2) *or isometric in two directions.*

*Proof.* By the given condition, $\tau$ is of the form (5.1) with all parameters real. The conclusion then follows from Theorem 5.1.

COROLLARY 5.5. (*Grząślewicz* [5]). *A contraction on real $l_p^2$ ($p \neq 1, 2, \infty$) is extreme if and only if it is of the form* (5.2) *or isometric in two directions.*

PROPOSITION 5.6. *Let $1 < p < \infty$ and let*

$$\tau = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$$

*where $a, c, d > 0$. Then $\tau$, as an operator on $l_p^2$, is of norm 1 if and only if $a, c, d < 1$ and*

$$\frac{1}{d}\left(\frac{1 - a^p}{c}\right)^{1/(p-1)} - \frac{c}{d} = \frac{cd^{1/(p-1)}}{1 - d^{p/(p-1)}} \ (= w), \tag{5.11}$$

*or equivalently*

$$c = (1 - a^p)^{1/p}\left(1 - d^{p'(p-1)}\right)^{1-1/p},$$

*in which case $\tau$ is isometric solely in the direction of* $(1, w)$.

*Proof.* Suppose $\|\tau\| = 1$. Clearly $a, c, d < 1$. $\tau$ has an isometric vector $(x, y) \neq (0,0)$, which by Lemma 2.10 satisfies

$$a^p \bar{x}^{p-1} + c(c\bar{x} + d\bar{y})^{p-1} = \bar{x}^{p-1}, \qquad d(c\bar{x} + d\bar{y})^{p-1} = \bar{y}^{p-1}.$$

Obviously $x \neq 0$, or else $(x, y) = (0, 0)$. Let $w = y/x$ and solve for it. The result is (5.11). Conversely the conditions imply that $f = (1, w) > (0, 0)$ satisfies (2.11) in Lemma 2.10 for $\tau$. By [11, Theorem 4], $\tau$ is a contraction isometric on $(1, w)$. So $\|\tau\| = 1$.

COROLLARY 5.7. *If*

$$\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

*is a contraction on* $l_p^2$, $1 < p < \infty$, *with exactly one element* 0, *then it is not extreme.*

*Proof.* We may assume $b = 0$ and $a, c, d > 0$ in $\tau$ (up to isometric factors). For $p = 2$, $\tau$ is not an isometry, and so is not extreme. For $p \neq 2$, $\tau$ is not extreme if $\|\tau\| < 1$. If $\|\tau\| = 1$, $(x, y) = (1 + w^p)^{-1/p}(1, w)$ gives the only isometric direction and $\tau$ is of the form (5.1) and, by a simple calculation, belongs to case (d)(i) of Theorem 5.1, by which $\tau$ is not extreme.

## 6. Extension of the characterization for $\mathscr{S}\mathscr{E}$

Let $\mathbf{E}$ and $\mathbf{F}$ be $L_p$ spaces, $1 < p < \infty$, $p \neq 2$. Denote by $\mathscr{S}'(\mathbf{E}, \mathbf{F})$ the class of those $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ such that

$$(6.1) \quad T_{BA^c} = 0 \quad \text{and} \quad T_{YA}, T^*_{XB^c} \text{ are disjunctive for some } A \in \mathscr{F},\ B \in \mathscr{G}.$$

Clearly $\mathscr{S}(\mathbf{E}, \mathbf{F}) \subset \mathscr{S}'(\mathbf{E}, \mathbf{F})$. We shall show (in Theorem 6.4) that Theorem 3.8 will remain true if $\mathscr{S}$ is replaced by $\mathscr{S}'$. First we lay out the principles by which the proof is effected.

Let $\phi \neq A \in \mathscr{F}$ and $\phi \neq B \in \mathscr{G}$. Let there be an $L_p$ space $\mathbf{G} = L_p(Z, \mathscr{H}, \lambda) \neq \{0\}$ and contractions

$$(6.2) \quad U\colon \mathbf{G} \to \mathbf{F}_B, \quad V\colon \mathbf{G} \to \mathbf{F}_{B^c}, \quad R\colon \mathbf{G}' \to \mathbf{E}'_A \quad \text{and} \quad S\colon \mathbf{G}' \to \mathbf{E}'_{A^c}.$$

With these fixed, for bounded measurable functions $\alpha, \beta, \gamma, \delta$ on $Z$, define

$$(6.3) \qquad \Omega(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} U \circ \alpha R^* & U \circ \beta S^* \\ V \circ \gamma R^* & V \circ \delta S^* \end{pmatrix} \in \mathscr{L}(\mathbf{E}, \mathbf{F}),$$

an operator matrix relative to the decompositions $\mathbf{E} = \mathbf{E}_A \oplus \mathbf{E}_{A^c}$ and $\mathbf{F} = \mathbf{F}_B \oplus \mathbf{F}_{B^c}$. Considering

$$\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as an operator on $l_p^2$, define $\Delta_p(a, b, c, d) = \|\tau\|$. $\Delta_p$ is a continuous function. We shall write $\Delta$ for $\Delta_p$.

LEMMA 6.1. *For any bounded measurable functions $\alpha, \beta, \gamma, \delta$ on $Z$,*

$$\|\Omega(\alpha, \beta, \gamma, \delta)\| \le \|\Delta(\alpha, \beta, \gamma, \delta)\|_\infty,$$

*and equality holds if $U$, $V$, $R$ and $S$ are isometries.*

*Proof.* Clearly L.H.S. ≤ R.H.S. Conversely, under the isometry assumption, by Lemma 2.10, each $f \in \mathbf{G}$ has an isometric preimage by $R^*$, namely

$$\left(\overline{Rf^{p-1}}\right)^{1/(p-1)} \in \mathbf{E}_A,$$

and similarly one by $S^*$. It follows, via approximation of $\alpha, \beta, \gamma, \delta$ by simple functions, that the reverse inequality, hence the equality, holds.

PROPOSITION 6.2. *If* $\|\Delta(\alpha, 0, \gamma, \delta\|_\infty \le 1$ *and* $\alpha, \gamma, \delta > 0$ a.e., *then* $\Omega(\alpha, 0, \gamma, \delta)$ *is not extreme.*

*Proof.* Let $T = \Omega(\alpha, 0, \gamma, \delta)$. We have $0 < \Delta \equiv \Delta(\alpha, 0, \gamma, \delta) \le 1$ a.e.

Case (1). $\{\Delta < 1\} \ne \varnothing$. Let $\alpha' = (1/\Delta - 1)\alpha$, etc. It is easy to see (cf. proof of Theorem 3.8, case (a)(i)) that $W \equiv \Omega(\alpha', 0, \gamma', \delta') \ne O$ and $T \pm W \in \mathscr{U}(\mathbf{E}, \mathbf{F})$, by Lemma 6.1, and $T$ is not extreme.

Case (2). $\Delta \equiv 1$. Without loss of generality we may assume $\alpha, \gamma, \delta > 0$ a.e. (cf. proof of Theorem 5.1). By Proposition 5.6, for positive real numbers $a, c, d$ satisfying $\Delta(a, 0, c, d) = 1$, $c$ is a continuous function of $(a, d)$, and

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \tau_s$$

as in (5.1), for unit vectors $(x, y), (u, v) = \tau_s(x, y) > (0, 0)$, a scalar $s > 0$, each depending continuously on $(a, d)$, and $\phi = 0$. By Corollary 5.7 and Theorem 5.1(a), there is $t \equiv s^* > s$, which is an upper semicontinuous function of $(a, d)$, such that $\tau_t$ is a contraction. Let $\hat{x}$, $\hat{y}$, etc. be the functions obtained by composing $x$, $y$, etc. with $(\alpha, \delta)$. Then they are measurable and

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} = \hat{\tau}_s.$$

Further, $w \equiv \min\{s, t - s\}(-v, u)^{p-1} \otimes (-y, x) \ne O$ and $\hat{\tau}_s \pm \omega$ are a.e. $l_p^2$ contractions. As in case (1), this and Lemma 6.1 imply that T is not extreme.

*Remark* 6.3. (i) By the same arguments, operator (6.3) with $\|\Delta(\alpha, \beta, \gamma, \delta\|_\infty \le 1$ is not extreme if $\alpha, \beta, \gamma, \delta > 0$ a.e., since a strictly positive operator on $l_p^2$ of norm 1 is isometric in only one direction [12], [16], and so is not extreme by Theorem 5.4.

(ii) The same conclusion holds if $\beta \equiv \delta \equiv 0$, $\alpha, \gamma \neq 0$ a.e. and $1 < p \leq 2$. (Modify the proof of Lemma 3.6, with $e \in \mathbf{G}$.)

THEOREM 6.4. *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be* $L_p$ *spaces*, $1 < p < \infty$, $p \neq 2$. *Then*

$$\mathscr{E}(\mathbf{E},\mathbf{F}) \cap \mathscr{S}'(\mathbf{E},\mathbf{F}) = \mathscr{E}(\mathbf{E},\mathbf{F}) \cap \mathscr{S}(\mathbf{E},\mathbf{F}).$$

*Proof.* Each contraction $T$ in $\mathscr{S}'(\mathbf{E},\mathbf{F}) \setminus \mathscr{S}(\mathbf{E},\mathbf{F})$ satisfies (6.1) with non-zero $T_{BA}$, $T_{B^cA}$ and $T_{B^cA^c}$. To prove the theorem, we need only show that such a $T$ is not extreme. We can further assume that those three sub-operators are all extensive and coextensive. This follows from Corollary 3.5 and the fact that, by (3.5), $T$ decomposes into a direct sum, of which one summand has such a triplet of sub-operators while the other is semidisjunctive. Now by Lemma 3.1, there exist coextensive isometries

$$(6.2') \qquad U\colon \mathbf{E}_A \to \mathbf{F}_B, \quad V\colon \mathbf{E}_A \to \mathbf{F}_{B^c} \quad \text{and} \quad Q\colon \mathbf{F}'_{B^c} \to \mathbf{E}'_{A^c},$$

such that

$$(6.3') \qquad T_{BA} = U \circ \alpha,\ T_{B^cA} = V \circ \gamma \text{ and } T^*_{A^cB^c} = Q \circ \eta$$

for measurable functions $0 < \alpha$, $\gamma < 1$ on $A$ and $0 < \eta < 1$ on $B^c$. Let the associated Boolean $\sigma$-homomorphism for $V$ be $\Phi$. Since $T_{B^cA}$ is codisjunctive, so is $V$. It is not hard to show that $\Phi$ is invertible, $V$ is coisometric, and $V^{-1}$ exists with associated $\sigma$-homomorphism $\Phi^{-1}$ (cf. [10, Proposition 4.1]). Hence

$$V^{*-1}\colon \mathbf{E}'_A \to \mathbf{F}'_{B^c} \quad \text{and} \quad S \equiv QV^{*-1}\colon \mathbf{E}'_A \to \mathbf{E}'_{A^c}$$

are isometries and

$$V^{-1}T_{B^cA^c} = V^{-1} \circ \eta Q^* = \delta V^{-1}Q^* = \delta S^*,$$

with $\delta = \Phi^{-1}\eta$, by (3.4). So with $\mathbf{G} = \mathbf{E}_A$ and $R$ the identity operator on $\mathbf{G}'$, $T = \Omega(\alpha, 0, \gamma, \delta)$, as defined in (6.3). By Lemma 6.1 and Proposition 6.2, $T$ is not extreme.

## 7. Counter-examples

An extreme contraction need not be of the types described in Proposition 2.6(b) (or its dual) and Theorem 2.8, at least when it is complex.

*Example* 7.1. The matrices

$$\tau = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \pm \frac{1}{2}it\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

where

$$t = \begin{cases} 1/\sqrt{p-1} & \text{when } 2 < p < \infty \\ \sqrt{p-1} & \text{when } 1 < p < 2, \end{cases}$$

are extreme $l_p^2$ contractions isometric only in the direction of (1, 1).

By Lemmas 2.7 and 2.10 we need only prove this for $2 < p < \infty$, and with $+$ taken for $\pm$, as the two cases differ by a factor $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have

$$\|\tau(1,1)\| = \|(1,1)\|$$

and

$$\|\tau(-1,1)\| = t\|(-1,1)\| < \|(-1,1)\|.$$

Fix an arbitrary $r > 0$. Let $f(r, \omega) = (1,1) + re^{i\omega}(-1,1)$. We are to prove that for all angles $\omega$,

$$\begin{aligned} D(r,\omega) &= \|f(r,\omega)\|^p - \|\tau f(r,\omega)\|^p \\ &= (1 + r^2 + 2r\cos\omega)^{p/2} + (1 + r^2 - 2r\cos\omega)^{p/2} \\ &\quad - (1 + t^2r^2 + 2tr\sin\omega)^{p/2} - (1 + t^2r^2 - 2tr\sin\omega)^{p/2} \\ &> 0. \end{aligned}$$

As $D(r,\omega) = D(r,-\omega) = D(r,\pi-\omega)$, we need only consider $0 \le \omega \le \pi/2$. Now

$$\begin{aligned} D_\omega = &-pr\sin\omega\left[(1 + r^2 + 2r\cos\omega)^{p/2-1} - (1 + r^2 - 2r\cos\omega)^{p/2-1}\right] \\ &-ptr\cos\omega\left[(1 + t^2r^2 + 2tr\sin\omega)^{p/2-1} - (1 + t^2r^2 - 2tr\sin\omega)^{p/2-1}\right] \\ &< 0 \end{aligned}$$

if $0 < \omega < \pi/2$. So for all $0 \le \omega < \pi/2$,

$$D(r,\omega) > D(r,\pi/2) = (1 + r^2)^{p/2}C(r),$$

where for all real $z$,

$$C(z) = 2 - B(z) - B(-z) \quad \text{and} \quad B(z) = \left[|1 + tz|/\sqrt{1 + z^2}\right]^p.$$

We have

$$B'(z) = W(z)A(z),$$

where

$$W(z) = p(1 + z^2)^{-p/2-1} \text{ and } A(z) = (1 + tz)^{p-1}(t - z).$$

Further

$$A'(z) = |1 + tz|^{p-2}[(p-1)(z-t) + (1 + tz)] = ptz|1 + tz|^{p-2}.$$

As $C(0) = 0$,

$$\begin{aligned} C(r) &= r[-B'(\xi) + B'(-\xi)] \\ &= rW(\xi)[A(\xi) - A(-\xi)] \\ &= rW(\xi)\xi[A'(\zeta) + A'(-\zeta)] \\ &= rW(\xi)\xi pt\zeta\left[(1 + t\zeta)^{p-2} - |1 - t\zeta|^{p-2}\right] \\ &> 0 \end{aligned}$$

for some $0 < \zeta < \xi < r$, by applying the mean value theorem twice.

Hence $\tau$ is a contraction isometric only in the direction of (1, 1).

Now $\tau$ is of the form (5.1), with $u = v = x = y = 2^{-1/p}$, $\phi = \pi/2$ and $s = t = s^{**}$ in the proof of Theorem 5.1, by (5.7), (5.7′) and (5.10′). By Corollary 5.2, $\tau$ is extreme.

*Question.* (5.2) is the limit of contractions isometric in two directions (see Remark 5.3(iv). Is this also true of Example 7.1?

*Conjecture.* Every extreme contraction $T$ between two $L_p$ spaces ($1 < p < \infty$, $p \neq 2$) with contractive $|T|$ is semidisjunctive.

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