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# ON THE TENSOR PRODUCT OF A CLASS OF NON-LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

BY

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## 0. Introduction

Let $f$ be a real valued function defined on $[0, \infty)$ which satisfies:
(i) $f(x) = 0$ if and only if $x = 0$;
(ii) $f$ is increasing;
(iii) $f(x + y) \leq f(x) + f(y)$;
(iv) $\lim_{x \to 0^+} f(x) = 0$.
It is clear that every such function is continuous. For every sequences $x = (x_n)$ we define

$$|x|_f = \sum_{n=0}^{\infty} f|x_n|.$$

The space $L(f)$ is the set of all real sequences $x = (x_n)$ such that $|x|_f < \infty$. One can easily show that $|x|_f$ defines a metric on $L(f)$.

It was shown in [1] that $(L(f), |\ \ |_f)$ is a complete metric space.

The space $(L(f), |\ \ |_f)$ is a topological vector space [1]. For more about $L(f)$ spaces we refer to [2], [3], [7]. The object of this paper is to characterize the isometries of $(L(f), |\ \ |_f)$ and to define the projective tensor product of $L(f)$ with itself, proving some results on the tensor product.

Throughout this paper, $N$ will denote the set of positive integers. If $X$ and $Y$ are topological vector spaces, $WL(X, Y)$ will denote the weakly continuous linear operators from $X$ into $Y$, and $B(X, Y)$ the continuous bilinear functional on $X \times Y$. The dual of a topological vector space $X$ will be denoted by $X^*$.

## 1. Isometries of $L(f)$

A continuous linear operator $F$: $L(f) \to L(f)$ will be called an isometry if

$$|F(x)|_f = |x|_f \quad \text{for all } x \in L(f).$$

Let $e_i$ be the sequence with 1 at the $i$th-coordinate and zero elsewhere.

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THEOREM 1.1. *Let $F$: $L(f) \to L(f)$ be an onto continuous linear operator. Then $F$ is an isometry if and only if there exists a permutation $\pi$ of $N$ such that $F(e_i) = \pm e_{\pi(i)}$ for all $i \in N$.*

*Proof.* If $F(e_i) = e_{\pi(i)}$, then for any $x \in L(f)$, $x = \sum_{i=1}^{\infty} x_i e_i$ and

$$|F(x)|_f = \sum_{i=1}^{\infty} f|x_{\pi(i)}| = \sum_{i=1} f|x_i| = |x|_f.$$

Thus $F$ is an isometry.

For the converse, let $F$: $L(f) \to L(f)$ be an isometric onto operator. Fix $i \in N$ and suppose that $F(e_i) = x = \sum_{n=1}^{\infty} x_n e_n$; then

$$|F(e_i)|_f = |e_i|_f = f(1) = \sum_{n=1}^{\infty} f|x_n|.$$

Since $F$ is onto, then for every $m \in N$ there exists $x(m) \in L(f)$, $x(m) = \sum_{k=1}^{\infty} x_k(m)$, such that $F(x(m)) = x_m e_m$. Since $F$ is 1-1 and continuous we get

$$F^{-1}(x) = \sum_{m=1}^{\infty} F^{-1}(x_m e_m) = \sum_{m=1}^{\infty} x(m) \in L(f).$$

Consequently $F(\sum_1^{\infty} x(m)) = x$ and $\sum_1^{\infty} x(m) = e_i$. Hence $\sum_1^{\infty} x_i(m) = 1$, noting that $(e_i)$ is a Schauler basis for $L(f)$. Set $y(m) = x_i(m) e_i$. Then

$$\begin{aligned} f(1) = f\left( \sum_{m=1}^{\infty} x_i(m) \right) &\leq \sum_{m=1}^{\infty} f|x_i(m)| \leq \sum_{m=1}^{\infty} |x(m)|_f \\ &= \sum_{m=1}^{\infty} f|x_m| = f(1). \end{aligned}$$

Consequently $f|x_k(m)| = 0$ for all $k \neq i$, for all $m$, and hence $x_k(m) = 0$ for all $k \neq i$, and $x(m) = y(m)$. Since $x \neq 0$, there exists some $j$ such that $x_j \neq 0$. Since $F(x(j)) = F(y(j))$, we get $x_j = x_i(j) x_j$ and $x_i(j) x_m = 0$ for all $m \neq j$. Since $x_j \neq 0$, it follows that $x_i(j) = 1$. Consequently $x_m = 0$ for all $m \neq j$, $x = x_j e_j$, and $|x|_f = f(|x_j|) = 1$. We claim that $|x_j| = 1$. To see that, assume if possible $|x_j| < 1$. So there exists a real number $a$ such that $f|x_j a| < f|a|$. But $F(ae_i) = ax_j e_j$ and $|ae_i|_f = |ax_j e_j|_f$. Thus $f|a| = f|ax_j|$, which is a contradiction. Similarly, if $|x_j| > 1$, there exists an a $\in (0,1)$ such that $f(a) < f|ax_j|$. But $F(ae_i) = ax_j e_j$, so $f(a) = f|ax_j|$, a contradiction. Hence $|x_j| = 1$. This completes the proof of the theorem.

DEFINITION 1.2. A sequence $(a_n)$ is called a multiplier for $L(f)$ if $x \cdot a = (x_n a_n) \in L(f)$ for all $x = (x_n) \in L(f)$. We write $M(L(f))$ for the space of all multipliers of $L(f)$.

THEOREM 1.3. *$M(L(f)) = l^\infty$, the space of bounded sequences.*

*Proof.* Let $a \in l^\infty$ and $x \in L(f)$. If $\lambda$ is an upper bound for $a$, then we can choose an integer $\hat{\lambda} \geq \lambda$ which is an upper bound for $a$. Consequently

$$\begin{aligned}|a\cdot x|_f &= \sum_0^\infty f|a_n x_n| \\ &\leq \sum_0^\infty f|\lambda x_n| \\ &\leq \sum_0^\infty f|\hat{\lambda} x_n| \\ &\leq \hat{\lambda}\sum_0^\infty f|x_n| \quad \text{(by the subadditivity of } f\text{)} \\ &= \hat{\lambda}|x|_f < \infty.\end{aligned}$$

Conversely. Let $a \in M(L(f))$. If possible assume that $a \notin l^\infty$. Thus there exists a subsequence $(a_{n_j})$ such that $|a_{n_j}| \to \infty$. With no loss of generality we assume $a_{nj} \neq 0$ for all $n_j$. Now, choose the subsequence $(a_{n_{j_k}})$ such that

$$f\left|\frac{1}{a_{n_{j_k}}}\right| < \frac{1}{2^{k-1}}.$$

Then the subsequence

$$y = \sum_k \frac{1}{a_{n_{j_k}}} e_{n_{j_k}} \in L(f),$$

but $a \cdot y \notin L(f)$. This is a contradiction. Hence $a \in l^\infty$.

## 2. Tensor product of $L(f)$ spaces

Let $L(f) \otimes L(f)$ be the algebraic tensor product of $L(f)$ with itself. Every element $\varphi \in L(f) \otimes L(f)$ has a representation $\varphi = \sum_{r=1}^n U_{r_n} \otimes V_r$. The element $\varphi$ can be considered as a double sequence

$$\varphi(i, j) = \sum_{r=1}^n U_r(i) V_r(j).$$

For $\varphi, \psi \in L(f) \otimes L(f)$, define

$$d(\varphi, \psi) = \inf\left\{ \sum_{r=1}^m |Q_r|_f \cdot |W_r|_f \right\},$$

where the infimum is taken over all representations of $\varphi - \psi$ in $L(f) \otimes L(f)$. One can easily check that $d$ defines a metric on $L(f) \otimes L(f)$, and we write $d(\varphi)$ for $d(\varphi, 0)$. The space $L(f) \otimes L(f)$ with the metric $d$ is not complete. We set $L(f) \hat{\otimes} L(f)$ for the completion.

THEOREM 2.1. *The space $L(f) \hat{\otimes} L(f)$ is a topological vector space.*

*Proof.* First we prove that $d$ is a quasi-norm on $L(f) \hat{\otimes} L(f)$. That is,
(i) $d(\varphi) = 0$ if and only if $\varphi = 0$,
(ii) $d(-\varphi) = d(\varphi)$,
(iii) $d(\varphi + \psi) \le d(\varphi) + d(\psi)$.
However, these follows easily from the properties of the metric $d$ and the function $f$.

By Proposition 1 of [6, p. 38], it remains only to show:
(i) If $\alpha_n \to 0$, then $d(\alpha_n \cdot \varphi) \to 0$ for all $\varphi \in L(f) \hat{\otimes} L(f)$.
(ii) If $d(\varphi_n) \to 0$, then $d(\alpha \varphi_n) \to 0$ for all scalars $\alpha$.

To prove (i), let $\varphi = \Sigma_{i=1}^{m} U_i \otimes V_i \in L(f) \hat{\otimes} L(f)$. Then

$$0 \le d(\alpha_n \varphi) \le \sum_{i=1}^{m} |\alpha_n U_i|_f \cdot |V_i|_f.$$

and

$$0 \le \lim_{n \to \infty} d(\alpha_n \varphi) \le \sum_{n=1}^{m} \lim_{n} |\alpha_n U_i|_f \cdot |V_i|_f = 0,$$

by Lemma 4 of [1].

Now, if

$$\varphi = \sum_{i=1}^{\infty} U_i \otimes V_i, \quad \sum_{i=1}^{\infty} |U_i|_f \cdot |V_i|_f < \infty,$$

then define $g_n(\mathrm{i}) = |\alpha_n U_i|_f \cdot |V_i|_f$. The Lebesgue dominated convergence theorem on $N$ with the counting measure, applied to the sequence of functions $g_n$, implies that

$$0 \le \lim_{n \to \infty} d(\alpha_n) \le \sum_{i=1}^{\infty} \lim_{n} |\alpha_n U_i|_f \cdot |V_i|_f = 0.$$

For (ii), let $\varphi_n \in L(f) \hat{\otimes} L(f)$, and $d(\varphi_n) \to 0$. Let $\hat{\alpha}$ be an integer such that $\hat{\alpha} > \alpha$. Then

$$0 \le d(\alpha \varphi_n) \le d(\hat{\alpha} \varphi_n) \le \hat{\alpha} d(\varphi_n) \quad \text{(by the subadditivity of } f\text{)}.$$

Hence $d(\alpha \varphi_n) \to 0$. This completes the proof of the theorem.

It should be remarked, that the topological tensor product of topological vector spaces is known only for the case of local convenity. The space $L(f)$ is not locally convex [7].

The dual of $L(f)$ was studied in [2], and it is proved there that $L(f)^*$ can be identified with $l^\infty$.

THEOREM 2.2. *The space* $[L(f) \hat{\otimes} L(f)]^*$ *can be identified with* $WL(L(f), L(f)^*)$.

*Proof.* Let $B(L(f) \times L(f))$ be the space of continuous bilinear functionals on $L(f) \times L(f)$. For each $\psi \in B(L(f) \times L(f))$ define $\hat{\psi} \in WL(L(f), L(f)^*)$ by $\hat{\psi}(U)(V) = \psi(U, V)$. Since $\psi$ is separately continuous, it follows that $\hat{\psi}$ is weakly continuous. To see that this correspondence is onto, let $\varphi \in WL(L(f), L(f)^*)$. Then the bilinear map $\varphi(U, V) = \varphi(U)(V)$ is separately continuous. Consequently [5 p. 171], $\varphi$ is continuous.

Now, we can identify $B(L(f) \times L(f))$ with $[L(f) \times L(f)]^*$ via the correspondence

$$F\colon B(L(f) \times L(f)) \to [L(f) \hat{\otimes} L(f)]^*, \qquad F(\psi)(U \otimes U) = \psi(U, V).$$

One can easily check that $F$ is 1-1 and onto. Consequently, $[L(f) \hat{\otimes} L(f)]^*$ is identified with $WL(L(f), L(f)^*)$. The proof is complete.

Let $K$ be the space of all functions $\varphi\colon N \times N \to R$. Define

$$e_{nm}(i, j) = \begin{cases} 0 & \text{if } (i, j) \neq (n, m) \\ 1 & \text{if } (i, j) = (n, m) \end{cases}.$$

Then every $\varphi \in K$ has a unique representation $\varphi = \sum_{n,m} a_{nm} e_{nm}$, and $\varphi(i, j) = a_{ij}$. Set $L(f \times f)$ to be the subspace of $K$ consisting of all $\varphi = \sum_{n,m} a_{nm} e_{nm}$ for which $\sum_{n,m} f|a_{nm}| < \infty$. If we define

$$|\varphi|_{f \times f} = \sum_{n,m} f|a_{nm}|,$$

then as in [1] and [7], one can prove:

THEOREM 2.3. *The space* $(L(f \times f), |\ \ |_{f \times f})$ *is a complete metric topological vector space.*

Now we prove:

THEOREM 2.4. *Let $f$ satisfy the additional condition*

$$f(xy) \le f(x) f(y) \ (x, y \ge 0).$$

*Then* $L(f) \hat{\otimes} L(f)$ *is isometrically isomorphic to* $L(f \times f)$.

First we prove the following:

LEMMA 2.5. *Every* $\varphi \in L(f) \otimes L(f)$ *has a unique representation*

$$\varphi = \sum_{n,m} a_{nm} e_{nm},$$

*and the series converges in the topology of the metric* $d$.

*Proof.* Let $\varphi = U \otimes V$. Since $(e_i)$ is a Schauder basis for $L(f)$, then

$$U = \sum_{i=0}^{\infty} \lambda_i e_i, \quad V = \sum_{j=0}^{\infty} \xi_j e_j,$$

and

$$|U|_f = \sum_j f|\lambda_j|, \quad |V|_f = \sum_j f|\xi_j|.$$

Hence

$$\varphi = \sum_{i,j} \lambda_i \xi_j e_i \otimes e_j = \Sigma \lambda_i \xi_j e_{ij}.$$

Set $P_n(U) = \Sigma_{i=0}^{n} \lambda_i e_i$, $P_m(V) = \Sigma_{j=0}^{m} \xi_j e_j$. Then

$$\begin{aligned} \varphi - \sum_{i,j=0}^{n,m} \lambda_i \xi_j e_i \otimes e_j &= \varphi - P_n(U) \otimes P_m(V) \\ &= U \otimes (V - P_m(V)) + (U - P_n(U)) \otimes P_m(V). \end{aligned}$$

Thus

$$d\left(\varphi - \sum_{i,j=0}^{n,m} \lambda_i \xi_j e_i \otimes e_j\right) \leq |U|_f \cdot |V - P_m(V)|_f + |U - P_n(U)|_f \cdot |P_m(V)|_f.$$

Since $(e_i)$ is a Schauder basis for $L(f)$ and $|P_m(V)|_f \leq |V|_f$, it follows that

$$\begin{aligned} &\lim_{n,m \to \infty} d\left(\varphi - \sum_{i,j=0}^{n,m} \lambda_i \xi_j e_i \otimes e_j\right) \\ &\quad \leq |U|_f \lim_m |V - P_m(V)|_f + |V|_f \lim_n |U - P_n(U)|_f \\ &\quad = 0. \end{aligned}$$

Hence the claim is true for every $\varphi = U \otimes V \in L(f) \otimes L(f)$, and consequently for every $\varphi = \Sigma_{r=1}^{n} U_r \otimes V_r \in L(f) \otimes L(f)$. This proves the lemma.

*Proof of the theorem* Consider the map

$$F: L(f\times f) \to L(f) \hat{\otimes} L(f),$$

$$F\left(\sum_{n,m=0}^{r,s} a_{nm}e_{nm}\right) = \sum_{n,m=0}^{r,s} a_{nm}e_n \otimes e_m.$$

Now,

$$(1) \qquad \left|\sum_0^{r,s} a_{nm}e_{nm}\right|_{f\times f} = \sum_0^{r,s} f|a_{nm}| = \sum_0^{r,s} |a_{nm}e_n|_f \cdot |e_m|_f$$

$$\geq d\left(\sum_0^{r,s} a_{nm}e_n \otimes e_m\right)$$

On the other hand, let $\varphi \in L(f) \otimes L(f)$,

$$\varphi = \sum_{i=1}^{k} U_i \otimes V_i = \sum_{i=1}^{s}\left(\sum_{k,m} \lambda_{ik}\xi_{im}e_k \otimes e_m\right)$$

$$= \sum_{k,m}\left(\sum_i \lambda_{ik}\xi_{im}\right)e_k \otimes e_m$$

$$= \sum_{k,m} b_{km}e_k \otimes e_m,$$

where $b_{km} = \sum_{i=1}^{s}\lambda_{ik}\xi_{im}$. By lemma 2.5, the last representation $\varphi$ is independent of the representation $\sum_{i=1}^{s}U_i \otimes V_i$.

For every $\varepsilon > 0$, one can choose a representation $\varphi = \sum_{i=1}^{s}U_i \otimes V_i$ such that

$$d(\varphi) \geq \sum_{i=1}^{s} |U_i|_f \cdot |V_i|_f - \varepsilon.$$

The proof of this is similar to the case of Banach space tensor products [4, p. 227].

Set $\varphi = \sum_{k,m} b_{km}e_{km}$. Then

$$|\varphi|_{f\times f} = \sum_{k,m} f|b_{km}|$$

$$\leq \sum_{k,m} f\left|\sum_i \lambda_{ik}\xi_{im}\right|$$

$$\leq \sum_i\left(\sum_k f|\lambda_{ik}|\right)\cdot\left(\sum_m f|\xi_{im}|\right)$$

$$= \sum_i |U_i|_f \cdot |V_i|_f$$

$$\leq d(\varphi) + \varepsilon.$$

Since $\varepsilon$ was arbitrary, we get

$$|\varphi|_{f\times f} \leq d(\varphi). \tag{2}$$

However, $F(\varphi) = \varphi$. Consequently, from (1) and (2) we get $d(F(\varphi)) = |\varphi|_{f\times f}$, and $F$ is an isometric linear map, whose range contains a dense subspace of $L(f) \hat{\otimes} L(f)$. Thus $F$ is isometric linear operator from $L(f\times f)$ onto $L(f) \hat{\otimes} L(f)$. This completes the proof of the theorem.

As a consequence of this theorem we get:

THEOREM 2.6. *Let $f(xy) \leq f(x)f(y)$ $(x, y \geq 0)$. Then*

$$M(L(f) \hat{\otimes} L(f)) = l^{\infty}(N \times N).$$

THEOREM 2.7. *Let $f(xy) \leq f(x)f(y)$ $(x, y) \geq 0)$. Then*

$$F\colon L(f) \hat{\otimes} L(f) \to L(f) \hat{\otimes} L(f)$$

*is an isometric onto operator if and only if there exists a permutation $\pi$ of $N \times N$ such that $F(e_{ij}) = F(e_{\pi(ij)})$.*

The proof follows from Theorem 2.4, together with Theorem 1.2 and Theorem 1.1.

*Closing Remarks* (i) We were not able to prove Theorem 2.4 without assuming $f(xy) \leq f(x)f(y)$.

(ii) It would be very interesting if one can define $L(f\times g)$ and prove that $L(f) \times L(g) \cong L(f \times g)$.

(iii) $f(x) = x^p$, $0 < p \leq 1$ is a class of functions which satisfy the condition $f(xy) \leq f(x)f(y)$.

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