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A SEVEN CONNECTED FINITE H-SPACE IS FOURTEEN CONNECTED

BY

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0. Introduction

In this note, the action of the Steenrod algebra on the mod 2 cohomology of a finite H-space is studied. One interesting question is to determine the first nonvanishing homotopy group for a finite H-space. Work of the author [4] showed that any 3-connected finite H-space is 6-connected. In this note we show that *any* 7-*connected finite H-space is in fact* 14-*connected*. The arguments are related to secondary cohomology operations and can be considered a continuation of the work done to prove the loop space conjecture [2], [5].

The original motivation for this work goes back to papers of Browder, Thomas and Zabrodsky [1], [7], [9]. Browder used the fact that Sq^1 maps even degree cohomology classes to decomposables for a finite H-space X. Using this observation he was able to prove a 1-connected H-space is 2-connected. Thomas [8] restricted himself to a smaller class of finite H-spaces, namely those with primitively generated mod 2 cohomology to prove a $2^i - 1$ connected, primitively generated finite H-space was in fact $2^{i+1} - 2$ connected. This result was quite spectacular, because it also described the action of the Steenrod algebra in quite simple terms. He was finally able to show that mod 2 primitively generated H-spaces have first nonvanishing homotopy in degrees 1, 3, 7 or 15 [7]. The only drawback was that not all finite H-spaces admit primitively generated mod 2 cohomology rings. In fact the exceptional group E_8 has $H^*(E_8; \mathbf{Z}_2)$ not primitively generated and the formulas given by Thomas for the action of the Steenrod algebra do not hold for E_8.

The present task, therefore, is to devise a more general method to attack finite H-spaces which do not have primitively generated mod 2 cohomology. Some of Thomas' results are still valid. For example we showed Sq^2 of a $4l + 1$ dimensional cohomology class is decomposable [4]. In this note we prove Sq^4 *of an* $8l + 3$ *dimensional cohomology class* is decomposable. These

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results appear to be the beginning of a pattern of the form

$$Sq^{2^i}QH^{2^i+2^{i+1}k-1}(X;\mathbf{Z}_2)=0$$

for X a finite H-space, $k>0$.

We also prove

$$\sigma^*\left(QH^{8l+3}(X;\mathbf{Z}_2)\right)\subseteq \operatorname{im} Sq^4.$$

In a previous paper [4] we showed

$$\sigma^*\left(QH^{4l+1}(X;\mathbf{Z}_2)\right)\subseteq \operatorname{im} Sq^2.$$

This may be part of a pattern of the form

$$\sigma^*\left(QH^{2^i+2^{i+1}k-1}(X;\mathbf{Z}_2)\right)\subseteq \operatorname{im} Sq^{2^i}\quad \text{for } k>0.$$

The results in this paper are by no means exhaustive, but hopefully serve to illustrate the methods used. In a later paper, the author will derive further primary results.

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1. Primary results and secondary operations

In this section results of some other papers are gathered here for later use. A secondary operation ψ_2 is defined here. Its main memorable characteristic is that it suspends to Sq^4 of a transpotence element. $Sq^4\psi_2$ will be related to other secondary operations. This will be a key element in our proof.

Unless otherwise noted all cohomology and homology will be understood to have $\mathbf{Z}_2$ coefficients.

We begin by *reserving the symbol X for a simply connected H-space with the following properties*:

Property 1. $QH^{\text{even}}(X)=0$.

Property 2. For $k>0$, $QH^{4k+1}(X)=Sq^{2k}QH^{2k+1}(X)$.

Property 3. $\Sigma_{R>0}QH^{4k+1}(X)+\Sigma_{k>0}QH^{8k+3}(X)$ is a finite dimensional vector space.

These properties hold for all finite simply connected H-spaces as has been shown in [2], [5], [4].

The following notational conventions will be used throughout the paper:

$$Q^* = QH^*(X; \mathbf{Z}_2) \qquad Q_* = QH_*(X; \mathbf{Z}_2)$$
$$P^* = PH^*(X; \mathbf{Z}_2) \qquad P^* = PH_*(X; \mathbf{Z}_2)$$
$$H^* = H^*(X; \mathbf{Z}_2) \qquad H_* = H_*(X; \mathbf{Z}_2).$$

Note that H^* is a Hopf algebra over the Steenrod algebra. Define

$$Q_2 = IH^*/(IH^*)^3.$$

Then the reduced coproduct induces a map of Steenrod modules

$$d\colon Q_2 \to Q^* \otimes Q^*.$$

If $x \in H^*$, denote the projection of x to Q_2 by $\{x\}$. We have the following lemma.

LEMMA 1.1. (a) *If $\bar{x} \in Q^{\text{odd}}$ then $\bar{x}$ has representative x with $d\{x\} = 0$.*
(b) *Suppose x is decomposable and has degree not congruent to two mod four. Then if $d\{x\} = 0$ then x is three fold decomposable. If $d\{x\} \neq 0$ then $d\{x\}$ lies in* $\operatorname{im}(1 + T)$ *where T is the twist map.*

Proof. By property 1 $Q^{\text{even}} = 0$. Therefore if $x \in H^{\text{odd}}$,

$$\bar{\Delta} x \in D \otimes H^* + H^* \otimes D$$

where D is the module of decomposables. This implies $d\{x\} = 0$ which proves (a).

To prove (b) note that if degree x is not congruent to two mod four then x is not a cup product square of a generator. Therefore modulo three fold decomposables x is a sum of terms $x_i' x_i''$ where x_i', x_i'' are odd degree generators. But

$$d\{x_i' x_i''\} = \bar{x}_i' \otimes \bar{x}_i'' + \bar{x}_i'' \otimes \bar{x}_i' \in \operatorname{im}(1 + T).$$

So either $d\{x\} = 0$ and x is three fold decomposable or $d\{x\} \in \operatorname{im}(1 + T)$.
Q.E.D.

We also would like to bring to the reader's attention the relationship between Q^* and the primitives of $H^*(\Omega X)$. Recall there is an Eilenberg Moore spectral sequence relating $H^*(X)$ and $H^*(\Omega X)$. We have

$$E_2 = \operatorname{Tor}_{H^*(X)}(\mathbf{Z}_2, \mathbf{Z}_2) \quad \text{and} \quad E_\infty = \operatorname{Gr} H^*(X).$$

According to [3], E_∞ is isomorphic as coalgebras to $H^*(\Omega X)$. But in our case $E_2 = E_\infty$ because $Q^{\text{even}} = 0$ so $H^*(X)$ is a tensor product of truncated polynomial and exterior algebras on generators of odd degree.

It follows that $\operatorname{Tor}_{H^*(X)}(\mathbf{Z}_2, \mathbf{Z}_2)$ is a tensor product of divided power and exterior coalgebras on primitives that are suspension or transpotence elements. We easily derive the following:

LEMMA 1.2. (a) *All primitives of $H^*(\Omega X)$ are either suspension or transpotence elements on generators of odd degree.*

(b) σ^*: $Q^{2l+1} \to PH^{2^l}(\Omega X)$ *is an isomorphism if l is even and is a monomorphism if l is odd.*

(c) *If $y \in PH^{4m-2}(\Omega X)$ is a transpotence element then express m as $m = 2^i n$ where n is odd. Then $y = \varphi_{2^{i+2}}(x)$ where* $\deg x$ *is n and x has height* 2^{i+2}.

We now build the universal example for a tertiary operation which will be used in Section 2. We first build the universal example for a certain transpotence element.

Our universal example will eventually be used to prove

$$\sigma^* Q^{8k+3} \subseteq Sq^4 PH^{8k-2}(\Omega X).$$

Express $k = 2^i l$ where l is odd. Let w_0: $K(\mathbf{Z}_2, l) \to K(\mathbf{Z}_2, 16k)$ be defined by $w_0^*(i_{16k}) = (i_l)^{2^{i+4}}$. Then w_0 is an infinite loop map. Let BE_0 be the fibre of Bw_0. Let $\bar{w}_0$: $K(\mathbf{Z}_2, 2k) \to K(\mathbf{Z}_2, 16k)$ be defined by

$$\bar{w}_0^*(i_{16k}) = i_{2k}^8.$$

Let $B\bar{E}_0$ be the fibre of $B\bar{w}_0$. We have a commutative diagram

$$\begin{array}{ccc}
K(\mathbf{Z}_2, 16k) & \longrightarrow & K(\mathbf{Z}_2, 16k) \\
\downarrow {\scriptstyle Bj_0} & & \downarrow {\scriptstyle B\bar{j}_0} \\
BE_0 & \xrightarrow{\bar{h}} & B\bar{E}_0 \\
\downarrow {\scriptstyle Bp_0} & & \downarrow {\scriptstyle B\bar{p}_0} \\
K(\mathbf{Z}_2, l+1) & \xrightarrow{h} & K(\mathbf{Z}_2, 2k+1) \\
& {\scriptstyle Bw_0} \searrow \quad \swarrow {\scriptstyle B\bar{w}_0} & \\
& K(\mathbf{Z}_2, 16k+1) &
\end{array}$$

We have $B\bar{w}_0^*(i_{16k+1}) = Sq^{8k} Sq^{4k} Sq^{2k} i_{2k+1}$. There exist elements

$$\bar{u}_0 \in H^{16k+5}(B\bar{E}_0), \quad \bar{u}_1 \in H^{16k+2}(B\bar{E}_0), \quad \bar{u}_2 \in H^{16k+4}(B\bar{E}_0)$$

with

$$Bj_0^*(\bar{u}_0) = Sq^4Sq^1i_{16k}, \quad Bj_0^*(\bar{u}_1) = Sq^2i_{16k}, \quad Bj_0^*(\bar{u}_2) = Sq^4i_{16k}.$$

We have

$$\bar{\Delta}\bar{u}_1 = Sq^{4k}Sq^{2k}B\bar{p}_0^*(i_{2k+1}) \otimes Sq^{4k}Sq^{2k}B\bar{p}_0^*(i_{2k+1})$$

where $\bar{u}_0$, $\bar{u}_2$ are primitive. Hence $Sq^4\bar{u}_2 + Sq^6\bar{u}_1 + Sq^3\bar{u}_0$ is primitive and in the kernel of Bj_0^*. We have

$$\Omega\bar{E}_0 \simeq K(\mathbf{Z}_2, 2k-1) \times K(\mathbf{Z}_2, 16k-2)$$

and

$$(\sigma^*)^2(\bar{u}_0) = \alpha_0 i_{2k-1} \otimes 1 + 1 \otimes Sq^4Sq^1i_{16k-2},$$
$$(\sigma^*)^2(\bar{u}_1) = \alpha_1 i_{2k-1} \otimes 1 + 1 \otimes Sq^2i_{16k-2},$$
$$(\sigma^*)^2(\bar{u}_2) = \alpha_2 i_{2k-1} \otimes 1 + 1 \otimes Sq^4i_{16k-2}.$$

Changing $\bar{u}_i$ by $B\bar{p}_0^*(\alpha_i i_{2k+1})$ we may assume

$$(\sigma^*)^2(\bar{u}_0) = 1 \otimes Sq^4Sq^1i_{16k-2},$$
$$(\sigma^*)^2(\bar{u}_1) = 1 \otimes Sq^2i_{16k-2},$$
$$(\sigma^*)^2(\bar{u}_2) = 1 \otimes Sq^4i_{16k-2}.$$

Then

$$(\sigma^*)^2[Sq^3\bar{u}_0 + Sq^6\bar{u}_1 + Sq^4\bar{u}_2] = 0.$$

Hence since σ': $QH^{\text{odd}}(\bar{E}_0) \to PH^{\text{even}}(\Omega\bar{E}_0)$ is monic,

$$\sigma^*[Sq^3\bar{u}_0 + Sq^6\bar{u}_1 + Sq^4\bar{u}_2] = 0$$

since it's odd degree decomposable. Now since σ^*: $QH^{16k+8}(B\bar{E}_0) \to PH^{16k+7}(\bar{E}_0)$ is monic it follows that

$$Sq^3\bar{u}_0 + Sq^6\bar{u}_1 + Sq^4\bar{u}_2 = Sq^{8k+4}B\bar{p}_0^*(\alpha i_{2k+1}) = [B\bar{p}_0^*(\alpha i_{2k+1})]^2$$

where $\alpha \in \mathscr{A}(2)$.

Define $u_i = \bar{h}^*(\bar{u}_i)$, $v_i = \sigma^*(u_i)$. Let ψ_i be the secondary operations defined by the v_i. We have proved:

PROPOSITION 1.3. *There exist elements $v_0, v_1, v_2 \in H^*(E_0)$ that are suspensions of elements u_0, u_1, u_2 with the following properties.*
(1) $Sq^3 v_0 + Sq^6 v_1 + Sq^4 v_2 = 0$.
(2) $\sigma^*(v_2) = Sq^4 \varphi_{2^{i+4}}(p_0^*(i_l))$.
(3) $Sq^3 u_0 + Sq^6 u_1 + Sq^4 u_2$ *is a fourth power.*

Proof. Property 3 implies property 1. $\sigma^*(v_2) = 1 \otimes Sq^4 i_{16k-2}$ and $1 \otimes i_{16k-2}$ represents $\varphi_{2^{i+4}}(p_0^*(i_l))$. Hence property 2 is satisfied.
Finally

$$Sq^3\bar{u}_0 + Sq^6\bar{u}_1 + Sq^4\bar{u}_2 = \left[B\bar{p}_0^*(\alpha i_{2k+1})\right]^2$$

and since α has odd degree,

$$\alpha h^*(i_{2k+1}) \in \alpha Sq^k H^{k+1}\big(K(\mathbf{Z}_2, l+1)\big) \subseteq \xi H^*\big(K(\mathbf{Z}_2, l+1)\big).$$

Hence $Sq^3 u_0 + Sq^6 u_1 + Sq^4 u_2$ is a fourth power. Q.E.D.

We now build the third stage of our Postnikov system. The Adem relations imply

$$\begin{aligned} (1.1)\qquad Sq^{8k+4} &= Sq^4 Sq^{8k} + Sq^{8k+2}Sq^2 + Sq^{8k+3}Sq^1, \\ Sq^{8k+2} &= Sq^4 Sq^{8k-2} + Sq^{8k}Sq^2, \\ Sq^2Sq^2 &= Sq^3Sq^1. \end{aligned}$$

Combining the above equations we obtain

$$(1.2)\quad Sq^{8k+4} = Sq^4\left[Sq^{8k} + Sq^{8k-2}Sq^2\right] + \left[Sq^{8k+3} + Sq^{8k}Sq^3\right]Sq^1.$$

For convenience let $\theta = Sq^{8k} + Sq^{8k-2}Sq^2$. Then we have

$$Sq^{8k+4} = Sq^4\theta + \left[Sq^{8k+3} + Sq^{8k}Sq^3\right]Sq^1.$$

Let

$$\begin{aligned} K &= E_0 \times K(\mathbf{Z}_2, 8k+3, 8k+1), \\ K_0 &= K(\mathbf{Z}_2, 16k+3, 16k+1, 16k+4, 8k+4, 8k+4) \end{aligned}$$

Let $w\colon K \to K_0$ be defined by

$$\begin{aligned} w^*(i_{16k+3}) &= \theta i_{8k+3} - v_2 \\ w^*(i_{16k+1}) &= v_1 - Sq^{8k} i_{8k+1} \\ w^*(i_{16k+4}) &= v_0 \\ w^*(i_{8k+4}) &= Sq^1 i_{8k+3} \\ w^*(i'_{8k+4}) &= Sq^3 i_{8k+1}. \end{aligned}$$

Then w is a loop map. Let E be the fibre of w:

$$\begin{array}{ccc} \Omega K_0 & & \\ \downarrow{\scriptstyle j} & & \\ E & & \\ \downarrow{\scriptstyle p} & & \\ K & \xrightarrow[w]{} & K_0 \end{array}$$

Consider the element $z \in H^*(BK_0)$,

$$\begin{aligned} z = {} & Sq^4 i_{16k+4} + Sq^6 i_{16k+2} + Sq^3 i_{16k+5} \\ & + (Sq^{8k+3} + Sq^{8k} Sq^3) i_{8k+5} + Sq^{8k+3} i'_{8k+5}. \end{aligned}$$

Then

$$\begin{aligned} (Bw)^*(z) &= Sq^4[\theta i_{8k+4} - u_2] + Sq^6[u_1 - Sq^{8k} i_{8k+2}] \\ &\quad + Sq^3 u_0 + [Sq^{8k+3} + Sq^{8k} Sq^3] Sq^1 i_{8k+4} \\ &\quad + Sq^{8k+3} Sq^3 i_{8k+2} \\ &= [Sq^4\theta + (Sq^{8k+3} + Sq^{8k} Sq^3) Sq^1] i_{8k+4} \\ &\quad + Sq^4 u_2 + Sq^6 u_1 + Sq^3 u_0 \\ &\quad + (Sq^6 Sq^{8k} + Sq^{8k+3} Sq^3) i_{8k+2} \\ &= Sq^{8k+4} i_{8k+4} + \text{a fourth power} \quad \text{(by Proposition 1.3)}. \end{aligned}$$

Therefore in the projective plane of E, P_2E, the inclusion

$$i_2\colon P_2E \to BE$$

takes $Bp^*(i_{8k+4})$ to an element truncated at height two. Hence by [5, Prop.

3.1], there exists a $v \in H^*(E)$ with $\bar{\Delta} v = u \otimes u$ where $u = p^*(i_{8k+3})$ and

$$j^*(v) = Sq^4 i_{16k+2} + Sq^6 i_{16k} + Sq^3 i_{16k+3} + \left(Sq^{8k+3} + Sq^{8k} Sq^3\right) i_{8k+3} + Sq^{8k+3} i'_{8k+3}.$$

By [9] we have:

PROPOSITION 1.4. *There exists an element $\sigma^* v \in PH^{16k+5}(\Omega E_0)$ with*

$$c(\sigma^* v) = \sigma^* u \otimes \sigma^* u$$

and

$$\Omega j^*(\sigma^* v) = Sq^4 i_{16k+1} + Sq^6 i_{16k-1} + Sq^3 i_{16k+2} + \left(Sq^{8k} Sq^3\right) i_{8k+2}.$$

2. Applications of the c_2-invariant

In this chapter, the three stage system E is used to prove

$$\sigma^* Q^{8k+3} \subseteq \operatorname{im} Sq^4.$$

By property 3, $\Sigma_{l>0} Q^{8l+3}$ is a finite dimensional vector space. Therefore, we may use downward induction. Assume that for $k' > k$, $\sigma^* Q^{8k'+3} \subseteq \operatorname{im} Sq^4$. Let $\bar{x} \in Q^{8k+3}$ have representative x with $d\{x\} = 0$. Then if $\theta \bar{x}$ is nontrivial, by induction

$$\sigma^*(\theta \bar{x}) = Sq^4 y.$$

Since degree $\sigma^*(\theta \bar{x}) = 16k + 2$ it is primitive indecomposable. Hence y may be chosen primitive indecomposable. It follows that y is either a suspension or transpotence element. In either case y is realizable by a map

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}_0} & E_0 \\ & \searrow^{f_0} & \downarrow p_0 \\ & & K(\mathbf{Z}_2, l) \end{array}$$

and

$$(\Omega \tilde{f}_0)^*(\sigma^* v_2) = Sq^4 y$$

by Proposition 1.3. Therefore $\tilde{f}_0^*(v_2)$ and $\theta \bar{x}$ suspend to the same element. Since σ^*: $Q^{\text{odd}} \to PH^{\text{even}}(\Omega X)$ is monic, $\tilde{f}_0^*(v_2) - \theta x$ is three-fold decomposable.

Similarly, if $\tilde{f}_0^*(v_1)$ is indecomposable, by Property 2,

$$\tilde{f}_0^*(v_1) = Sq^{8k}x_{8k+1} + \text{three-fold decomposables}.$$

By Lemma 1.1, Sq^1x and Sq^3x_{8k+1} are three-fold decomposable. Finally, the Cartan formula for $\bar{\Delta}\tilde{f}_0^*(v_0)$ (see [5]) implies that if D is the module of decomposables, then

$$\bar{\Delta}\tilde{f}_0^*(v_0) \in D \otimes H^* + H^* \otimes D + \operatorname{im} Sq^4Sq^1$$

since $\bar{\Delta}\tilde{f}_0^*(i_l) \in D \otimes H^* + H^* \otimes D$. But $Sq^1H^* \subseteq D$.

We conclude $d\{\tilde{f}_0^*(v_0)\} = 0$. By Lemma 1.1, $\tilde{f}_0^*(v_0)$ is also three-fold decomposable.

If $P_2\Omega X$ is the projective plane of ΩX, since all three-fold products vanish on $H^*(P_2\Omega X)$ it follows that there is a commutative diagram

$$\begin{array}{ccccccc} & & & \nearrow & E & & \\ & & & & \downarrow & & \\ P_2\Omega X & \to & X & \xrightarrow[f]{} & K & \xrightarrow[w]{} & K_0 \end{array}$$

where $f^*(i_{8k+3}) = x$, $f^*(i_{8k+1}) = x_{8k+1}$. This yields a diagram:

$$\begin{array}{ccccc} & & \Omega E & & \\ & \overset{\tilde{f}}{\nearrow} & \downarrow & & \\ X & \xrightarrow[\Omega f]{} & \Omega K & \to & \Omega K_0 \end{array}$$

By [4, equation 2.2], we have $\sigma^*x \otimes \sigma^*x \in (Sq^4 + Sq^6 + Sq^3 + Sq^{8k}Sq^3)[F_2' \otimes \operatorname{im}\sigma^* + \operatorname{im}\sigma^* \otimes F_2' + PH^*(\Omega X) \otimes PH^*(\Omega X)]$ where F_2' is a submodule of $\operatorname{im}\sigma^*$ + 2-fold products of elements of $\operatorname{im}\sigma^*$. Since σ^*x is indecomposable and $H^*(\Omega X)$ is even dimensional this implies

$$\sigma^*x \otimes \sigma^*x \in [Sq^4 + Sq^6](PH^*(\Omega X) \otimes PH^*(\Omega X)).$$

Since $\bar{x} \notin \operatorname{im} Sq^2$ and $PH^{4l}(\Omega X) = \sigma^*Q^{4l+1}$ by Lemma 1.2, it follows that $\sigma^*x \in Sq^4PH^*(\Omega X)$. This completes the inductive step and proves:

THEOREM 2.1. $\sigma^*Q^{8k+3} \subseteq Sq^4PH^*(\Omega X)$.

COROLLARY 2.2. $Sq^4Q^{8k+3} = 0$.

Proof.

$$\begin{aligned} \sigma^*Sq^4Q^{8k+3} &\subseteq Sq^4Sq^4PH^{8k-2}(\Omega X) \\ &\subseteq Sq^6Sq^2PH^{8k-2}(\Omega X) \\ &= 0 \end{aligned}$$

since $Sq^2PH^{4l}(\Omega X) = 0$. Q.E.D.

THEOREM 2.3. *If X is 7-connected then X is 14-connected.*

Proof. By [2], [5] the first nonvanishing homotopy group is torsion free of odd degree. By the Hurewicz theorem if $0 < l$ is the lowest degree where $\pi_l(X)$ is nontrivial, then $H^l(X)$ is nontrivial. If $14 > l > 7$ then by properties 1 and 2, $l = 11$. By Theorem 2.1, $\sigma^* Q^{11} = Sq^4 PH^6(\Omega X)$. But ΩX is 6-connected so $Q^{11} = 0$. We conclude that if $l > 7$ then $l \geq 14$. Q.E.D.

PROPOSITION 2.4. *$Q^{11} = Sq^4 Q^7$ and $Q^{19} = Sq^4 Q^{15}$.*

Proof. Since X is two connected, the first transpotence element in $H^*(\Omega X)$ of degree $8k - 2$ is in degree greater than or equal to 22. Hence

$$\sigma^* Q^{11} = Sq^4 PH^6(\Omega X) = Sq^4 \sigma^* Q^7$$

and

$$Q^{11} = Sq^4 Q^7.$$

Similarly,

$$\sigma^* Q^{19} = Sq^4 PH^{14}(\Omega X) = Sq^4 \sigma^* Q^{15}$$

and

$$Q^{19} = Sq^4 Q^{15}.$$

Q.E.D.

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