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REMARKS ON THE EXISTENCE AND UNIQUENESS OF UNBOUNDED VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS[1]

BY

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This paper deals with existence and uniqueness questions for solutions of general first-order Hamilton-Jacobi equations. The development of the theory of "viscosity solutions" has resulted in existence and uniqueness results of substantial generality for solutions which are uniformly continuous (or UC) on $\mathbf{R}^N$: we refer the reader to M.G. Crandall and P.L. Lions [6] and M.G. Crandall, L.C. Evans and P.L. Lions [4] for the main properties of viscosity solutions including definitions, uniqueness for bounded uniformly continuous (or BUC) solutions and existence in model cases; P.L. Lions [18], [19], P.E. Souganidis [22], Barles [1] for existence of BUC solutions; H. Ishii [13]–[15], M.G. Crandall and P.L. Lions [8]–[11] for existence and uniqueness of UC solutions; P.L. Lions [18] for the relevance of viscosity solutions to deterministic optimal control theory and L.C. Evans and P.E. Souganidis [12] concerning differential games.

However, for Hamiltonians such as those which occur in control theory or differential games, dealing only with BUC or UC value functions requires somewhat stringent assumptions. It is our goal here to broaden the scope of the theory and to point out relations between structure properties of the Hamiltonian and naturally associated classes of viscosity solutions in which existence and uniqueness holds.

For the sake of simplicity we will consider two model problems, namely the stationary problem

$$(\mathrm{SP})_f \qquad u + H(x, Du) + f(x) = 0 \text{ in } \mathbf{R}^N,$$

which we have indexed by an "inhomogeneous" term $f \in C(\mathbf{R}^N)$ for later convenience, and the Cauchy problem

$$(\mathrm{CP}) \qquad \begin{aligned} u_t + H(x, t, Du) &= 0 \text{ in } \mathbf{R}^N \times]0, T], \\ u(x,0) &= \varphi(x) \text{ in } \mathbf{R}^N, \end{aligned}$$

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in which the Hamiltonians $H(x, p)$, $H(x, t, p)$ will always be assumed to be at least continuous. In these problems, all the functions involved are real-valued functions of the indicated arguments, $x \in \mathbf{R}^N$ and Du stands for the gradient of u in the space variables x; $Du = (u_{x_1}, \dots, u_{x_N})$. The methods we present below allow one to treat more general equations, for example, more general dependencies on u of the form $H(x, u, Du) = 0$ in place of $(\mathrm{SP})_f$, but we will not pursue this straightforward matter here.

We next introduce a condition which will often be assumed and then illustrate the nature of our results with some examples. Here and later, B_R denotes the ball of radius R centered at the origin and $|\ \ |$ is the Euclidean norm on $\mathbf{R}^N$. A nondecreasing, continuous and subadditive function m: $[0, \infty) \to [0, \infty)$ satisfying $m(0) = 0$ will be called a *modulus* and a mapping

$$\sigma: [0, \infty) \times [0, \infty) \to [0, \infty)$$

for which $r \to \sigma(r, R)$ is a modulus for $R > 0$ will be called a *local modulus*. If

$$H \in C(\mathbf{R}^N \times [0, T] \times \mathbf{R}^N)$$

is a Hamiltonian, we say H satisfies (U) (for "uniqueness") if there is a modulus m such that

$$\text{(U)} \qquad \begin{aligned} &H(y, t, \lambda(x - y)) - H(x, t, \lambda(x - y)) \\ &\quad \le m(\lambda|x - y|^2 + |x - y|) \\ &\qquad\qquad \text{for } x, y \in \mathbf{R}^N, \lambda \ge 0, \text{ and } t \in [0, T]. \end{aligned}$$

We are also interested in the local version of (U); that is there is a local modulus σ such that

$$\text{(LU)} \qquad \begin{aligned} &H(y, t, \lambda(x - y)) - H(x, t, \lambda(x - y)) \\ &\quad \le \sigma(\lambda|x - y|^2 + |x - y|, R) \\ &\qquad\qquad \text{for } R > 0,\ x, y \in B_R, \lambda \ge 0 \text{ and } t \in [0, T]. \end{aligned}$$

For example, the Hamiltonian $H(x, p) = \sin(x^2)p$ in $\mathbf{R} \times \mathbf{R}$ satisfies (LU) but not (U). Existence and uniqueness results using (U) and other hypotheses are given in Crandall and Lions [8]–[11] and Ishii [15] where one can find further commentary on this condition. Examples given below (see also [11, Section 5]) show that it does not guarantee uniqueness by itself—other assumptions are needed. A generalization of (U) is given in [9], [10], but we will not employ this generality here in either its global or local form, as this would obscure the ideas and may easily be done by the reader once the ideas are made clear. We will use the conditions (U) and (LU) in the stationary problem with the obvious interpretation if H is independent of t.

Example 1. Assume that for some $\mu > 0$ H satisfies

$$|H(x,p) - H(x,q)| \le \mu|p-q|, \quad \forall x,p,q \in \mathbf{R}^N. \tag{1}$$

Assume moreover, that H satisfies (LU). Then there is at most one viscosity solution $u \in C(\mathbf{R}^N)$ of $(\mathrm{SP})_f$ satisfying

$$\lim_{|x|\to\infty} u(x)\exp(-|x|/\mu) = 0. \tag{2}$$

If H also satisfies

$$\int_0^\infty \sup_{|x|\le r} |H(x,0)| e^{-r/\mu}\, dr < \infty \tag{3}$$

then $(\mathrm{SP})_0$ has a viscosity solution $u \in C(\mathbf{R}^N)$ satisfying (2).

Example 2. Assume that there is a C and $\theta \in (0,1)$ such that

$$|H(x,p) - H(x,q)| \le C|p-q|^\theta \quad \text{for } p,q \in \mathbf{R}^N \text{ and } x,y \in \mathbf{R}^N, \tag{4}$$

and H satisfies (LU). Then there exists a unique viscosity solution u of $(\mathrm{SP})_f$ in $C(\mathbf{R}^N)$.

Observe that in this case no restriction is needed on the behaviour of the solution for uniqueness nor on the growth of f for existence.

Example 3. Let H satisfy (LU). Assume, moreover, that there is a modulus γ such that

$$|H(x,t,p) - H(x,t,q)| \le \gamma(|p-q|(1+|x|)) \quad \text{for } x,q,p \in \mathbf{R}^N,\ t \in [0,T]. \tag{5}$$

Then for any $\varphi \in C(\mathbf{R}^N)$ there exists a unique viscosity solution $u \in C(\mathbf{R}^N \times [0,T])$ of (CP) (i.e., the equation is satisfied in the viscosity sense on $\mathbf{R}^N \times (0,T]$ and $u(x,0) = \varphi(x)$ in $\mathbf{R}^N$). Again, no restrictions are made at infinity.

In the text we present many more results of this kind, including existence results in cases where nonuniqueness is possible and the existence of minimal solutions. We also mention that the uniqueness statement in Example 1 and some particular cases covered by Example 3 were first obtained by H. Ishii [13] by a somewhat more complicated argument than that given herein. For problems arising in control theory, results analogous to those of Example 1 are to be found in P.L. Lions [20] (as particular cases of optimal stochastic control situations). Some of the results given below were announced in M.G. Crandall and P.L. Lions [7].

Finally we would like to mention that many of the results presented below may be adapted to the case in which the equations are set in an infinite dimensional Banach space V (instead of $\mathbf{R}^N$). This can be done by combining the arguments outlined herein with those given in M.G. Crandall and P.L. Lions [9], [10], [11]. Moreover, as mentioned above, the role of $\lambda(x-y)$ in (U) and (LU) may be replaced by more general quantities as in these works.

I. Lipschitz Hamiltonians and the stationary problem

In this section we will assume that H satisfies (1) (or variants) and (LU) and we will be interested in the existence and uniqueness of viscosity solutions of $(\mathrm{SP})_f$. Let us first observe that some limitations on the growth of solutions have to be imposed in order to have uniqueness. Indeed, the simple linear equation

$$-\mu u' + u = 0 \quad \text{in } \mathbf{R} \tag{6}$$

has the distinct C^1 (and hence viscosity) solutions $u \equiv 0$ and $u \equiv \exp(x/\mu)$. In view of this example, it is natural to impose the following conditions on sub and supersolutions as is done below:

$$\varlimsup_{|x|\to\infty} u(x)\exp(-|x|/\mu) \le 0, \tag{7}$$

$$\varliminf_{|x|\to\infty} v(x)\exp(-|x|/\mu) \ge 0. \tag{8}$$

In the following statement of our main result, $r^+ = \max(r,0)$.

THEOREM I.1. *Let H satisfy* (1), (LU) *and $f \in C(\mathbf{R}^N)$.*

Uniqueness. Let $u, v \in C(\mathbf{R}^N)$ satisfy (7) *and* (8). *Let H satisfy* (LU). *Assume that u is a viscosity subsolution of* $(\mathrm{SP})_0$ *and that v is a viscosity supersolution of* $(\mathrm{SP})_f$ *in $\mathbf{R}^N$. Then for all $x \in \mathbf{R}^N$,*

$$(u-v)^+(x) \le \int_0^\infty \sup\{f(y)^+ : |x-y| \le \mu t\} e^{-t}\, dt. \tag{9}$$

Existence. Let $H(x,0)$ also satisfy (3). *Then there exists a unique viscosity solution $u \in C(\mathbf{R}^N)$ of* $(\mathrm{SP})_0$ *satisfying* (2).

Remark. A uniqueness result of a similar form was first obtained by H. Ishii [13] by a different method. Observe also that the right hand side of (9) is bounded from above by $\sup_{\mathbf{R}^N} f^+$.

Proof of the uniqueness. Let $x_0 \in \mathbf{R}^N$ and let $R < \infty$. We are going to compare u and v on $B(x_0, R)$, the open ball of radius R centered at x_0. This is achieved by use of the next lemma.

LEMMA I.1. *Let $u, v \in C(\mathbf{R}^N)$ be, respectively, viscosity sub and supersolutions of* $(\mathrm{SP})_0$ *and* $(\mathrm{SP})_f$. *Let H satisfy* (1) *and* (LU). *Then we have*

$$(10)\qquad \begin{aligned}(u(x_0) - v(x_0))^+ &\le \int_0^{R/\mu} \sup\{f^+(y) \colon |y - x_0| \le \mu t\} e^{-t}\,dt \\ &\quad + \max\Big(\sup_{\partial B(x_0, R)} (u - v)^+,\ \sup_{B(x_0, R)} f^+\Big) e^{-R/\mu}.\end{aligned}$$

We first complete the proof of (9) and then prove the lemma. We may of course assume that the right hand side of (10) is finite; this implies that

$$(11)\qquad \Big(\sup_{B(x_0, R_n)} f^+\Big)\exp(-R_n/\mu) \to 0$$

for some sequence $R_n \to \infty$. On the other hand, in view of (7)–(8),

$$\Big(\sup_{\partial B(x_0, R)} (u - v)^+\Big)\exp(-R/\mu) \to 0 \quad \text{as } R \to \infty.$$

Therefore choosing $R = R_n$ in (10) and sending $n \to \infty$, we deduce (9).

We now prove Lemma I.1: We set

$$K = \max\Big(\sup_{\partial B(x_0, R)} (u - v)^+,\ \sup_{B(x_0, R)} f^+\Big)$$

and introduce the function

$$\begin{aligned} w(x) &= K \exp\{(|x - x_0| - R)/\mu\} \\ &\quad + (1/\mu)\exp(|x - x_0|/\mu) \int_{|x - x_0|}^{R} \sup\{f^+(y) \colon |y - x_0| \le s\} e^{-s/\mu}\,ds.\end{aligned}$$

One easily checks that w is the viscosity solution of

$$\begin{aligned} -\mu|Dw| + w &= \sup\{f^+(y) \colon |y - x_0| \le |x - x_0|\} \quad \text{in } B(x_0, R) \\ w &= K \quad \text{on } \partial B(x_0, R).\end{aligned}$$

Next, we claim that $v + w$ is a viscosity supersolution of $(\mathrm{SP})_0$ in $B(x_0, R)$. Formally this is clear since

$$\begin{aligned} H(x, D(v + w)) &\ge H(x, Dv) - \mu|Dw| \\ &\ge -(v + w) - f - \mu|Dw| + w \ge -(v + w).\end{aligned}$$

To justify this one first replaces $|x - x_0|$ by $(\delta^2 + |x - x_0|^2)^{1/2}$ in the definition of w, makes the corresponding estimate (now valid since w_δ is C^1) at points of superdifferentiability of $v + w$ and then passes to the limit as $\delta \to 0$.

Observing that $v + w \geq u$ on $\partial B(x_0, R)$ and applying the comparison results of [8], we deduce that $v + w \geq u$ in $B(x_0, R)$, which implies (10).

Proof of the existence. For $R > 0$ let φ_R be a smooth function on $\mathbf{R}^N$ supported in B_{2R} and satisfying $0 \leq \varphi_R \leq 1$, $\varphi_R \equiv 1$ on B_R and $|D\varphi_R| \leq 2/R$, and then define $H_R(x, p) = \varphi_R(x)H(x, p)$. We consider the approximate problem

$$u_R + H_R(x, Du_R) = 0.$$

The Hamiltonian H_R satisfies (1). Moreover

$$\begin{aligned} &H_R(y, \lambda(x-y)) - H_R(x, \lambda(x-y)) \\ &\quad = (\varphi_R(y) - \varphi_R(x))(H(y, \lambda(x-y))) - H(y,0)) \\ (12) \qquad &\quad + \varphi_R(x)((H(y, \lambda(x-y)) - H(x, \lambda(x-y))) \\ &\quad + (\varphi_R(x) - \varphi_R(y))H(y,0). \end{aligned}$$

Now the first term on the right is at most

$$2(\mu/R)|x-y|\,|\lambda(x-y)|$$

and if $|x - y| \leq 1$, the second is at most

$$2\sigma(\lambda|x-y|^2 + |x-y|, 2R+1),$$

where σ is the local modulus of H from (LU), and the third term may obviously be estimated by a multiple of $|x - y|$. It follows that H_R satisfies (U). Since also $H_R \in \mathrm{UC}(\mathbf{R}^N \times B_K)$ for all $K > 0$, the existence results of [8] apply and there are viscosity solutions $u_R \in \mathrm{BUC}(\mathbf{R}^N)$ of $u_R + H_R(x, Du_R) = 0$. We now use the comparison result Lemma I.1 to compare U_R and the solution $w \equiv 0$ of $w + H_R(x, Dw) - H_R(x, 0) = 0$ and let the radius in (10) tend to ∞ to conclude that

$$(13) \qquad |u_R(x)| \leq \frac{1}{\mu}e^{|x|/\mu}\int_{|x|}^{\infty} \sup\{|H(y,0)| : |y| \leq s\}\, e^{-s/\mu}\, ds.$$

Let $R, R', R_0, R_1 > 0$ and $R, R' \geq R_1 + R_0$. Then for $x \in B_{R_0}$, H_R and $H_{R'}$ agree on $B(x, R_1)$ and we may use Lemma I.1 again with $u, v = u_R, u_{R'}$ to conclude that

$$|u_R(x) - u_{R'}(x)| \leq \sup_{\partial B(x, R_1)} |u_R - u_{R'}| e^{-R_1/\mu}.$$

Using (13) in this estimate we finally deduce that

$$|u_R(x) - u_{R'}(x)|$$
$$\leq \frac{2}{\mu}\exp(R_0/\mu)\int_{(R_1-R_0)}^{\infty} \sup\{|H(y,0))|: |y| \leq s\} e^{(-s/\mu)}\, ds.$$

Because $H(x,0)$ satisfies (3), we conclude that the u_R form a Cauchy net in $C(B_{R_0})$ as $R \to \infty$ for any $R_0 > 0$. By the standard stability results for viscosity solutions ([6]), we deduce that u_R converges uniformly on bounded sets to some viscosity solution u of $(\mathrm{SP})_0$. In addition, letting $R \to \infty$ in (13), we see that u satisfies (2).

Remark. Considering the equation $-\mu|Du| + u = f(x)$ in $\mathbf{R}^N$, one sees (again) that uniqueness is false without the condition $u(x)\exp(-|x|/\mu) \to 0$ as $|x| \to \infty$ (take $f \equiv 0$, $u(x) = \exp|x|/\mu$ or $u = 0$) and then that existence in this class may fail if f does not satisfy (3).

We consider next a slightly more general situation where (1) is replaced by

$$(14) \qquad |H(x,p) - H(x,q)| \leq \Phi(|x|)|p-q|, \quad \forall x, p, q \in \mathbf{R}^N$$

where Φ is continuous, increasing, $\Phi(0) \geq 0$ and Φ satisfies

$$(15) \qquad \int_1^{\infty} \frac{ds}{\Phi(s)} = +\infty$$

Then the arguments used above are easily adapted to prove the following result:

THEOREM I.2. *Let H satisfy* (LU), (14) *and* $f \in C(\mathbf{R}^N)$.

Uniqueness. Let $u, v \in C(\mathbf{R}^N)$ satisfy respectively

$$(16) \quad \overline{\lim_{|x|\to\infty}}\, u(x)\exp\left\{-\int_1^{|x|}\frac{d\sigma}{\Phi(\sigma)}\right\} \leq 0 \qquad \underline{\lim_{|x|\to\infty}}\, v(x)\exp\left\{-\int_1^{|x|}\frac{d\sigma}{\Phi(\sigma)}\right\} \geq 0.$$

Let u be a viscosity subsolution of $(\mathrm{SP})_0$ *and v a viscosity supersolution of* $(\mathrm{SP})_f$. *Then for all $x \in \mathbf{R}^N$,*

$$(u-v)^+(x) \leq \int_{|x|}^{\infty} \sup\{f^+(y): |y| \leq t\}\exp\left(-\int_{|x|}^{t}\frac{ds}{\Phi(s)}\right)\frac{dt}{\Phi(t)}.$$

Existence. *In addition, let*

$$\int_1^\infty \sup\{|H(y,0)| : |y| \le t\} \exp\left(-\int_1^t \frac{ds}{\Phi(s)}\right) \frac{dt}{\Phi(t)} < \infty. \tag{17}$$

Then there exists a unique viscosity solution $u \in C(\mathbf{R}^N)$ *of* $(\mathrm{SP})_0$ *satisfying*

$$\lim_{|x|\to\infty} u(x) \exp\left\{-\int_1^{|x|} \frac{ds}{\Phi(s)}\right\} = 0. \tag{18}$$

II. Uniformly continuous hamiltonians and the stationary problem

We now turn to the case when H satisfies (4) or, more generally,

$$|H(x,p) - H(x,q)| \le \gamma(|p-q|), \quad \forall p, q \in \mathbf{R}^N \tag{19}$$

where γ is a modulus. We assume that the inverse ν of γ satisfies

$$\int_1^\infty \frac{ds}{\nu(s)} < \infty, \quad \int_0^1 \frac{ds}{\nu(s)} = +\infty. \tag{20}$$

Of course if $\gamma(r) = Cr^\alpha$ with $C \ge 0$, $0 < \alpha < 1$, then (20) holds. The main result below asserts existence and uniqueness without any conditions at infinity: such results may be expected in view of the trivial case $H(x,p) \equiv H(x)$.

THEOREM II.1. *Let H satisfy* (LU), (19), (20) *and let* $f \in C(\mathbf{R}^N)$.

Uniqueness. *Let* $u, v \in C(\mathbf{R}^N)$ *be, respectively, viscosity sub and supersolutions of* $(\mathrm{SP})_0$ *and* $(\mathrm{SP})_f$. *Then*

$$\sup_{\mathbf{R}^N} (u-v)^+ \le \sup_{\mathbf{R}^N} f^+. \tag{21}$$

Existence. *There exists a unique viscosity solution of* $(\mathrm{SP})_f$.

Remark. Of course, one could formulate results which unify Theorem I.1 and Theorem II.1. Roughly speaking, if (20) does not hold (as is the case if $\gamma(r) = \mu r$), then $H(x,0)$, u, v have to satisfy certain growth conditions at infinity which are revealed by an examination of the proofs we present.

Sketch of proof. We will only prove (21), which follows from a simple application of the lemma below. The existence is also obtained in a similar way to the above by use of the lemma. Let $R > 0$ and denote by w_R the function defined by

$$(22) \qquad \int_{w_R(|x|)}^{+\infty} \frac{ds}{\nu(s)} = R - |x| \quad \text{for } 0 \le |x| \le R.$$

One checks easily that w_R is a viscosity solution of

$$(23) \quad -\gamma(|Dw_R|) + w_R = 0 \quad \text{in } B_R, \quad w_R(x) \to +\infty \quad \text{as } |x| \to R- .$$

Therefore, one has:

LEMMA II.1. *With the notations and assumptions of Theorem II.*1, *for all* $x_0 \in \mathbf{R}^N$, $R > 0$,

$$(24) \quad (u - v)^+(x) \le w_R(|x - x_0|) + \sup_{B(x_0, R)} f^+, \quad \forall x \in B(x_0, R).$$

Next, observe that in view of the explicit formula (22) we have: $w_R(|x|) \to 0$ as R, $|x| \to \infty$, If $R - |x| \to \infty$. In particular w_R converges uniformly to 0 on compact sets. This combined with (24) yields (21).

Remark. One sees in the above proof the basic role played by the solution w_R of (23), which is an HJ equation with infinite boundary conditions. In a different context, H. Brezis [3] uses a similar method to obtain uniqueness results without growth restrictions. Finally, let us observe that for more general nonlinear partial differential equations, the possibility of prescribing infinite boundary conditions as in (23) is studied in J.M. Lasry and P.L. Lions [16], [17]. In particular, using the results and methods of [16], [17] one sees that for any (say smooth) bounded open set Ω and bounded continuous function f on Ω there exists a unique viscosity solution of

$$-\mu(|Dw|) + w = f(x) \quad \text{in } \Omega, \quad w \to +\infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to +\infty.$$

In addition, w is locally Lipschitz on Ω and if $\Omega_n \uparrow \mathbf{R}^N$ the corresponding solutions w_n converge to 0 uniformly on compact sets.

We next formulate (without proof) an extension of (19),

$$(25) \quad |H(x, p) - H(x, q)| \le \gamma(\Phi(|x|)|p - q|), \quad \forall x, p, q \in \mathbf{R}^N,$$

where γ satisfies (20), Φ is continuous, increasing, $\Phi(0) \ge 0$ and Φ satisfies (15).

THEOREM II.2. *Let* (20) *and* (15) *hold*, *let* $f \in C(R^N)$, *and suppose* H *satisfies* (LU) *and* (25).

Uniqueness. *Let $u, v \in C(\mathbf{R}^N)$ be, respectively, viscosity sub and supersolutions of* $(\mathrm{SP})_0$ *and* $(\mathrm{SP})_f$ *with $f \in C(\mathbf{R}^N)$. Then* (21) *holds.*

Existence. *There exists a viscosity solution of* $(\mathrm{SP})_f$.

III. Power-like Hamiltonians and the stationary problem

Motivated by the case of a Hamiltonian of the form $H(x, p) = \lambda |p|^m$ with $m > 1$, $\lambda > 0$, we consider the case of a Hamiltonian satisfying:

$$(26) \quad |H(x, p) - H(x, q)| \le \left\{C_0|p|^{m-1} + C_0|q|^{m-1} + C\right\}|p - q|, \quad \forall x, p, q \in \mathbf{R}^N$$

for some constants $C_0, C > 0$. We will consider viscosity sub and supersolutions which are Lipschitz locally in $\mathbf{R}^N$ and satisfy

$$(27) \quad |Du| \le C_1|x|^{(m'-1)} + C, \; |Dv| \le C_2|x|^{(m'-1)} + C \quad \text{a.e. on } \mathbf{R}^N$$

where $C, C_1, C_2 > 0$ and $m' = m/(m-1)$.
Then the main comparison result is:

THEOREM III.1. *Let* (26) *hold. Let u and v be locally Lipschitz on $\mathbf{R}^N$ and, respectively, viscosity sub and supersolutions of* $(\mathrm{SP})_0$ *and* $(\mathrm{SP})_f$ *with $f \in C(\mathbf{R}^N)$. Assume that* (26) *and* (27) *hold and*

$$C_0\left(C_1^{m-1} + C_2^{m-1}\right) < 1/m'.$$

Then

$$\sup_{\mathbf{R}^N}(u - v)^+ \le \sup_{\mathbf{R}^N} f^+.$$

Remark. First of all we could replace $|p|^m$ by more general convex, increasing functions $\Phi(|p|)$. In fact, the proof below uses only (27) and the fact that if

$$p \in D^+u(x), q \in D^-v(y), \quad \text{and } |x - y| \le 1$$

then

$$(28) \qquad |H(x, p) - H(x, q)| \le (C + \theta|x|)|p - q|$$

for some $C > 0$, $\theta \in (0, 1/m')$. Clearly (26) and (27) imply (28) if

$$\theta > C_0\left(C_1^{m-1} + C_2^{m-1}\right).$$

The assumption that (28) holds with $\theta \in (0, 1/m')$ is nearly optimal in view of the following example: If $H(x, p) = -(1/m')|p|^m$ then $u \equiv 0$ solves $(\mathrm{SP})_0$ while $u(x) = (1/m')|x|^{m'}$ also solves $(\mathrm{SP})_0$. Observe that (28) holds with $\theta = 1/m'$.

Proof. Again, the proof is quite similar to those sketched before. One chooses

$$\theta \in \left(C_0\left(C_1^{m-1} + C_2^{m-1}\right), 1/m'\right)$$

for which (28) holds and considers the solution w_R of

$$-(C + \theta|x|)|Dw_R| + w_R = 0 \quad \text{on } B_R, \quad w_R|_{\partial B_R} = 1,$$

given by

$$w_R(x) = \left(\frac{C + \theta\left(|x|^2 + 1\right)^{1/2}}{C + \theta(R^2 + 1)^{1/2}}\right)^{1/\theta}.$$

Considering the maximum of the function

$$u(x) - v(y) - \frac{|x - y|^2}{\varepsilon} - w_R(x) \max_{\partial B_R}(u - v)^+$$

on $B_R \times B_R$ and letting $\varepsilon \downarrow 0$ we deduce

$$(u - v)^+(x) \le \max_{B_R} f^+ + w_R(x) \max_{\partial B_R}(u - v)^+ \quad \text{on } B_R.$$

Finally, observing that $|u(x)|$, $|v(x)| \le C + C|x|^{m'}$ and that $1/\theta > m'$ we deduce that for bounded x, $w_R(x) \max_{\partial B_R}(u - v)^+ \to 0$ as $R \to \infty$. Thus, we conclude by letting R go to $+\infty$.

We now conclude this section with an existence result corresponding to the above uniqueness result.

THEOREM III.2. *We assume* (26) *and*

(29) $\quad H(x, p) - H(x, 0) \ge \alpha|p|^m, \forall(x, p) \in \mathbf{R}^N,$ *for some* $\alpha > 0$

(30) $\quad |H(x, 0)| = o\left(|x|^{m'}\right)$ *as* $|x| \to \infty$.

Then there exists a unique locally Lipschitz viscosity solution u of $(\mathrm{SP})_0$ *with the property that for all* $\varepsilon > 0$ *there is a* C_ε *such that*

$$|Du(x)| \le \varepsilon|x|^{(m'-1)} + C_\varepsilon \quad \text{a.e. } on\ \mathbf{R}^N.$$

Remark. It is possible to replace (29) by

(29′) $H(x, p) - H(x,0) \leq -\alpha|p|^m, \forall(x, p) \in \mathbf{R}^N$, for some $\alpha > 0$.

Proof. We first build subsolutions of $(\mathrm{SP})_0$. For each $\varepsilon > 0$ with C_ε denoting a positive constant to be determined below we set

$$u_\varepsilon(x) = -\varepsilon\frac{1}{m'}|x|^{m'} - C_\varepsilon.$$

Then u_ε satisfies

$$\begin{aligned} H(x, Du_\varepsilon) + u_\varepsilon &\leq C|Du_\varepsilon|^m + C + H(x,0) + u_\varepsilon \\ &\leq C\varepsilon^m|x|^{m'} + C + o(|x|^{m'}) - \frac{\varepsilon}{m'}|x|^{m'} - C_\varepsilon \end{aligned}$$

and thus we may choose C_ε large enough so that for ε small enough u_ε is a viscosity subsolution of $(\mathrm{SP})_0$. It is clearly enough to show that there exists a viscosity solution of $(\mathrm{SP})_0$ satisfying $u \geq u_\varepsilon$. Indeed the equation and (29) yield that u is locally Lipschitz and

$$\alpha|Du|^m \leq -H(x,0) - u \leq o(|x|^{m'}) + \frac{\varepsilon}{m'}|x|^{m'} + C_\varepsilon$$

and thus for ε small enough u satisfies (27) with C_1 arbitrary small. Applying the uniqueness result, we are then able to conclude. Finally, the existence of a viscosity solution of $(\mathrm{SP})_0$ above u_ε is a very special case of the results proved in Section V.

IV. Remarks on the existence of BUC and UC solutions of the stationary problem

In this section $(\mathrm{SP})_0$ will simply be denoted by (SP). It was proved in M.G. Crandall and P.L. Lions [8] (see also H. Ishii [15]) that there exists a unique viscosity solution u of (SP) in $\mathrm{UC}(\mathbf{R}^N)$ provided H satisfies (U) and

(31) $$H \in \mathrm{BUC}(\mathbf{R}^N \times B_R) \quad \text{for } R > 0.$$

Related results were previously obtained by G. Barles [1] in the class BUC and then H. Ishii [13] in the class UC under more restrictive assumptions.

In the papers, H. Ishii [15] and M.G. Crandall and P.L. Lions [9], [10], [11] the requirement (31) is replaced by a (somewhat confusing) array of more general substitutes which we will not detail here. In particular, the substitutes separate various roles of (31) as they pertain to uniqueness questions and estimates on moduli of continuity. It is our goal here to obtain existence in

$\mathrm{UC}(\mathbf{R}^N)$ or $\mathrm{BUC}(\mathbf{R}^N)$ without (31). In order to do so we will need to supplement (U) in some way, and we do so quite simply. The simplicity of the assumptions, compared to the works mentioned above, is possible because we give up uniqueness.

We will use the following assumptions: Either

$$H(x,0) \in \mathrm{BUC}(\mathbf{R}^N) \tag{32}$$

or, for some $\theta \in (0,1)$ and modulus μ,

$$H(y, \lambda(x-y)) - H(x, \lambda(x-y)) \leq \theta\lambda|x-y|^2 + \mu(|x-y|), \quad \forall x, y, p \in \mathbf{R}^N. \tag{33}$$

(This is just one way to insure the existence of global supersolutions of an associated problem—see [10].)

The existence result is:

THEOREM IV.1. *Let H satisfy* (U).

(1) *If* (32) *holds, then there exists a viscosity solution $u \in \mathrm{BUC}(\mathbf{R}^N)$ of* (SP).

(2) *If* (33) *holds, then there exists a viscosity solution $u \in \mathrm{UC}(\mathbf{R}^N)$ of* (SP). *In addition, if $\mu(r) \leq Cr^\alpha$ for some $C > 0$ and $\alpha \in (0,1)$, then this solution lies in $C^{0,\alpha}(\mathbf{R}^N)$.*

Example IV.1. If $H(x,p) = -\theta(x,p) - |x|$, $x, p \in \mathbf{R}^N$. The above result shows that if $\theta \in (0,1)$ there exists a viscosity solution u (which belongs to $C^{0,1}(\mathbf{R}^N)$). In view of the results of the preceding sections, u is unique. It is given by

$$u(x) = \frac{1}{1-\theta}|x|.$$

Now, for $\theta = 1$, if there exists a viscosity solution u of (SP), then using the relations between linear equations in viscosity form and integral equalities proved in M.G. Crandall and P.L. Lions [4], we deduce that for all $x \in \mathbf{R}$, $t \geq 0$,

$$u(x) = u(e^t x)e^{-t} + t|x|. \tag{34}$$

Thus, if u were a uniformly continuous viscosity solution of (SP) for $\theta = 1$, (34) would yield

$$t|x| \leq C|x| + C + \{C|e^t x| + C\}e^{-t} \leq C|x| + C,$$

for all $t > 0$, which is impossible. Therefore for $\theta = 1$ there does not exist a solution of (SP) in $\mathrm{UC}(\mathbf{R}^N)$.

Example IV.2. Take $H(x, p) = -x\sqrt{|p|}$ for $x, p \in \mathbf{R}$. Then $u \equiv 0$ is a solution of (SP) and

$$\tilde{u}(x) = \frac{x}{1 + |x|}$$

is also a Lipschitz continuous viscosity solution of (SP). Observe that H satisfies (32). Observe also that H satisfies (33) for all $\theta \in (0,1)$ with $\mu(r) = C_\theta r$. Thus we do not have uniqueness in $\mathrm{BUC}(\mathbf{R}^N)$ or even in $W^{1,\infty}(\mathbf{R}^N)$.

Proof of Theorem IV.1. Step 1. We treat the case when (32) holds. Let $M = \sup_{\mathbf{R}^N}|H(x,0)|$. Truncating H by $\pm M$, we may assume without loss of generality that H is bounded by M provided we prove the existence of a viscosity solution in $\mathrm{BUC}(\mathbf{R}^N)$ bounded by M. Then, let $\varphi_R \in D(\mathbf{R}^N)$ satisfy $\varphi_R \equiv 1$ on B_R, $0 \le \varphi_R \le 1$, and $|D\varphi_R| \le 1/R$. We consider $H_R(x, p) = \varphi_R(x)H(x, p)$. Clearly, $H_R \in \mathrm{BUC}(\mathbf{R}^N \times B_R)$, $|H_R| \le M$ and (35)

$$\begin{aligned} (35) \qquad & H_R(y, \lambda(x-y)) - H_R(x, \lambda(x-y)) \\ & \le m(\lambda|x-y|^2 + |x-y|) + (M/R)|x-y|. \end{aligned}$$

Hence, by the results of [8], there exists a unique viscosity solution u_R of

$$(36) \qquad H_R(x, Du_R) + u_R = 0 \quad \text{in } \mathbf{R}^N$$

and $|u_R| \le M$ in $\mathbf{R}^N$. Next, we go through the proof of M.G. Crandall and P.L. Lions [8] to estimate the modulus of continuity of u_R and we observe that since $|u_R(x) - u_R(y)| \le 2M$ for $|x-y| = 1$ and (35) holds, we obtain a uniform modulus of continuity on u_R. Therefore, u_R (or a subsequence) converges uniformly on bounded sets to a $u \in \mathrm{BUC}(\mathbf{R}^N)$ which is a viscosity solution of (SP) and which satisfies $|u| \le M$.

Step 2. We treat the case when (33) holds. For $R < \infty$ we introduce

$$H_R(x, p) = \varphi_R(x)\mathrm{Max}[\mathrm{Min}(H(x, p), R), -R],$$

where φ_R is defined as in Step 1. Then we have

$$\begin{aligned} (37) \qquad & H_R(y, \lambda(x-y)) - H_R(x, \lambda(x-y)) \\ & \le \theta\lambda|x-y|^2 + \mu(|x-y|) + |x-y|. \end{aligned}$$

Hence, there exists a viscosity solution $u_R \in \mathrm{BUC}(\mathbf{R}^N)$ of (36).

Next we go through the estimates on the modulus of continuity of u in [8] and we observe (using (37) and the subadditivity of μ) that we have

$$\begin{aligned}
&H_R\left(x, C\frac{x-y}{|x-y|}\right) - H_R\left(y, C\frac{x-y}{|x-y|}\right) + \varepsilon + C|x-y| \\
&\quad \geq -C\theta|x-y| + \varepsilon + C|x-y| - \mu(|x-y|) \\
&\quad \geq \varepsilon + C(1-\theta)|x-y| - \mu(|x-y|) \\
&\quad \geq 0
\end{aligned}$$

on $\mathbf{R}^N \times \mathbf{R}^N$ if C is large enough. Hence, for all $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$|u_R(x) - u_R(y)| \leq \varepsilon + C_\varepsilon|x-y|, \quad \forall (x, y) \in \mathbf{R}^N$$

and thus u_R has a modulus of continuity uniform in R.

Next, we consider a maximum point $\bar{x}$ of $u_R(x) - \frac{1}{2}|x|^2$. We have, using the uniform modulus,

$$|\bar{x}|^2 \leq u_R(\bar{x}) - u_R(0) \leq C(1 + |\bar{x}|)$$

for some C independent of R and thus $|\bar{x}| \leq C$ (where C will denote various constants). Since u_R is a viscosity solution of (36),

$$H_R(\bar{x}, \bar{x}) + u_R(\bar{x}) \leq 0$$

and we finally deduce

$$u_R(0) \leq u_R(\bar{x}) + C \leq -H_R(\bar{x}, \bar{x}) + C \leq C.$$

Similarly one proves that $u_R(0)$ is bounded from below independently of R.

Therefore, u_R (or a subsequence) converges uniformly on compact sets to some $y \in \mathrm{UC}(\mathbf{R}^N)$ which is a viscosity solution of (SP) by the standard stability properties of viscosity solutions.

Remark. In fact, the proof in Step 2 still works if we replace (33) by (U) provided that (38)

$$\varlimsup_{r \to 0+} m(r)r^{-1} < 1. \tag{38}$$

V. Existence of minimal solutions of the stationary problem

In this section we consider Hamiltonians $H(x, p) \in C(\mathbf{R}^N \times \mathbf{R}^N)$ which satisfy

(39) $H(x, p) - H(x, 0) \to +\infty$ as $|p| \to \infty$ uniformly for x bounded.

THEOREM V.1. *We assume* (39) *and that there exists a viscosity subsolution* $\underline{u} \in C(\mathbf{R}^N)$ *of* $(\mathrm{SP})_0$. *Then there exists a viscosity solution* $u \in C(\mathbf{R}^N)$ *of* $(\mathrm{SP})_0$ *satisfying* $u \geq \underline{u}$ *which is minimal in the sense that if* v *is another viscosity solution of* $(\mathrm{SP})_0$ *satisfying* $v \geq \underline{u}$ *then* $v \geq u$.

Remark. (i) In view of (39), any subsolution of $(\mathrm{SP})_0$ is locally Lipschitz on $\mathbf{R}^N$.

(ii) $(\mathrm{SP})_0$ may not have a viscosity subsolution as is shown by the following example: let $H(x, p) = (1/2)|p|^2 + |x|^2$ and assume that v is a viscosity subsolution, that is

$$v + \frac{1}{2}|Dv|^2 \leq -|x|^2$$

in the viscosity sense. Since we clearly must have $v \leq -|x|^2$ it follows that for large n we may choose points x_n of least modulus so that $v(x_n) = -n^2$ and these points satisfy $|x_n| \leq n$. It follows that v varies by at least $-(n+1)^2 + n^2 = -2n - 1$ on the part of the ray through the origin and x_{n+1} which joins x_{n+1} and the sphere $|x| = n$. Moreover, since $|x_n| \leq n$, for any $\delta > 0$ we have $|x_{n+1}| - |x_n| \leq 1 + \delta$ infinitely often and we assume that this is satisfied for the n's we deal with below. It follows that the least Lipschitz constant for u on any part of the annulus $A = \{x: |x_n| < |x| < |x_{n+1}|\}$ containing the ray mentioned above is at least $(2n+1)/(1+\delta)$. Therefore the superdifferential D^+u of u has values of at least this modulus on this annulus (because a bound on values of D^+ is a Lipschitz constant [4]). Let $y \in A$ and $p \in D^+u(y)$ satisfy

$$|p| \geq (2n+1)/(1+\delta).$$

Since u is a subsolution we conclude that

$$-(n+1)^2 + ((2n+1)/(1+\delta))^2 \leq u(y) + |p|^2 \leq -|y|^2 \leq 0$$

which is impossible if $\delta < 1$ (and δ is at our disposal) and n is large.

Using more refined arguments, we can give a sharper nonexistence analysis for a class of examples of this sort. Consider the problem

$$u + \frac{1}{m}|Du|^m = f(x)$$

where $f(x) \in C(\mathbf{R}^N)$, $m > 1$ and

$$\limsup_{|x| \to +\infty} f(x)|x|^{-m'} < -(m')^{-(1+m')}.$$

where $m' = m/(m-1)$. We claim there exists no viscosity (sub) solution $\underline{u}$ of this problem. Indeed, subtracting if necessary some large constant from $\underline{u}$ we may assume that

$$f(x) \in -\lambda|x|^{m'} \quad \text{on } \mathbf{R}^N \text{ with } \lambda > (m')^{-(1+m')}.$$

Then if there were to exist $\underline{u}$, a viscosity subsolution of $(\mathrm{SP})_0$, the equation would yield $\underline{u} \le f$ on $\mathbf{R}^N$. Thus if $f^n \in \mathrm{BUC}(\mathbf{R}^N)$, $f^n \le 0$, $f^n \equiv f$ on $\overline{B}_n$, $f^n \downarrow_n f$ on $\mathbf{R}^N$ and if u^n is the viscosity solution (in $\mathrm{BUC}(\mathbf{R}^N)$) of

$$\frac{1}{m}|Du^n|^m + u^n = f^n \quad \text{in } \mathbf{R}^N$$

then we know (cf. P.L. Lions [18]) that u^n is given by

$$u^n(x) = \inf\left\{\int_0^\infty \left\{f^n(X_t) + \frac{1}{m'}|\dot{X}_t|^{m'}\right\} e^{-t}dt \colon X_t \in C^1([0,\infty[,\mathbf{R}^N), x_0 = x\right\}.$$

Choosing $|x| = 1$, $X_t = xe^{t/m'}$ if $t \le t_n = m'\log n$, $x_t = x_{t_n}$ or $t \ge t_n$ we find

$$u^n(x) \le \int_0^{t_n}\left\{-\lambda e^t + \left(\frac{1}{m'}\right)^{1+m'} e^t\right\}e^{-t}\,dt = -t_n\left\{\lambda - \left(\frac{1}{m'}\right)^{1+m'}\right\}$$

and thus $u^n \downarrow -\infty$ for $|x| = 1$. On the other hand since $u^n \in \mathrm{BUC}$, $\underline{u} \le f$ and thus $\underline{u} \to -\infty$ as $|x| \to \infty$, we have $\underline{u} \le u^n$ for $|x|$ large and by the standard comparison results $\underline{u} \le u^n$ on $\mathbf{R}^N$. The contradiction shows that there is no subsolution.

Proof of Theorem V.1. We first consider the problem

$$(40) \qquad H(x, Du) + u = 0 \quad \text{in } B_R, \quad u = \underline{u} \quad \text{on } \partial B_R$$

for $R < \infty$. Clearly, **u** is a viscosity subsolution of (40) and since H satisfies (39), we may apply the existence result of P.L. Lions [19] (see also G. Barles [2]) to deduce the existence of a viscosity solution u_R of (40): in addition u_R is Lipschitz continuous, $u_R \ge \underline{u}$ on $\overline{B}_R$, and thus $u_R(x)$ increases with R. Then, for any fixed R_0 and $R \ge R_0$, we deduce from (39) and (40)

$$u_R \le -\inf\{H(x,p) \colon |x| \le R_0, p \in \mathbf{R}^N\} \quad \text{on } \overline{B}_{R_0}.$$

Therefore u_R is uniformly bounded in R on bounded sets of $\mathbf{R}^N$. Using once more (39), (40) we deduce that u_R is bounded in $W^{1,\infty}(B_{R_0})$ for any fixed $R_0 < \infty$. Therefore u_R converges to a limit u uniformly on bounded sets and u is a viscosity solution of (SP). Finally, if v is any viscosity supersolution of

$(\mathrm{SP})_0$ above $\underline{u}$, it is in particular a viscosity supersolution of (40) and $v \geq \underline{u}$ on ∂B_R. Therefore, $v \geq u_R$ on $\overline{B}_R$ and letting R go to $+\infty$, we deduce $v \geq u$ in $\mathbf{R}^N$; that is, u is the minimal solution above $\underline{u}$.

VI. The Cauchy problem without conditions at infinity

We now consider the Cauchy problem (CP). Our main result is the

THEOREM VI.1. *Let H satisfy* (5).

Uniqueness. Let H satisfy (LU) *and let* $u, v \in C(\mathbf{R}^N \times [0, T])$ *be, respectively, a viscosity subsolution of*

$$u_t + H(x, t, Du) = 0 \quad \text{in } \mathbf{R}^N \times (0, T] \tag{41}$$

and a viscosity supersolution of

$$v_t + H(x, t, D_x v) + f(x, t) = 0 \quad \text{in } \mathbf{R}^N(0, T]. \tag{41$'$}$$

Then for all $t \in [0, T]$ *we have*

$$\sup_{R^N} (u(\cdot, t) - v(\cdot, t))^+ \leq \sup_{R^N} (u(\cdot, 0) - v(\cdot, 0))^+ + \int_0^t \sup_{R^N} f(\cdot, s)^+ \, ds. \tag{42}$$

Existence. Let H satisfy (LU). *If* $\varphi \in C(\mathbf{R}^N)$, *then there exists a unique viscosity solution* $u \in C(\mathbf{R}^N \times [0, T])$ *of* (CP).

Remark. We could replace $1 + |x|$ in (5) by $\Phi(|x|)$ where Φ is continuous, positive, increasing and $\int^\infty ds/\Phi(s) = \infty$.

Sketch of proof. Since most of the proof of Theorem VI.1 is a straightforward adaptation of proofs presented in the first three sections, we will only prove the uniqueness part for $f = 0$. The main point is the following lemma:

LEMMA VI.1. *With the notations and the assumptions of part* (1) *of Theorem VI.*1, *if* $f \equiv 0$. *Then for all* $x \in \mathbf{R}^N$, $t \in [0, T]$ *and* $\varepsilon > 0$, *we have*

$$\begin{aligned}(u - v)^+ (x, t) \\ \leq \sup\{(u - v)^+ (y, 0) \colon |y| \leq (1 + |x|)e^{C_\varepsilon t} - 1\} + 2\varepsilon t\end{aligned} \tag{43}$$

where C_ε *is a constant large enough such that* $\gamma(r) \leq \varepsilon + C_\varepsilon r$.

Proof. To prove Lemma VI.1, we fix $x_0 \in \mathbf{R}^N$, $t_0 \in (0, T]$ and we are going to prove (43) with $x = x_0$, $t = t_0$. To this end, we set $R = \mathrm{Log}(1 + |x_0|) + C_\varepsilon t_0$ and we consider

$$w_\delta(x, t) = \exp\left\{\frac{1}{\delta}\left[\mathrm{Log}(1 + |x|) + C_\varepsilon t - R\right]\right\}$$

where $\delta > 0$. One checks easily that w_δ is a viscosity solution of

$$\frac{\partial w_\delta}{\partial t} - C_\varepsilon(1 + |x|)|Dw_\delta| = 0 \quad \text{in } \mathbf{R}^N \times]0, \infty[$$

and $w_\delta \geq 0$. Furthermore, if $\Delta_h = \{(x, t) \in \mathbf{R}^N \times (0, T),\ 1 + |x| < \exp(R + h - C_\varepsilon t)\}$ for $h \geq 0$, we observe that $w_\delta \to +\infty$ on $\partial\Delta_h \cap (\mathbf{R}^N \times [0, T))$ as $\delta \to 0+$. Next, using (5), we deduce from the usual comparison argument that for any fixed h and for all δ small enough

$$(u - v)(x, t) - 2\varepsilon t - w_\delta(x, t) \leq \sup\left\{(u - v)^+(y, 0) \colon |y| \leq e^{R+h} - 1\right\}$$

for all $(x, t) \in \Delta_h$. Then, remarking that $w_\delta(x, t) \to 0$ as $\delta \to 0+$ if $(x, t) \in \Delta_0$, we deduce that

$$(u - v)(x, t) \leq 2\varepsilon t + \sup\left\{(u - v)^+(y, 0) \colon |y| \leq e^R - 1\right\}, \quad \forall (x, t) \in \overline{\Delta}_0.$$

We conclude by observing that $(x_0, t_0) \in \overline{\Delta}_0$ and that $e^R - 1 = (1 + |x_0|)e^{C_\varepsilon t_0} - 1$.

VII. Remarks on the Cauchy Problem

We wish to present in this section the analogues of the results given in Sections III, IV. We begin with the results corresponding to Section IV.

THEOREM VII.1. *Let H satisfy* (U) *and let $\varphi \in \mathrm{UC}(\mathbf{R}^N)$. Then there exists a viscosity solution u of* (CP) *which is uniformly continuous on $\mathbf{R}^N$, uniformly for $t \in [0, T]$.*

Remarks. (i) If $H(x, t, 0)$ is bounded on $\mathbf{R}^N \times [0, T]$, then we may choose u so that it is also bounded.

(ii) If (U) is strengthened to

$$H(y, t, \lambda(x - y)) - H(x, t, \lambda(x - y)) \leq C\lambda|x - y|^2 + C|x - y|^\alpha$$

for some $C > 0$, $\alpha \in (0,1]$ and $\varphi \in C^\alpha(\mathbf{R}^N)$ then we may choose the solution u so that it is Hölder continuous in x of exponent α, uniformly in $t \in [0, T]$.

Proof. We follow the approximation procedure introduced in the proof of Theorem IV.1 in Section IV and it will suffice to explain how to obtain enough a priori estimates which depend only on m in (U). First of all, we claim that one may obtain an estimate on the modulus of continuity of $u(\cdot, t)$ uniform in t, which depends only on m. Indeed, following the proof in [8], we just have to exhibit convenient supersolutions of

$$w_t + H(x, t, D_x w) - H(y, t, -D_y w) = 0 \quad \text{on } \mathbf{R}^N \times (0, T[.$$

Given $\varepsilon > 0$ choose $\delta > 0$, $C_\varepsilon \geq 1$ and C_δ such that $m(\delta) \leq \varepsilon$,

$$|\varphi(x) - \varphi(y)| \leq \varepsilon + C_\varepsilon|x - y|,$$

and $m(r) \leq C_\delta r$ for $r \geq \delta$. Put $K = \max(2C_\delta, 1)$ and

$$w(x, y) = \varepsilon(1 + t) + C_\varepsilon|x - y|e^{Kt}.$$

One easily justifies the following computation in the viscosity sense:

$$\begin{aligned} w_t + H(x, t, D_x w) - H(y, t, -D_y w) &\geq \varepsilon + KC_\varepsilon|x - y|e^{Kt} \\ &\quad - m\big((1 + C_\varepsilon e^{Kt})|x - y|\big). \end{aligned}$$

Furthermore the right-hand side is nonnegative if $|x - y|(1 + C_\varepsilon e^{Kt}) \leq \delta$ since $\varepsilon \geq m(\delta)$ and it is nonnegative if $|x - y|(1 + C_\varepsilon e^{Kt}) \geq \delta$ since in this case

$$\begin{aligned} m\big((1 + C_\varepsilon e^{Kt})|x - y|\big) &\leq C_\delta\big(1 + C_\varepsilon e^{Kt}\big)|x - y| \leq 2C_\delta C_\varepsilon e^{Kt}|x - y| \\ &\leq KC_\varepsilon e^{Kt}|x - y|. \end{aligned}$$

We conclude that the (approximations of) u satisfy

$$|u(x, t) - u(y, t)| \leq \varepsilon(1 + T) + C_\varepsilon e^{Kt}|x - y|$$

where K and C_ε depend only on ε and m as in (U). This yields the modulus of continuity with respect to x uniform in t as claimed.

To obtain a modulus of continuity in t for x bounded, we may argue as follows: Fix $x \in B_R$ and $\varepsilon > 0$ and set

$$m_\varepsilon(t) = \max_{y \in \mathbf{R}^N}\left(u(y, t) - \frac{1}{2\varepsilon}|y - x|^2\right).$$

Since u is uniformly continuous in the space variables, m_ε is well defined and continuous on $[0, T]$. We next claim that there is a C (depending on R and ε)

such that

$$m'_\varepsilon \le C \quad \text{on } (0, T) \tag{44}$$

in the viscosity (and, by linearity, distribution—see [4]) sense. Thus

$$m_\varepsilon(t+s) \le m_\varepsilon(t) + Cs \quad \text{for } 0 \le t \le t+s \le T. \tag{45}$$

Indeed, if $\psi \in C^1((0, T))$ and $m_\varepsilon - \psi$ has a maximum at $\bar{t} \in (0, T)$, we choose $\bar{x}$ which maximizes

$$u(\cdot, t) - (1/2\varepsilon)|\cdot - x|^2$$

and then we have

$$\psi'(\bar{t}) + H\left(\bar{x}, \bar{t}, \frac{1}{\varepsilon}(x - \bar{x})\right) \le 0. \tag{46}$$

However, the uniform continuity in x already proved provides the estimate

$$\frac{1}{2\varepsilon}|\bar{x} - x|^2 \le u(\bar{x}, t) - u(x, t) \le C(1 + |\bar{x} - x|)$$

so $|\bar{x} - x| \le C\varepsilon^{1/2}$ with (another) constant C independent of t, x, R, ε. Since H is bounded on bounded sets, (46) implies (44). In particular, choosing $\varepsilon = 1$ we deduce that

$$u(x, t) \le C_R + m_1(0) \le C_R + \varphi(x)$$

for $t \in [0, T]$ and $x \in B_R$, where C_R denotes various constants depending on R. A lower bound is obtained in the same way.

Next, we see that for $x \in B_R$ and $0 \le t \le t + s \le T$ and χ a modulus in x uniform in t

$$\begin{aligned} u(x, t+s) &\le C_{\varepsilon, R}s + m_\varepsilon(t) \\ &\le C_{\varepsilon, R}s + u(x, t) + \sup_{\lambda \ge u}\left(\chi(\lambda) - \lambda^2/2\varepsilon\right) \end{aligned} \tag{47}$$

where $C_{\varepsilon, R}$ denotes various constants depending on ε and R. Choosing ε and then s small shows that $(u(x, t+s) - u(x, t))^+$ is small for $s \ge 0$ small. A lower bound is obtainable in a similar way, establishing the continuity in t.

Our next and final result concerns the analogue of the results of Section III. We will assume that for some $m > 1$ H satisfies

$$|H(x, t, p) - H(x, t, q)| \le C_0(1 + |p|^{m-1} + |q|^{m-1})|p - q| \tag{48}$$

for all $x, p, q \in \mathbf{R}^N$, $t \in [0, T]$.

We will also consider viscosity sub and supersolutions $u, v \in C(\mathbf{R}^N \times [0, T])$ of the equation

$$u_t + H(x, t, Du) = 0$$

which are locally Lipschitz continuous on $\mathbf{R}^N \times (0, T]$ and satisfy

$$(49) \qquad |Du| \leq C(1 + |x|)^{m'-1} t^{-\mu}, \quad |Dv| \leq C(1 + |x|)^{m'-1} t^{-\mu},$$

for some $0 \leq \mu < m' - 1$.

THEOREM VII.2. *Let H satisfy* (LU) *and* (48). *Let u, v satisfy the preceding conditions. Then we have*

$$\sup_{\mathbf{R}^N \times [0, T]} (u - v)^+ = \sup_{\mathbf{R}^N} (u(\cdot, 0) - v(\cdot, 0))^+.$$

Remark. If $H(x, t, p) = |p|^m/m$ and if $\varphi \in C(\mathbf{R}^N)$ satisfies

$$(50) \qquad \lim_{|x| \to \infty} \varphi(x)|x|^{-m'} = 0,$$

then, using the Lax-Oleinik formula (see P.L. Lions [18]), we see that

$$u(x, t) = \inf_{y \in \mathbf{R}^N} \left\{ \varphi(y) + \frac{1}{m'}|x - y|^{m'} t^{1-m'} \right\}$$

defines a viscosity solution of (CP) in $C(\mathbf{R}^N \times [0, T])$ for all $T < \infty$. Furthermore u is locally Lipschitz in $\mathbf{R}^N \times (0, \infty)$ and

$$|Du(x, t)| \leq C_T(1 + |x|)^{m'-1} t^{-1/m}, \quad \forall T < \infty \text{ a.e. in } \mathbf{R}^N \times (0, T)$$

for all $T < \infty$. Similar existence results for more general Hamiltonians may be obtained using the regularizing effects proved in P.L. Lions [21].

Sketch of proof. Since we have already sketched many similar proofs, we just outline the arguments. We first observe that combining (48) and (49) we get

$$|H(x, t, p) - H(x, t, q)| \leq C(1 + |x|) t^{-\theta} |p - q|$$

for $p \in D_x^+ u(x, t)$, $q \in D_y^- v(y, t)$ and $|x - y| \leq 1$ where $0 < \theta < 1$. Next, we consider for $\delta > 0$, $R > 0$,

$$w_\delta(x, t) = \exp\left\{ \frac{1}{\delta} \left(\operatorname{Log}\left(1 + \left(|x|^2 + 1\right)^{1/2}\right) + \frac{C}{2(1 - \theta)} t^{1-\theta} - R \right) \right\}$$

and we check that w_δ is a viscosity solution of

$$2\frac{\partial w_\delta}{\partial t} - C(1 + |x|)t^{-\theta}|Dw_\delta| \geq 0 \quad \text{in } \mathbf{R}^N \times (0, \infty).$$

Let Q_R be the region in which $\delta \operatorname{Log}(w_\delta) \leq 1$. Studying maxima of functions of the form

$$u(x, t) - v(y, t) - \frac{|x - y|^2}{\varepsilon} - w_\delta(x, t) - at$$

over the set $(x, t), (y, t) \in Q_R$, we find in the usual way that there is a $\delta(R)$ with $\delta(0+) = 0$ such that

$$u(x, t) - v(x, t) \leq \sup_{\mathbf{R}^n} (u(\cdot, 0) - v(\cdot, 0))^+ + w_{\delta(R)}(x, t)$$

for $(x, t) \in Q_R$. Let $R \to \infty$ to complete the proof.

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