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FINITE ABELIAN GROUPS THAT CAN ACT FREELY ON $(S^{2n})^k$

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1. Introduction

In this article we shall prove the following:

MAIN THEOREM. *Suppose that G is a finite abelian group and X is a finite CW-complex, the homotopy type of $(S^{2n})^k$. Assume further that G acts freely and cellularly on X. Then $G \cong \prod_{j=i}^{m} Z_2\ell_j$ and $k \geq \sum_{j=i}^{m} 2^{\ell_j - 1}$.*

This theorem generalizes the main theorem of [3].

Theorems of this type (restrictions on finite groups that can act freely on products of spheres) have been investigated by Conner [2] and Carlsson [1]. Their results are concerned primarily with elementary abelian p-group actions on products of spheres and use the Serre spectral sequence for the fibration $X \to X/G \to BG$ as a tool. The methods of this paper are completely different. We exploit the fact that for even dimensional spheres the Euler characteristic, $\chi(X)$, is not zero. Using this and the Lefshetz fixed point theorem we can reduce the proof of the theorem to a problem in representation theory which we then solve.

The main theorem is sharp in the following sense. An action of $\mathbf{Z}_{2m}$ on $(S^{2n})^m$ is given by

$$T(x_1, \dots, x_m) = (x_2, \dots, x_m, -x_1).$$

This action is free if, and only if, m is a power of 2 [4]. In this way we may build up a free action of $\prod_{j=1}^{m} \mathbf{Z}_2\ell_j$ on $(S^{2n})^k$ if $k = \sum_{j=1}^{m} 2^{\ell_j - 1}$.

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2. Preliminaries

We assume throughout the rest of the paper that X is a finite complex with the homotopy type of $(S^{2n})^k$.

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PROPOSITION. *If G is a finite group acting freely and cellularly on X then G is a 2-group and there is a faithful representation of G in $GL(k, \mathbf{F})$ for any field $\mathbf{F}$ whose characteristic is not 2.*

The main theorem will follow from:

THEOREM. *If $G \cong \prod_{j=1}^{m} \mathbf{Z}_{2^{\ell_j}}$ admits a faithful representation in $GL(k, \mathbf{Q})$ then $k \geq \sum_{j=1}^{m} 2^{\ell_j - 1}$.*

We will prove this theorem in the next section.

Proof of the proposition. The Euler characteristic of X is $\chi(X) = 2^k$. The first conclusion follows from the equation $\chi(X/G) \cdot |G| = \chi(X)$.

Let g be an element of G. If the induced map

$$g^*\colon H^{2n}(X; \mathbf{F}) \to H^{2n}(X; \mathbf{F})$$

is the identity map then g^* is the identity on $H^*(X; \mathbf{F})$ and the Lefshetz number $L(g^*) = \chi(X) = 2^k$ is nonzero if the characteristic of $\mathbf{F}$ is not 2. So, by the Lefshetz fixed point theorem, the G-module $H^{2n}(X; \mathbf{F})$ is a faithful k-dimensional representation. Q.E.D.

We will use the following lemma in the proof of the above theorem:

LEMMA. *If G is a finite abelian 2-group, $G \cong \mathbf{Z}_{2^\ell} \times G_1$, and 2^ℓ is the maximal order among cyclic subgroups of G then the kernel of any epimorphism $G \to \mathbf{Z}_{2^\ell}$ is isomorphic to G_1.*

Proof. Let $p\colon G \to \mathbf{Z}_{2^\ell}$ be the epimorphism and let T denote a generator of $\mathbf{Z}_{2^\ell}$. Since 2^ℓ was the maximal order in G every element of $p^{-1}(T)$ has order 2^ℓ. Choose one such element $g \in p^{-1}(T)$. Define a splitting $s\colon \mathbf{Z}_{2^\ell} \to G$ by $s(T) = g$. This will give us a direct product decomposition $G \cong \mathbf{Z}_{2^\ell} \times K$, where $K = \ker(p)$. By the structure theorem for finite abelian groups we may conclude that $K \cong G_1$. Q.E.D.

3. Proof of the theorem

Assume G is a finite abelian 2-group and $\rho\colon G \to GL(k, \mathbf{Q})$ is a faithful representation. Let 2^ℓ be the maximal order among the cyclic subgroups of G. We may write $G \cong \mathbf{Z}_{2^\ell} \times G_1$. If we extend $\mathbf{Q}$ to $\mathbf{Q}(\zeta)$, where ζ is a primitive 2^ℓ-th root of unity, the representation space, V, splits as a direct sum of 1-dimensional representations:

$$\mathbf{Q}(\zeta) \otimes V \cong \bigoplus_{j=1}^{k} W_j.$$

Let C_l denote a cyclic subgroup of G, of order 2^ℓ. One of the W_j's must be a faithful C_l-module. We may suppose this is W_1. Since

$$\dim_{\mathbf{Q}} W_1 = \dim_{\mathbf{Q}} \mathbf{Q}(\zeta) = 2^{\ell-1},$$

we may regard W_1 as a $2^{\ell-1}$-dimensional rational G-representation given by a homomorphism

$$\rho_1 \colon G \to GL(2^{\ell-1}, \mathbf{Q}),$$

and the map $V \to W_1$ given by the composition

$$V \to \mathbf{Q}(\zeta) \otimes V \cong \bigoplus_{j=1}^{k} W_j \xrightarrow{\text{proj}} W_1$$

is an epimorphic G-map. Since ρ_1 is faithful on C_l, its image is isomorphic to $\mathbf{Z}_{2^\ell}$. Let K denote the kernel of ρ_1. By the above lemma we may conclude that $K \cong G_1$. Let $W_1^\perp$ be a G-complimentary subspace to W_1 in V. The action of K on W_1 is trivial, so K must act faithfully on $W_1^\perp$. Furthermore $\dim_{\mathbf{Q}} W_1^\perp = k - 2^{\ell-1}$. Now proceed inductively on the faithful representation $G_1 \cong K \to GL(k - 2^{\ell-1}, \mathbf{Q})$. Q.E.D.

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