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A CRITERION FOR DELOOPING THE FIBRE OF THE SELF-MAP OF A SPHERE WITH DEGREE A POWER OF A PRIME

BY

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Fix, once and for all, p to be an odd prime, and n and j to be strictly positive integers. Let F be the homotopy fibre of the self-map of S^{2n-1} of degree p^j (i.e.,

$$F \to S^{2n-1} \xrightarrow{p^j} S^{2n-1}$$

is a fibration up to homotopy). Notice that F is its own localization at p. The sphere S^{2n-1} itself, localized at p, deloops if and only if n divides $p-1$. In [2], the second author showed that for certain values of p, n and j, the fibre F deloops. The deloopings are of the form $BG(\mathbf{F}_q)^+_{(p)}$ where $G(\mathbf{F}_q)$ is the universal Chevalley group of some exceptional Lie type over the finite field $\mathbf{F}_q$, q a power of a prime different from p. Here "+" denotes Quillen's "plus construction" (see [6]) and (p) denotes localization at the prime p. In all these cases n divides $p-1$. The main result of this paper is the following more general theorem:

THEOREM I. *F is a loop space if and only if n divides $p-1$.*

We divide the paper into two, essentially separate, parts. In the first part, when n divides $p-1$ we give two methods for constructing a delooping. One of these deloopings is of the form $(BG^+)_{(p)}$ where G is the special linear group of a finite field. In the second part we show that if a delooping exists, then n divides $p-1$.

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Part I

LEMMA II. *If n divides $p - 1$ then F is a loop space.*

Proof. If $n = 1$, then F consists of p^j discrete points, and hence is trivially a loop space (e.g., $F = \Omega B(\mathbf{Z}/p^j)$). Thus we may assume that $n > 1$. From the argument in chapter 9 of [2], we see that if X is a space such that

(a) X is simply connected,

(b) $H^*(X; \mathbf{Z}/p) = \Lambda[u] \otimes S[v]$ where $|u| = 2n - 1$ and $|v| = 2n$, and

(c) $\beta_j(u) = v$,

then $\Omega(X_{(p)}) \simeq F$. Here Λ denotes the exterior algebra, S denotes the symmetric algebra, $|\ \ |$ denotes the cohomological dimension and β_j denotes the jth order Bockstein.

In [1] it is shown that spaces that satisfy conditions (b) and (c) always exist whenever n divides $p - 1$. Two constructions are used.

For the first construction, [4] shows that if m and q are natural numbers such that

(d) q is a power of a prime different from p,

(e) n is the order of q in $(\mathbf{Z}/p)^*$, the multiplicative group of units in $\mathbf{Z}/p$,

(f) $n \leq m < 2n$, and

(g) $\nu_p(q^n - 1) = j$ (where $\nu_p(s) = \sup\{t : p^t | s\}$),

then $BGL_m(\mathbf{F}_q)$ will satisfy (b) and (c). Such m and q can always be found by a classical theorem of Dirichlet. To obtain a space so that requirement (a) also holds, we simply note that if we add the conditions

(h) $GCD(q - 1, m) = 1$, and

(i) $m \geq 3$,

to conditions (d), (e), (f) and (g), then $BSL_m(\mathbf{F}_q)$ has the same cohomology as $BGL_m(\mathbf{F}_q)$ and $SL_m(\mathbf{F}_q)$ is perfect. Thus $(BSL_m(\mathbf{F}_q))^+$ will satisfy (a), (b) and (c). It is easily seen that the same theorem of Dirichlet guarantees that m and q satisfying (d)–(i) exist.

The second construction comes from considering the semi-direct product

$$\mathbf{Z}/p^j \rightarrowtail G \twoheadrightarrow \pi,$$

where π is a subgroup of $(\mathbf{Z}/p^j)^*$, the group of units in the ring $\mathbf{Z}/p^j$, and hence acts on $\mathbf{Z}/p^j$. We choose π to be the subgroup generated by σ, an element of order n, such that $\bar{\sigma}$, its image in $(\mathbf{Z}/p)^*$, also has order n. This can always be done since $n | p - 1$. The Serre spectral sequence for this extension thus collapses to

$$H^*(G; \mathbf{Z}/p) = H^*(\mathbf{Z}/p^j; \mathbf{Z}/p)^{\pi}$$

because the order of π is prime to p. This shows that BG satisfies (b) and (c).

To satisfy (a), consider the cofibre of the map

$$S^1 \cup_n e^2 \xrightarrow{f} BG,$$

where f is chosen so that the normal subgroup of G generated by $f_\#(\pi_1(S^1 \cup_n e^2))$ is all of G. The cofibre satisfies condition (a) by the Seifert Van-Kampen theorem and it satisfies (b) and (c) by the long exact sequence in cohomology for a cofibration. It is easy to see that such an f exists because $\bar{\sigma} \neq \bar{1}$ in $\mathbf{Z}/p$ (since $n > 1$).

Thus lemma II follows. Q.E.D.

Part II

In this part of the paper we present a series of lemmas, which prove the converse of lemma II. This is trivial if $n \leq 2$. We thus assume, from now on, that n is at least 3, and that F is a loop space. The last assumption is equivalent to being a topological group up to homotopy.

LEMMA III. *$H^*(F; \mathbf{Z}/p^t) = \Gamma[x] \otimes \Lambda[y]$ where $|x| = 2n - 2$, $|y| = 2n - 1$ and $t \leq j$. (Here Γ denotes the divided polynomial algebra and Λ the exterior algebra.) Also, $\beta_j(x) = y$, for $t = 1$.*

Proof. $\mathbf{Z}/p^t$ coefficients are suppressed throughout the proof. The Serre spectral sequence in cohomology for the fibration,

$$F \to S^{2n-1} \xrightarrow{p^j} S^{2n-1},$$

degenerates to the Wang long exact sequence

$$\cdots \to H^i(S^{2n-1}) \to H^i(F) \xrightarrow{d} H^{i-(2n-2)}(F) \to H^{i+1}(S^{2n-1}) \to \cdots.$$

Here d acts as a derivation with respect to the cup product on $H^*(F)$. This shows that

$$H^i(F) \underset{\cong}{\xrightarrow{d}} H^{i-(2n-2)}(F)$$

is an isomorphism for $i > 2n - 1$. The low end of the sequence is

$$H^{2n-2}(S^{2n-1}) \to H^{2n-2}(F) \xrightarrow{d} H^0(F) \to H^{2n-1}(S^{2n-1})$$
$$\to H^{2n-1}(F) \xrightarrow{d} H^1(F).$$

From the long exact sequence in homotopy for the fibration, we have

$$\pi_*(F) = \begin{cases} 0 & \text{if } * < 2n-2 \\ \mathbf{Z}/p^j & \text{if } * = 2n-2. \end{cases}$$

Thus, by the Hurewicz theorem and the universal coefficient theorem, the end of the sequence becomes

$$0 \to [\mathbf{Z}/p = H^{2n-2}(F)] \xrightarrow{d} [\mathbf{Z}/p = H^0(F)]$$

$$\to \mathbf{Z}/p \to H^{2n-1}(F) \xrightarrow{d} [0 = H^1(F)].$$

Thus,

$$H^{2n-2}(F) \underset{\cong}{\xrightarrow{d}} H^0(F)$$

is an isomorphism and $H^{2n-1}(F) = \mathbf{Z}/p^t$. This shows that

$$H^*(F) = \begin{cases} \mathbf{Z}/p^t & \text{if } * \geq 0,\ * \neq 1 \text{ and } * \equiv 0 \text{ or } 1 \bmod 2n-2 \\ 0 & \text{otherwise} \end{cases}$$

and that

$$d\colon H^i(F) \to H^{i-(2n-2)}(F)$$

is an isomorphism for $i \neq 2n-1$, $i \geq 2n-2$. The multiplicative structure of $H^*(F)$ is now seen to be as stated from the fact that d is a derivation. Finally, the fact that $\pi_{2n-2}(F) = \mathbf{Z}/p^j$ implies that $\beta_j(x) = y$ when $t = 1$. Q.E.D.

Since F is a topological group, $H_*(F; \mathbf{Z}/p)$ is a Hopf algebra with Pontrjagin product, and coproduct dual to the cup product. The essential idea, suggested by M. Hopkins, is that the coproduct severely limits the possibilities for the product. For the rest of the paper, $\mathbf{Z}/p$ coefficients are understood.

LEMMA IV. *Under the Pontrjagin product*

$$H_*(F) = \frac{S[a_0, a_1, a_2, \cdots]}{< (a_i^p - \lambda_i a_{i+1})\colon i = 0, 1, \cdots >} \otimes \Lambda[b]$$

where $|a_i| = p^i(2n-2)$, $|b| = 2n-1, \lambda_i = 0$ *or* 1 *and each* a_i *and* b *is primitive. If* $j \neq 1$ *then all the* $\lambda_i = 1$ *(so that* $H_*(F) = S[a_0] \otimes \Lambda[b]$*).*

Proof. From lemma III we know that as a $\mathbf{Z}/p$-vector space $H_*(F)$ has a basis $\{\tilde{x}_i, \tilde{y}_i\}$ dual to $\{x^{[i]}, x^{[i]}y\}$. Also, the coproduct is

$$\Delta(\tilde{x}_i) = \sum_{t=0}^{i} \binom{i}{t} \tilde{x}_t \otimes \tilde{x}_{i-t}$$

and

$$\Delta(\tilde{y}_i) = \sum_{t=0}^{i} \binom{i}{t} (\tilde{x}_t \otimes \tilde{y}_{i-t} + \tilde{y}_t \otimes \tilde{x}_{i-t}).$$

If we can show that $\tilde{y}_0^2 = 0$, $\tilde{x}_m \tilde{y}_0 = \tilde{y}_m = \tilde{y}_0 \tilde{x}_m$ and, for $c_i \neq p-1$, $\tilde{x}_m \tilde{x}_{p^i} = \tilde{x}_{m+p^i}$, where the p-adic expansion for m is $c_0 + c_1 p + \cdots + c_k p^k$, then the first part of the lemma will be proven by letting $a_i = \tilde{x}_{p^i}$ and $b = \tilde{y}_0$. The first of these follows since $H_{2(2n-1)}(F) = 0$. For the second, note that $\tilde{x}_m \tilde{y}_0 = \lambda \tilde{y}_m$ for some $\lambda \in \mathbf{Z}/p$. Applying Δ to both sides and comparing the coefficient of the $\tilde{x}_m \otimes \tilde{y}_0$ term we have $\lambda = 1$. Similarly $\tilde{y}_m = \tilde{y}_0 \tilde{x}_m$. For the last we again know that $\tilde{x}_m \tilde{x}_{p^i} = \lambda \tilde{x}_{m+p^i}$ for some $\lambda \in \mathbf{Z}/p$. Comparing the $\tilde{x}_m \otimes \tilde{x}_{p^i}$ terms after applying Δ we obtain

$$\tilde{x}_m \otimes \tilde{x}_{p^i} + \binom{m}{p^i} \tilde{x}_{m-p^i} \tilde{x}_{p^i} \otimes \tilde{x}_{p^i} = \lambda \binom{m+p^i}{p^i} \tilde{x}_m \otimes \tilde{x}_{p^i}.$$

Recall that if $s = e_0 + e_1 p + \cdots + e_k p^k$ and $t = f_0 + f_1 p + \cdots + f_k p^k$ are p-adic expansions, then

$$\binom{s}{t} \equiv \binom{e_0}{f_0} \binom{e_1}{f_1} \cdots \binom{e_k}{f_k} \mod p.$$

Thus

$$\binom{m}{p^i} \equiv c_i \quad \text{and} \quad \binom{m+p^i}{p^i} \equiv c_i + 1.$$

So for $c_i = 0$ it is immediate that $\lambda = 1$, and for $c_i \neq 0$ we have $\lambda = 1$ because $\tilde{x}_{m-p^i} \tilde{x}_{p^i} = \tilde{x}_m$ by induction.

Finally, if $j \neq 1$ we consider the ring map $H_*(F; \mathbf{Z}/p^2) \to H_*(F)$. Using the same name for elements in both rings we will show that

$$\tilde{x}_{p^i} \tilde{x}_{p^i[p-1]} \equiv \tilde{x}_{p^{i+1}} \mod p,$$

for $\mathbf{Z}/p^2$ coefficients and so the final statement in the lemma will follow. Again, for some λ in $\mathbf{Z}/p^2$, $\tilde{x}_{p^i} \tilde{x}_{p^i(p-1)} = \lambda \tilde{x}_{p^{i+1}}$. Applying Δ and comparing coefficients of the $\tilde{x}_p \otimes \tilde{x}_{p^i(p-1)}$ term, we get

$$1 + \binom{p^i(p-1)}{p^i} \equiv \lambda \binom{p^{i+1}}{p^i} \quad \mathrm{mod}\ p^2.$$

Since

$$\binom{p^{i+1}}{p^i} \equiv p \text{ and } \binom{p^i(p-1)}{p^i} \equiv p - 1 \quad \mathrm{mod}\ p^2$$

we have $\lambda \equiv 1 \bmod p$. Q.E.D.

Lemma V gives the basic argument which shows that n divides $p - 1$ in almost all cases. The remainder of the paper disposes of the few exceptions not covered by it.

LEMMA V. *If $j \neq 1$ then n divides $p - 1$.*

Proof. From Lemma IV we have $H_*(F) = S[a_0] \otimes \Lambda[b]$ with $|a_0| = 2n - 2$, $|b| = 2n - 1$ and both a_0 and b primitive. From the Rothenberg-Steenrod spectral sequence [5],

$$\mathrm{Ext}^{s,t}_{H_*(F)}(\mathbf{Z}/p, \mathbf{Z}/p) \Rightarrow H^*(BF),$$

we have $E_2 = \Lambda[u] \otimes S[v]$, where $u \in E_2^{1,2n-2}$ and $v \in E_2^{1,2n-1}$. For dimensional reasons (since $d_r\colon E_r^{p,q} \to E_r^{p+r,q+1-r}$) u and v must be infinite cycles. Thus, the fact that d_r is a derivation implies that the spectral sequence collapses. Hence,

$$H^*(BF) = \Lambda[u] \otimes S[v]$$

with $|u| = 2n - 1$, $|v| = 2n$. It is shown in [1] that this implies n divides $p - 1$. The argument is that since β acts trivially on $H^*(BF)$ (because $j \neq 1$) and $\mathcal{P}^n v = v^p \neq 0$, we must have $\mathcal{P}^1$ acting non-trivially (by considering secondary operations of [3]). Since $\mathcal{P}^1$ raises dimension by $2(p - 1)$ it follows that $2n$ divides $2(p - 1)$. Q.E.D.

When $j = 1$, two problems arise. The first is that $H_*(F)$ might not equal

$$S[a_0] \otimes \Lambda[b].$$

The second is that since $\beta \neq 0$ the argument using secondary operations breaks down. The latter problem is solved by Aguadé who shows, in [1], that if

$$H^*(BF) = \Lambda[u] \otimes S[v]$$

through dimension $2np$, with $|u| = 2n - 1$, $|v| = 2n$ and $\beta(u) = v$, then n divides $p - 1$. To see this consider the Adem relation

$$(*) \qquad \mathscr{P}^1\beta\mathscr{P}^{n-1} = (n-1)\beta\mathscr{P}^n + \mathscr{P}^n\beta$$

applied to u. The right side becomes $v^p \neq 0$ so that $\beta\mathscr{P}^{n-1}(u)$ of degree $2np - 2(p-1)$ is non-zero, so that its degree is also a multiple of $2n$ showing that $n | p - 1$. Thus the only remaining case is when $j = 1$ and

$$H^*(BF) \neq \Lambda[u] \otimes S[v] \quad \text{for } * \leq 2np.$$

Now $H_*(F) = S[a_0] \otimes \Lambda[b]$ for $* < p^2(2n-2)$ provided that $a_0^p \neq 0$ and then

$$H^*(BF) = \Lambda[u] \otimes S[v] \quad \text{for } * \leq 2np.$$

We therefore need only prove the following:

LEMMA VI. *If $j = 1$ and $a_0^p = 0$ in $H_*(F)$ then n divides $p - 1$.*

Proof. From lemma IV we know that

$$H_*(F) = \frac{S[a_0, a_1]}{\langle a_0^p \rangle} \otimes \Lambda[b] \quad \text{for } * < p^2(2n-2).$$

Thus the Rothenberg-Steenrod spectral sequence has

$$E_2 = \Lambda[u, w] \otimes S[v, z] \quad \text{for total degree} < p^2(2n-2).$$

The bi-degrees of u, w, v and z are

$$(1, 2n-2), (1, p(2n-2)), (1, 2n-1) \text{ and } (2, p(2n-2))$$

respectively. Again, u and v are clearly seen to be infinite cycles. A simple arithmetic computation shows that if w is not an infinite cycle, then n divides $p - 1$. One can also see that z is an infinite cycle by noting that it must be the image of w under a (possibly higher order) Bockstein. Thus, the spectral sequence collapses in a range, and we have

$$H^*(BF) = \Lambda[u, w] \otimes S[v, z] \quad \text{for } * < p^2(2n-2) - 1,$$

with the degrees of u, w, v, z being the total degrees listed above. The key point is that we know $H^*(BF)$ for $* \leq 2np$. Applying the Adem relation $*$ above to u we get $\mathscr{P}^1\beta\mathscr{P}^{n-1}(u) = v^p \neq 0$ so that $\mathscr{P}^{n-1}(u) \neq 0$. If $\mathscr{P}^{n-1}(u) = uv^m$ for some m then it's immediate that $n | p - 1$. Hence we may assume that $\mathscr{P}^{n-1}(u) = w$. Also $\beta(w) = z$. Now consider the Serre spectral sequence, in cohomology, for the fibration

$$(F \simeq \Omega BF) \to (* \simeq PF) \to BF.$$

Here x and y transgress to u and v respectively under d_{2n-1} and d_{2n}. Since transgression, τ, and the Steenrod algebra commute, we have $\tau\mathscr{P}^{n-1}(x) = \mathscr{P}^{n-1}(u) = w$. But $\mathscr{P}^{n-1}(x) = x^p = 0$, so that w must be hit before the transgression gets a chance. A simple arithmetic calculation shows that all candidates to hit w are zero by E_{2n}. Thus, we have a contradiction. Q.E.D.

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