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ARENS SEMI-REGULARITY OF THE ALGEBRA OF COMPACT OPERATORS

BY

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1. Introduction

Quite a number of important Banach algebras occurring in functional analysis or harmonic analysis are not Arens regular, i.e., the two canonical extensions of the multiplication given on A to the bidual A^{**} are not identical. As J. Pym puts it in [18]: "In practice, regularity appears to be the exception rather than the rule." For example, the group algebra $L^1(G)$ of any infinite locally compact Hausdorff group G or the algebra $K_0(X)$ of operators uniformly approximable by operators of finite rank, for a non-reflexive Banach space X, are not (Arens) regular [21], [22].

Among the Banach algebras possessing a two-sided bounded approximate identity, there is a subclass for which the Arens products still behave in a reasonable way although they need not to be identical. This class of so-called (Arens) semi-regular Banach algebras was introduced and described to some extent in [9]. As V. Losert and H. Rindler have shown, $L^1(G)$ is semi-regular if and only if G is discrete or abelian [15]. Further, $K_0(X)$ is semi-regular if X^*, the dual of X, possesses the Radon-Nikodym property [9]. (When discussing semi-regularity of $K_0(X)$, X^* always is assumed to possess the bounded approximation property so as to ensure that $K_0(X)$ contains a bounded approximate identity.)

These examples suggest that the notion of semi-regularity might be quite appropriate in the sense that it corresponds to important "classical" properties of the objects involved in applications.

It is the purpose of this paper to describe in some detail the relation between properties of X and X^* respectively (such as the Radon-Nikodym property) and semi-regularity of $K_0(X)$. Sections 1–5 contain the results valid for a general Banach space; in Section 6, the space $C(K)$ of continuous, complex-valued functions on a compact topological space K and spaces $L^1(\mu)$ of equivalence classes of integrable functions are considered.

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solution of the problem of semi-regularity of $K_0(C(K))$ and $K_0(L^1(\mu))$ respectively.

2. Notation and terminology

Basically, the notation is the same as in [9]. In this section we summarize the most important items.

Frequently, we neglect the canonical embedding ι_X (or simply ι) of a Banach space X into its bidual X^{**}; we rather consider X as a subset of X^{**}. For the definition of the *(bounded) approximation property* (a.p. and b.a.p. respectively) of a Banach space X, see [5, p. 76] for example; for the definition of the *Radon-Nikodym property* (RNp), see [6, p. 61]. $\mathscr{L}(X)$ denotes the algebra of bounded linear operators on X, equipped with the uniform norm. We write $\bigcirc X$ for the closed unit ball of the Banach space X.

If X is a Banach space and $f \in \mathscr{L}(X^{**})$ then we denote by $f^\flat$ (f flat—$\flat$ stands for lowering the order of duality) the operator in $\mathscr{L}(X^*)$ defined by

$$f^\flat := (\iota_X)^* \circ f^* \circ \iota_{X^*} = \iota^* \circ f^* \circ \iota.$$

$f \mapsto f^\flat$ is a projection of norm 1 from $\mathscr{L}(X^{**})$ onto the "subspace" $\mathscr{L}(X^*)$ (via $g \mapsto g^*$), i.e., $g^{*\flat} = g$ for $g \in \mathscr{L}(X^*)$. The kernel of $f \mapsto f^\flat$ consists exactly of those operators which vanish on ιX, i.e., those satisfying $f\iota = 0$:

$$\begin{aligned} f^\flat = 0 &\leftrightarrow \langle x, (\iota^* \circ f^* \circ \iota)x' \rangle = 0 \quad (\text{for all } x \in X,\ x' \in X^*) \\ &\leftrightarrow \langle (f \circ \iota)x, \iota x' \rangle = 0 \\ &\leftrightarrow \langle x', (f \circ \iota)x \rangle = 0 \\ &\leftrightarrow (f \circ \iota)x = 0 \quad (\text{for all } x \in X). \end{aligned}$$

Let X be a Banach space, $x \in X$, $x' \in X^*$. The tensor $x \otimes x'$ defines an element of $\mathscr{L}(X)$ by

$$(x \otimes x')(y) := \langle y, x' \rangle x \quad (y \in X).$$

The linear span of all operators of this form is denoted by $F(X)$. $K_0(X)$ is defined as the closure of $F(X)$ in $\mathscr{L}(X)$. For the definition *of the projective tensor product* $X \hat{\otimes} Y$ of two Banach spaces X and Y; see [5, p. 54] for example. The mapping from $X \otimes X^*$ into $\mathscr{L}(X)$ defined above induces a linear contraction from $X \hat{\otimes} X^*$ into $\mathscr{L}(X)$; its image, equipped with the quotient norm, is denoted by $N(X)$. Elements of $N(X)$ are called nuclear operators.

Let $f \in \mathscr{L}(X)$; f is called integral if there exists a constant $C > 0$ such that

$$\left| \sum_{i=1}^{n} \langle x_i, f(x_i') \rangle \right| \le C \left\| \sum_{i=1}^{n} x_i \otimes x_i' \right\|$$

for all $\sum_{i=1}^{n} x_i \otimes x_i' \in X \otimes X^*$, where $\|\cdot\|$ denotes the norm in $\mathscr{L}(X)$ of the corresponding operator. The infimum over all the possible constants C is called the *integral norm* of f. The space $I(X)$ of all integral operators is a Banach space when equipped with the integral norm (cf. [5]). Every nuclear operator is integral, its integral norm being dominated by its nuclear norm.

The dual of $K_0(X)$ can be identified with $I(X^*)$ by [5, II.2.9., lemma]. On the dense subspace $F(X)$ of $K_0(X)$, $f \in I(X^*)$ acts according to

$$\langle a, f \rangle := \operatorname{trace}(a^* \circ f) = \operatorname{trace}(f \circ a^*) \quad (a \in F(X))$$

Here, *trace* denotes the linear functional on $X^* \otimes X^{**}$ induced by the canonical bilinear form $(y', y'') \to \langle y', y'' \rangle$. If X^* has the a.p., the above definition makes sense even for $a \in K_0(X)$ since, in this case, trace is well-defined on $N(X^*)$ [5, II.3.4.]. $a^* \circ f$ and $f \circ a^*$ are elements of $N(X^*)$ for any Banach space by [6, VIII.4.12].

Like the dual of any Banach algebra A, $K_0(X)^* = I(X^*)$ becomes an A-A-bimodule by defining

$$\langle a, fb \rangle := \langle ba, f \rangle, \quad \langle a, bf \rangle := \langle ab, f \rangle \quad (a, b \in A,\ f \in A^*)$$

In the present case, the explicit form of the action of A on A^* can be computed by taking a from the dense subalgebra $F(X)$ of $K_0(X)$:

$$\langle a, fb \rangle = \langle ba, f \rangle = \operatorname{trace}(a^* \circ b^* \circ f) = \langle a, b^* \circ f \rangle,$$
$$\langle a, bf \rangle = \langle ab, f \rangle = \operatorname{trace}(f \circ b^* \circ a^*) = \langle a, f \circ b^* \rangle,$$

i.e.,

$$fb = b^* \circ f, \quad bf = f \circ b^*.$$

Let us point out that the above definitions and calculations are meaningful even for $b \in \mathscr{L}(X)$.

Let A be a Banach algebra. A *bounded left* (resp. *right*) *approximate identity* is a bounded net $(e_\lambda)_{\lambda \in \Lambda}$ in A such that $\|e_\lambda a - a\| \to 0$ (resp. $\|ae_\lambda - a\| \to 0$) for every $a \in A$. If $(e_\lambda)_\lambda$ is a bounded left and right approximate identity simultaneously then it is called a (*two-sided*) *bounded approximate identity* (b.a.i.).

If V is a left module over the Banach algebra A (i.e., there is given a continuous bilinear operation on $A \times V$ satisfying $a(bv) = (ab)v$, $a, b \in A$, $v \in V$) then a *right multiplier* of V is a continuous linear operator T: $A \to V$ satisfying $T(ab) = aT(b)$ for every $a, b \in A$. The space of right multipliers on V is denoted by $M_r(V)$. The space $M_l(W)$ of *left multipliers* on a right Banach module W over A is defined analogously. If V is an A-A-bimodule (i.e. a left and right A-module simultaneously, satisfying $(av)b = a(vb)$ for $a, b \in A$, $v \in V$) then the space $M(V)$ of *double multipliers* on V consists of all pairs (S,T), $S \in M_l(V)$, $T \in M_r(V)$, satisfying

$$aS(b) = T(a)b \quad (a, b \in A).$$

The projections of $M(V)$ to $M_l(V)$ (resp. $M_r(V)$) are denoted by π_1 (resp. π_2).

By the adjoints of the left and right actions of A on itself, A^* and A^{**} become A-A-bimodules. For the multiplier spaces of $V = A^{**}$, it is convenient to consider them as subspaces of $\mathscr{L}(A^*)$ and $\mathscr{L}(A^*) \times \mathscr{L}(A^*)$ respectively by passing from operators R: $A \to A^{**}$ to the respective transposed operators R^t: $A^* \to A^*$. The defining relations

$$S(ab) = S(a)b,\ T(ab) = aT(b),\ aS(b) = T(a)b \quad (a, b \in A)$$

then become

$$S^t(af) = aS^t(f),\ T^t(fa) = T^t(f)a,\ \langle b, S^t(fa)\rangle = \langle a, T^t(bf)\rangle$$
$$(a, b \in A,\ f \in A^*).$$

Let A be a Banach algebra. The first and second *Arens product* on its bidual, denoted by $F \cdot G$ and $F \times G$ respectively ($F, G \in A^{**}$), both extend the multiplication given on A and are defined as follows: Let $a, b \in A$, $f \in A^*$, $F, G \in A^{**}$ and (according to [1], [2]) let

$$\begin{aligned} \langle b, fa\rangle &:= \langle ab, f\rangle, & \langle b, af\rangle &:= \langle ba, f\rangle,\\ \langle a, Ff\rangle &:= \langle fa, F\rangle, & \langle a, fF\rangle &:= \langle af, F\rangle,\\ \langle f, F\cdot G\rangle &:= \langle Gf, F\rangle, & \langle f, F\times G\rangle &:= \langle fF, G\rangle. \end{aligned}$$

A is called (*Arens*) *regular* if $F \cdot G = F \times G$ for all $F, G \in A^{**}$. A *mixed unit* is an element E of A^{**} which is a right unit for the first and a left unit for the second Arens product, simultaneously. A^{**} has a mixed unit if and only if A has a b.a.i. (for example, see [7, 1.41].). If E is a particular mixed unit, the set of all mixed units is $E + \langle A^*A + AA^*\rangle^{\perp}$. If E is a mixed unit then ε: $S \mapsto S^{**}(E)$ and η: $T \mapsto T^{**}(E)$ are topological embeddings of $M_l(A)$ and $M_r(A)$ respectively into A^{**} (they are even algebra homomorphisms in the appropriate sense; cf. [7, 4.42]). Therefore (π_1 and π_2 also being topological embeddings in this case),

$$\varepsilon\pi_1\colon (S,T) \mapsto S^{**}(E) \quad \text{and} \quad \eta\pi_2\colon (S,T) \mapsto T^{**}(E)$$

both embed $M(A)$ into A^{**}. A is called *(Arens-) semi-regular* if $\varepsilon\pi_1 = \eta\pi_2$ for every possible choice of the mixed unit E (equivalently, if $F \cdot G = F \times G$ on certain subspaces of A^{**}; cf. [9]). All regular and all commutative Banach algebras are semi-regular [9].

3. The relation between $K_0(X)^{**}$ and $\mathscr{L}(X^{**})$

This section presents the technical aspects of the relation between $K_0(X)^{**}$ and $\mathscr{L}(X^{**})$. The material is more or less well-known, yet scattered through the literature (for example, see [9] or [16].) For the convenience of the reader and as a sound basis for further work we shall give a rather systematic account.

The study of the relation between $K_0(X)^{**}$ and $\mathscr{L}(X^{**})$ is motivated by the following fact:

If X is a reflexive Banach space possessing the a.p. (and hence the b.a.p. [6, VIII. 4.2]) then the bidual of $K_0(X)$ is isometrically isomorphic to $\mathscr{L}(X^{**}) = \mathscr{L}(X)$ (for example, see [5, V. 3.10]). $K_0(X)$ is regular in this case, both Arens products agreeing with composition of operators [22].

For the case of a general Banach space X, let π denote the canonical mapping from $X^* \hat{\otimes} X^{**}$ to $I(X^*)$ (actually, the image of π is $N(X^*)$). Then its adjoint π^* is a map

$$\pi^*\colon K_0(X)^{**} \to \mathscr{L}(X^{**}).$$

The properties of π^* as a bounded linear operator depend on properties of X as follows:

3.1. THEOREM. (i) *π^* has w^*-dense image iff X^* has the a.p.*

(ii) *π^* is onto iff X^* has the b.a.p.*

(iii) *π is a (topological) embedding iff every integral operator $f\colon X^* \to X^*$ is nuclear.*

(iv) *π^* is an isomorphism if and only if X^* has the b.a.p. and $I(X^*) = N(X^*)$.*

(i)–(iv) follow easily from (slight modifications of) II.3.4. and II.3.9. in [5].

We shall see below that $\mathscr{L}(X^{**})$ even is isomorphic to a complemented subspace of $K_0(X)^{**}$ if X^* has the b.a.p.

The following proposition describes the mappings occurring in the definition of the Arens products (cf. Section 2), specialized to $A = K_0(X)$.

3.2. PROPOSITION. *Let X be a Banach space, let $a \in K_0(X)$, $f \in I(X^*) \cong K_0(X)^*$, $F \in K_0(X)^{**}$. Then the following relations hold*:

(i) $fa = a^* \circ f$, $af = f \circ a^*$,

(ii) $Ff = \iota^* \circ f^{**} \circ (\pi^* F)^* \circ \iota = [(\pi^* F) \circ f^*]^\flat$,
$fF = \iota^* \circ (\pi^* F)^* \circ f^{**} \circ \iota = [f^* \circ (\pi^* F)]^\flat = (\pi^* F)^\flat \circ f$

(iii) $\pi^*(F \cdot G) = (\pi^* F) \circ (\pi^* G)$, $\pi^*(F \times G) = (\pi^* F)^{\flat *} \circ (\pi^* G)$

Substituting $f = \pi(x' \otimes x'')$ $(x' \in X^*,\ x'' \in X^{**})$ *into (i) and (ii) yields*

(i)′ $\pi(x' \otimes x'')a = \pi(a^* x' \otimes x'')$, $a\pi(x' \otimes x'') = \pi(x' \otimes a^{**} x'')$,

(ii)′ $F\pi(x' \otimes x'') = \pi(x' \otimes (\pi^* F)x'')$,
$\pi(x' \otimes x'')F = \pi((\pi^* F)^\flat x' \otimes x'')$

Proof. (i) This has been shown already in the previous section.

(i)′ An immediate consequence of (i).

(ii). Let $a = x \otimes x' \in K_0(X)$ $(x \in X,\ x' \in X^*)$, $f \in I(X^*)$, $F \in K_0(X)^{**}$. *Then*

$$\begin{aligned}
\langle a, Ff \rangle &= \langle fa, F \rangle = \langle a^* \circ f, F \rangle \\
&= \langle (x' \otimes \iota x) \circ f, F \rangle = \langle \pi(x' \otimes f^* \iota x), F \rangle \\
&= \langle x' \otimes f^* \iota x, \pi^* F \rangle = \langle x', (\pi^* F) f^* \iota x \rangle \\
&= \langle (\pi^* F) f^* \iota x, \iota x' \rangle = \langle x, \iota^* f^{**} (\pi^* F)^* \iota x' \rangle \\
&= \langle a, \iota^* f^{**} (\pi^* F)^* \iota \rangle; \\
\langle a, fF \rangle &= \langle af, F \rangle = \langle f \circ a^*, F \rangle \\
&= \langle f \circ (x' \otimes \iota x), F \rangle = \langle \pi(fx' \otimes \iota x), F \rangle \\
&= \langle fx' \otimes \iota x, \pi^* F \rangle = \langle fx', (\pi^* F) \iota x \rangle \\
&= \langle x', f^* (\pi^* F) \iota x \rangle = \langle f^* (\pi^* F) \iota x, \iota x' \rangle \\
&= \langle x, \iota^* (\pi^* F)^* f^{**} \iota x' \rangle = \langle a, \iota^* (\pi^* F)^* f^{**} \iota \rangle.
\end{aligned}$$

(ii)′

$$\begin{aligned}
F\pi(x' \otimes x'') &= \iota^* (\iota x' \otimes \iota x'') (\pi^* F)^* \iota \\
&= \pi((\iota^* \iota x') \otimes (\iota^* (\pi^* F)^{**} \iota x'')) \\
&= \pi(x' \otimes (\iota^* \iota (\pi^* F) x'')) \\
&= \pi(x' \otimes (\pi^* F) x''); \\
\pi(x' \otimes x'') F &= \iota^* (\pi^* F)^* (\iota x' \otimes \iota x'') \iota \\
&= \pi((\iota^* (\pi^* F)^* \iota x') \otimes \iota^* \iota x'') \\
&= \pi\big((\pi^* F)^\flat x' \otimes x''\big).
\end{aligned}$$

(iii) Let $x' \in X^*$, $x'' \in X^{**}$, $F, G \in K_0(X)^{**}$. Then

$$\begin{aligned}
\langle x' \otimes x'', \pi^*(F \cdot G)\rangle &= \langle \pi(x' \otimes x''), F \cdot G\rangle \\
&= \langle G\pi(x' \otimes x''), F\rangle \\
&= \langle \pi(x' \otimes (\pi^*G)x''), F\rangle \\
&= \langle x' \otimes (\pi^*G)x'', \pi^*F\rangle \\
&= \langle x', (\pi^*F)\circ(\pi^*G)x''\rangle \\
&= \langle x' \otimes x'', (\pi^*F)\circ(\pi^*G)\rangle; \\
\langle x' \otimes x'', \pi^*(F \times G)\rangle &= \langle \pi(x' \otimes x''), F \times G\rangle \\
&= \langle \pi(x' \otimes x'')F, G\rangle \\
&= \left\langle \pi\left((\pi^*F)^{\mathrm{b}}x' \otimes x''\right), G\right\rangle \\
&= \left\langle (\pi^*F)^{\mathrm{b}}x' \otimes x'', \pi^*G\right\rangle \\
&= \left\langle (\pi^*F)^{\mathrm{b}}x', (\pi^*G)x''\right\rangle \\
&= \left\langle x', (\pi^*F)^{\mathrm{b}*} \circ (\pi^*G)x''\right\rangle \\
&= \left\langle x' \otimes x'', (\pi^*F)^{\mathrm{b}*} \circ (\pi^*G)\right\rangle. \square
\end{aligned}$$

Recall that $(\pi^*F)^{\mathrm{b}*} = 0$ if and only if π^*F vanishes on X (resp. ιX), i.e., if and only if F vanishes on (the closed linear span of) AA^* ($A = K_0(X^*)$) or, equivalently, on $\pi(X^* \otimes \iota X)$.

In a more or less explicit form, π^* appears in the work of several authors (e.g., [1], [9], [3], [16]). In the sequel, we shall present two more aspects of π^*: Beside the fact that this map allows to relate $K_0(X)^{**}$ to $\mathscr{L}(X^{**})$, it is also crucial when considering extensions of the canonical representation of $A = K_0(X)$ on X to representations of A^{**} on X^{**} and, finally, π^* relates A^{**} to the multiplier spaces of A and A^{**}.

The basic idea of constructing representations of A^{**} on X^{**} is due to R. Arens [1, 2.8]. For a left Banach module X over the Banach algebra A, consider $a \in A$, $F \in A^{**}$, $x \in X$, $x' \in X^*$, $x'' \in X^{**}$ and define $x'a$,

$x'F \in X^*$; $xx', x''x' \in A^*$; $F \cdot x''$, $F \times x'' \in X^{**}$ by

$$\begin{aligned} \langle x, x'a\rangle &:= \langle ax, x'\rangle, & \langle a, xx'\rangle &:= \langle ax, x'\rangle, \\ \langle a, x''x'\rangle &:= \langle x'a, x''\rangle, & \langle x, x'F\rangle &:= \langle xx', F\rangle, \\ \langle x', F\cdot x''\rangle &:= \langle x''x', F\rangle, & \langle x', F\times x''\rangle &:= \langle x'F, x''\rangle. \end{aligned}$$

All these operations are bilinear contractions. $F \cdot x''$ (resp. $F \times x''$) yield representations of A^{**} (equipped with the first (resp. second) Arens product) on X^{**}, the so-called first (resp. second) Arens representation. If $A = K_0(X)$, π^* is but the first Arens representation. This follows from the easily verified relations

$$\begin{aligned} x'a &= a^*(x'), & xx' &= \pi(x' \otimes \iota x), \\ x''x' &= \pi(x' \otimes x''), & x'F &= (\pi^*F)^{\flat}x' \\ F\cdot x'' &= (\pi^*F)x'', & F\times x'' &= (\pi^*F)^{\flat *}x''. \end{aligned}$$

A third aspect of π^* concerns the relation between A^{**} and multiplier spaces of A and A^{**} respectively ($A = K_0(X)$). First of all, these multiplier spaces can be expressed in terms of $\mathscr{L}(X)$, $\mathscr{L}(X^*)$, and $\mathscr{L}(X^{**})$ respectively. By [7, 3.5, 3.18, 3.23], we have

$$M_l(A) \cong \mathscr{L}(X), \quad M_r(A) \cong \mathscr{L}(X^*), \quad M(A) \cong \mathscr{L}(X)$$

where the isomorphisms have the following form:

$$\begin{aligned} &\sigma\colon \mathscr{L}(X) \to M_l(A), \quad \sigma_b(a) = b \circ a, \quad (b \in \mathscr{L}(X)), \\ &\tau\colon \mathscr{L}(X^*) \to M_r(A), \quad \iota \circ \tau_g(a) = a^{**} \circ g^* \circ \iota, \quad (g \in \mathscr{L}(X^*)), \\ &\omega\colon \mathscr{L}(X) \to M(A), \quad \omega_b = (\sigma_b, \tau_{b^*}), \quad (b \in \mathscr{L}(X)), \end{aligned}$$

For the multiplier spaces of A^{**}, we have

$$M_l(A^{**}) \cong \mathscr{L}(X^*), \quad M_r(A^{**}) \cong \mathscr{L}(X^{**}), \quad M(A^{**}) \cong \mathscr{L}(X^{**})$$

by the isomorphisms

$$\begin{aligned} &\sigma\colon \mathscr{L}(X^*) \to M_l(A^{**}), \quad \sigma_g(f) = g \circ f \quad (g \in \mathscr{L}(X^*)) \\ &\tau\colon \mathscr{L}(X^{**}) \to M_r(A^{**}), \quad \tau_h(f) = (h \circ f^*)^{\flat} \quad (h \in \mathscr{L}(X^{**})) \\ &\omega\colon \mathscr{L}(X^{**}) \to M(A^{**}), \quad \omega_h = (\sigma_{h^{\flat}}, \tau_h) \quad (h \in \mathscr{L}(X^{**})). \end{aligned}$$

The results concerning $M_l(A^{**})$ and $M_r(A^{**})$ respectively are contained in [8]. The representation of $M(A^{**})$ by $\mathscr{L}(X^{**})$ seems to be new and is demonstrated below.

The notation is chosen in such a way that the following diagrams commute:

$$\begin{array}{ccc} \mathscr{L}(X) & \xrightarrow{\sigma} & M_l(A) \\ \downarrow & & \downarrow \\ \mathscr{L}(X^*) & \xrightarrow{\sigma} & M_l(A^{**}) \end{array},$$

$$\begin{array}{ccc} \mathscr{L}(X^*) & \xrightarrow{\tau} & M_r(A) \\ \downarrow & & \downarrow \\ \mathscr{L}(X^{**}) & \xrightarrow{\tau} & M_r(A^{**}) \end{array},$$

$$\begin{array}{ccc} \mathscr{L}(X) & \xrightarrow{\omega} & M(A) \\ \downarrow & & \downarrow \\ \mathscr{L}(X^{**}) & \xrightarrow{\omega} & M(A^{**}) \end{array}.$$

The vertical arrows indicate the mapping which assigns to every operator its first (resp. second) adjoint.

Next, we establish that ω maps $\mathscr{L}(X^{**})$ isomorphically on $M(A^{**})$.

3.3. PROPOSITION. *Let X be any Banach space; let $A = K_0(X)$. Then*

$$\omega\colon \mathscr{L}(X^{**}) \to M(A^{**}), \quad \omega_h := (\sigma_{h^\flat}, \tau_h) \quad (h \in \mathscr{L}(X^{**}))$$

defines an isometric isomorphism of $\mathscr{L}(X^{**})$ onto $M(A^{**})$.

Proof. Let $(S, T) \in M(A^{**})$; this means $S \in M_l(A^{**})$, $T \in M_r(A^{**})$ and

$$(*) \qquad \langle a_2, S(fa_1)\rangle = \langle a_1, T(a_2 f)\rangle \quad (a_1, a_2 \in A,\ f \in A^*).$$

S is of the form σ_g ($g \in \mathscr{L}(X^*)$), T is of the form τ_h ($h \in \mathscr{L}(X^{**})$). We have to show that $(*)$ is satisfied if and only if $g = h^\flat$. Since the operators $y \otimes y'$ ($y \in X$, $y' \in X^*$) span A, $(*)$ is equivalent to

$$\langle z \otimes z', S(f(y \otimes y'))\rangle = \langle y \otimes y', T((z \otimes z'))f)\rangle$$
$$(y, z \in X,\ y', z' \in X^*)$$

Now

$$\begin{aligned}
\langle z \otimes z', S(f(y \otimes y'))\rangle &= \langle z \otimes z', S((y \otimes y')^* \circ f)\rangle \\
&= \langle z \otimes z', S((y' \otimes \iota y) \circ f)\rangle \\
&= \langle z \otimes z', S(y' \otimes f^* \iota y)\rangle \\
&= \langle z \otimes z', \sigma_g(y' \otimes f^* \iota y)\rangle \\
&= \langle z \otimes z', g \circ (y' \otimes f^* \iota y)\rangle \\
&= \langle z \otimes z', (gy') \otimes (f^* \iota y)\rangle \\
&= \langle z, gy'\rangle\langle z', f^* \iota y\rangle \\
&= \langle z, gy'\rangle\langle fz', \iota y\rangle \\
&= \langle z, gy'\rangle\langle y, fz'\rangle \\
&= \langle gy', \iota z\rangle\langle y, fz'\rangle \\
&= \langle y', g^* \iota z\rangle\langle y, fz'\rangle; \\
\langle y \otimes y', T((z \otimes z')f\rangle &= \langle y \otimes y', T(f \circ (z \otimes z')^*)\rangle \\
&= \langle y \otimes y', T(f \circ (z \otimes \iota z))\rangle \\
&= \langle y \otimes y', T(fz' \otimes \iota z)\rangle \\
&= \langle y \otimes y', \tau_h(fz' \otimes \iota z)\rangle \\
&= \langle y \otimes y', (h \circ (fz' \otimes \iota z)^*)^\flat\rangle \\
&= \langle y \otimes y', (h \circ (\iota z \otimes \iota fz'))^\flat\rangle \\
&= \langle y \otimes y', (h \iota z \otimes \iota fz')^\flat\rangle \\
&= \langle y \otimes y', fz' \otimes h \iota z\rangle \\
&= \langle y, fz'\rangle\langle y', h \iota z\rangle.
\end{aligned}$$

Thus $(*)$ *is equivalent to* $g^*\iota = h\iota$. *However,* $(g^* - h)\iota = 0$ *if and only if* $(g^* - h)^\flat = 0$, *i.e.,* $g = h^\flat$. $\square$

Now we are in a position to describe (for the present case $A = K_0(X)$) the explicit form of the general canonical mappings

$$\lambda\colon A^{**} \to M_l(A^{**}), \quad \rho\colon A^{**} \to M_r(A^{**}), \quad \delta\colon A^{**} \to M(A^{**})$$

(cf. [7, Chapter 4], where λ and ρ are denoted by ψ and φ, respectively). If $F \in A^{**}$, λ_F, ρ_F, δ_F are defined by

$$\lambda_F(f) = fF, \quad \rho_F(f) = Ff, \quad \delta_F = (\lambda_F, \rho_F).$$

From this it is immediate by Proposition 3.2. that

$$\begin{aligned}
&\lambda_F(f) = fF = (\pi^*F)^\flat \circ f = \sigma_{(\pi^*F)^\flat}(f), \\
&\rho_F(f) = (\pi^*F \circ f^*)^\flat = \tau_{\pi^*F}(f), \\
&\delta_F(f) = (\lambda_F, \rho_F) = (\sigma_{(\pi^*F)^\flat}, \tau_{\pi^*F}) = \omega_{\pi^*F}, \quad (f \in A^*,\ F \in A^{**})
\end{aligned}$$

i.e., we have the commutative diagrams

$$\begin{tikzcd}
& M_l(A^{**}) \\
A^{**} \arrow[ur, "\lambda"] \arrow[dr, "\flat\circ\pi^*"'] & \\
& \mathscr{L}(X^*) \arrow[uu, "\sigma"']
\end{tikzcd},$$

$$\begin{tikzcd}
& M_r(A^{**}) \\
A^{**} \arrow[ur, "\rho"] \arrow[dr, "\pi^*"'] & \\
& \mathscr{L}(X^{**}) \arrow[uu, "\tau"']
\end{tikzcd},$$

$$\begin{tikzcd}
& M(A^{**}) \\
A^{**} \arrow[ur, "\delta"] \arrow[dr, "\pi^*"'] & \\
& \mathscr{L}(X^{**}) \arrow[uu, "\omega"']
\end{tikzcd}.$$

If we consider multiplier spaces of A instead of A^{**}, we have to restrict λ, ρ, δ to suitable subspaces of A^{**}. Let $A_\lambda^{**} := \{F \in A^{**} \mid FA \subseteq A\}$, $A_\rho^{**} := \{F \in A^{**} \mid AF \subseteq A\}$, $A_\delta^{**} := A_\lambda^{**} \cap A_\rho^{**}$. Then

$$\begin{aligned}
A_\lambda^{**} &= \lambda^{-1}(M_l(A)) = (\pi^*)^{-1}(\mathscr{L}(X)), \\
A_\rho^{**} &= \rho^{-1}(M_r(A)) = (\pi^*)^{-1}(\mathscr{L}(X^*)), \\
A_\delta^{**} &= \delta^{-1}(M(A)) = (\pi^*)^{-1}(\mathscr{L}(X)).
\end{aligned}$$

This yields the commutative diagrams

$$\begin{tikzcd} & M_l(A) \\ A^{**}_\lambda \arrow[ur, "\lambda"] \arrow[dr, "\pi^*"'] & \\ & \mathscr{L}(X) \arrow[uu, "\sigma"'] \end{tikzcd} ,$$

$$\begin{tikzcd} & M_r(A) \\ A^{**}_\rho \arrow[ur, "\rho"] \arrow[dr, "\pi^*"'] & \\ & \mathscr{L}(X^*) \arrow[uu, "\tau"'] \end{tikzcd} ,$$

$$\begin{tikzcd} & M(A) \\ A^{**}_\delta \arrow[ur, "\delta"] \arrow[dr, "\pi^*"'] & \\ & \mathscr{L}(X) \arrow[uu, "\omega"'] \end{tikzcd} .$$

3.4. Proposition. *Let $A = K_0(X)$, $E \in A^{**}$.*

(a) *E is a right unit with respect to the first Arens product if and only if $\pi^*E = \mathrm{id}_{X^{**}}$.*

(b) *E is a left unit with respect to the second Arens product if and only if $(\pi^*E)^\flat = \mathrm{id}_{X^*}$, i.e., if and only if $\pi^*E - \mathrm{id}_{X^{**}}$ vanishes on ιX.*

Proof. (a) By w^*-continuity, E is a right unit with respect to the first Arens product if and only if E acts as right identity on A (for example, see [7, 1.40]), i.e., $\rho_E = \mathrm{id}_A$. Applying the isomorphism τ^{-1} gives the desired result.

(b) Proved similarly using the isomorphism σ^{-1}. □

3.5. Corollary. *For $E \in K_0(X)^{**}$, the following conditions are equivalent*:

(i) *E is a mixed unit.*

(ii) *$\pi^*E = \mathrm{id}_{X^{**}}$.*

(iii) *$\delta_E = (\mathrm{id}_A, \mathrm{id}_A)$ $(A = K_0(X))$.*

(iv) *E extends the trace functional from $\pi(X^* \otimes X^{**})$ to $I(X^*)$.*

Proof. By the preceding proposition, (i) implies (ii). (ii) is equivalent to (iii) since ω: $\mathscr{L}(X^{**}) \to M(A^{**})$ is an isomorphism by Proposition 3.3 and $\omega \circ \pi^* = \delta$. Condition (iii) states that E acts as two-sided unit on A which—by [7, 1.41]—implies (i). Finally, (iv) is equivalent to (ii): The trace functional on $\pi(X^* \otimes X^{**})$ corresponds to $\mathrm{id}_{X^{**}} \in (X^* \hat{\otimes} X^{**})^*$ and π^* "restricts" E from $I(X^*)$ to $X^* \hat{\otimes} X^{**}$, the latter containing $X^* \otimes X^{**}$ as a dense subspace. □

4. Multiplier spaces as complemented subspaces of $K_0(X)^{**}$

Let $A = K_0(X)$. Then $\pi^*: A^{**} \to \mathscr{L}(X^{**})$ is onto if and only if A has a bounded approximate right identity (observe that $\tau \circ \pi^* = \rho$ and apply [7, 4.39 (B)]). This, in turn, is equivalent to X^* possessing the b.a.p. [8, Theorem 4]. It follows from the general theory of Banach modules that even more is true in this case: The multiplier spaces of $K_0(X)^{**}$ (resp. $K_0(X)$) are not only quotients of $K_0(X)^{**} = A^{**}$ but they are even isomorphic to complemented subspaces of A^{**} (resp. A_λ^{**}, A_ρ^{**}, A_δ^{**}): For, if X^* has the b.a.p. then X also does; consequently, A has a two-sided b.a.i. and 4.14. and 4.17. of [7] apply. In the present case, these theorems yield isomorphisms

$$K_0(X)^{**} \cong \mathscr{L}(X^{**}) \oplus N(X^*)^{\perp},$$

$$K_0(X)^{**} \cong \mathscr{L}(X^*) \oplus [\pi(X^* \otimes \iota X)]^{\perp},$$

$$[K_0(X)^{**}]_\lambda \cong \mathscr{L}(X) \oplus [\pi(X^* \otimes \iota X)]^{\perp},$$

$$[K_0(X)^{**}]_\rho \cong \mathscr{L}(X^*) \oplus N(X^*)^{\perp},$$

$$[K_0(X)^{**}]_\delta \cong \mathscr{L}(X) \oplus N(X^*)^{\perp}.$$

Observe that $N(X^*)^{\perp} = [\pi(X^* \otimes X^{**})]^{\perp}$.

The embeddings ε: $M_l(A^{**}) \to A^{**}$ and η: $M_r(A^{**}) \to A^{**}$ occurring in [7, 4.14 and 4.17], have the form

$$\varepsilon(S) = S^*(E) \quad (S \in M_l(A^{**})),$$

$$\eta(T) = T^*(E) \quad (T \in M_r(A^{**})),$$

where E is a mixed unit in A^{**}. They extend the respective mappings ε: $M_l(A) \to A^{**}$ and η: $M_r(A) \to A^{**}$ defined in Section 2. To be precise, ε and η depend on E; however, we shall omit subscripts such as ε_E and η_E whenever this does not lead to confusion.

The general relations

$$\lambda\varepsilon = \mathrm{id}_{M_l(A^{**})}, \quad \rho\eta = \mathrm{id}_{M_r(A^{**})}$$

([cf. [7, 4.14]) yield

$${}^b\pi^*\varepsilon\sigma = \mathrm{id}_{\mathscr{L}(X^*)}, \quad \pi^*\eta\tau = \mathrm{id}_{\mathscr{L}(X^{**})}.$$

In the present case $A = K_0(X)$, even more is true:

4.1. PROPOSITION. *Let X be a Banach space such that X^* has the b.a.p.; let $A = K_0(X)$. Then*
(i) $\pi^*\varepsilon\sigma_g = g^*$ $(g \in \mathscr{L}(X^*))$;
(ii) $\delta\varepsilon\pi_1\omega_h = \omega_{h^{b*}}$ $(h \in \mathscr{L}(X^{**}))$;
(iii) $\delta\eta\pi_2 = \mathrm{id}_{M(A^{**})}$.

Proof. (i) To prove this stronger version of ${}^b\pi^*\varepsilon\sigma = \mathrm{id}_{\mathscr{L}(X^*)}$, let $z' \in X^*$, $z'' \in X^{**}$, $g \in \mathscr{L}(X^*)$. Then

$$\begin{aligned}\langle z' \otimes z'', \pi^*\varepsilon\sigma(g)\rangle &= \langle \pi(z' \otimes z''), \varepsilon\sigma(g)\rangle \\ &= \langle \pi(z' \otimes z''), \sigma_g^*(E)\rangle \\ &= \langle \pi(gz' \otimes z''), E\rangle \\ &= \langle gz \otimes z'', \pi^*E\rangle \\ &= \langle gz', z''\rangle \\ &= \langle z', g^*z''\rangle \\ &= \langle z' \otimes z'', g^*\rangle.\end{aligned}$$

(ii) and (iii) The formulas preceding the proposition, together with (i), imply

$$\begin{aligned}\delta\varepsilon\pi_1\omega_h &= \omega\pi^*\varepsilon\sigma_{h^b} = \omega_{h^{b*}};\\ \delta\eta\pi_2\omega_h &= \omega\pi^*\eta\tau_h = \omega_h. \square\end{aligned}$$

Part (iii) of the preceding proposition means that the isomorphism

$$K_0(X)^{**} \cong \mathscr{L}(X^{**}) \oplus N(X^*)^{\perp}$$

can also be interpreted as

$$A^{**} \cong M(A^{**}) \oplus \langle A^*A + AA^*\rangle^{\perp} \quad (A = K_0(X))$$

where $\eta\pi_2$ is the embedding of $M(A^{**})$ into A^{**}. This need not be true for a general Banach algebra, as the transposed algebra of $K_0(X)$ shows: Here, $\varepsilon\pi_1$—being not injective—plays the former role of $\eta\pi_2$.

For the sake of completeness, we mention the following fact: Restricting ε, η, π_1, π_2 to the respective multiplier spaces of A yields

$$\lambda\varepsilon = \mathrm{id}_{M_r(A)}, \qquad \delta\varepsilon\pi_1 = \mathrm{id}_{M(A)},$$

$$\rho\eta = \mathrm{id}_{M_l(A)}, \qquad \delta\eta\pi_2 = \mathrm{id}_{M(A)},$$

valid for *any* Banach algebra A with b.a.i. by [7, 4.17] and [9, Lemma 3] respectively.

5. Semi-regularity of $K_0(X)$

It is not true in general that $K_0(X)$ is semi-regular. A first counterexample has been constructed in [9] by taking $X = Y \oplus Y$ where Y is the space of continuous functions on an infinite compact Hausdorff group G (or $Y = L^1(G)$). It is natural to ask for which Banach spaces X the algebra $K_0(X)$ is semi-regular.

We begin by stating a condition equivalent to semi-regularity resp. the property that $S^{**}(E) = T^{**}(E)$ for a particular $(S, T) \in M(K_0(X))$ and a particular mixed unit E.

5.1. LEMMA. *Let* $(S, T) := \omega_b = (\sigma_b, \tau_{b^*}) \in M(A)$ $(A = K_0(X),\ b \in \mathscr{L}(X))$; *let E be a mixed unit in* $K_0(X)^{**}$. *Then*

$$S^{**}(E) = T^{**}(E)$$

if and only if

$$\langle b^* \circ f - f \circ b^*, E \rangle = 0$$

for every $f \in I(X^*) \cong K_0(X)^*$.

Proof. Clearly, $S^{**}(E) - T^{**}(E) = 0$ iff $\langle f, S^{**}(E) - T^{**}(E) \rangle = 0$ for all $f \in I(X^*)$. Moreover,

$$\begin{aligned} \langle f, S^{**}(E) - T^{**}(E) \rangle &= \langle \sigma_b^*(f) - (\tau_{b^*})^*(f), E \rangle \\ &= \langle \sigma_{b^*}(f) - \tau_{b^{**}}(f), E \rangle \\ &= \langle b^* \circ f - f \circ b^*, E \rangle \end{aligned}$$

by the considerations preceding Theorem 3.3. □

Lemma 5.1, together with Corollary 3.5., immediately yields:

5.2. THEOREM. *Let X be a Banach space such that X^* has the b.a.p. Then $K_0(X)$ is semi-regular if and only if every extension E of the trace functional from* $\pi(X^* \otimes X^{**})$ *to* $I(X^*)$ *satisfies*

$$(*) \qquad \langle b^* \circ f, E \rangle = \langle f \circ b^*, E \rangle$$

for all $f \in I(X^*)$, $b \in \mathscr{L}(X)$.

Condition $(*)$ in the preceding theorem is satisfied for any Banach space if $b \in K_0(X)$ or $f \in N(X^*)$; it is even true for any $f \in I(X^*)$ if b is weakly compact, by [11].

The following result also is an application of Lemma 5.1.

5.3. THEOREM. *Let X be a Banach space such that X^* possesses the b.a.p. and $K_0(X)$ is semi-regular. If Y is a complemented subspace of X then Y^* also has the b.a.p. and $K_0(Y)$ is semi-regular.*

Proof. That Y^* has the b.a.p. is immediate. Let X be isomorphic to $Y \oplus Y_1$. Since X^* is isomorphic to $Y^* \times Y_1^*$, $I(Y^*)$ can be considered as a complemented subspace of $I(X^*)$. Denote the embedding and projection by ι_0 and π_0 respectively. Let E be a mixed unit in $K_0(Y)^{**}$ and E_1 a mixed unit in $K_0(X)^{**}$. We claim that $E_2 = E_1 - \pi_0^*\iota_0^*(E_1) + \pi_0^*(E)$ is also a mixed unit and extends E. To this end, let $f \in I(X^*)$ and $b \in K_0(X)$. Then fb $(= b^* \circ f)$ and $\pi_0(fb)$ are nuclear. Therefore, by Corollary 3.5.,

$$\begin{aligned}\left\langle fb, \pi_0^*\iota_0^*(E_1) - \pi_0^*(E)\right\rangle &= \left\langle \iota_0\pi_0(fb), E_1\right\rangle + \left\langle \pi_0(fb), E\right\rangle \\ &= \operatorname{trace}\left(\iota_0\pi_0(fb)\right) - \operatorname{trace}\left(\pi_0(fb)\right) \\ &= 0,\end{aligned}$$

since ι_0 assigns to $g \in N(Y^*)$ the operator $\left(\begin{smallmatrix} g & 0 \\ 0 & 0\end{smallmatrix}\right)$. A similar computation for bf instead of fb establishes that E_2 is a mixed unit (cf. Section 2). Moreover, $\iota_0^*(E_2) = E$, $\iota_0^*\pi_0^*$ being the identity map on $K_0(Y)^{**}$.

Now let $f \in I(Y^*)$, $b \in \mathscr{L}(Y)$. For the operators

$$\tilde{f} = \iota_0 f = \begin{pmatrix} f & 0 \\ 0 & 0\end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b & 0 \\ 0 & 0\end{pmatrix}$$

we have $\langle \tilde{b}^* \circ \tilde{f} - \tilde{f} \circ \tilde{b}^*, E_2\rangle = 0$. This implies

$$\begin{aligned}\langle b^* \circ f - f \circ b^*, E\rangle &= \left\langle b^* \circ f - f \circ b^*, \iota_0^*(E_2)\right\rangle \\ &= \left\langle \iota_0(b^* \circ f - f \circ b^*), E_2\right\rangle \\ &= \langle \tilde{b}^* \circ \tilde{f} - \tilde{f} \circ \tilde{b}^*, E_2\rangle \\ &= 0,\end{aligned}$$

which shows $K_0(Y)$ to be semi-regular by Lemma 5.1. □

From now on, we make the general assumption that X^* has the b.a.p. whenever semi-regularity of $K_0(X)$ is discussed. In this case, $K_0(X)$ equals the algebra of compact operators on X [5, p. 82].

Concerning the relation between the Radon-Nikodym property and semi-regularity of $K_0(X)$, the following implications are known:

5.4. PROPOSITION. (i) *If X^* has the RNp then $I(X^*) = N(X^*)$.* [6, VI.4.8. and VIII.2.10].

(ii) *If $I(X^*) = N(X^*)$ then $K_0(X)$ is semi-regular, provided that X^* has the b.a.p.* [9, p. 49ff.]

More precisely, assuming $I(X^*) = N(X^*)$, we have the following situation: π^*, being an isomorphism in this case, allows to consider the Arens products to be defined also on $\mathscr{L}(X^{**})$. Proposition 3.2 (iii) now reads:

$$h_1 \cdot h_2 = h_1 \circ h_2, \quad h_1 \times h_2 = h_1^{\flat *} \circ h_2 \quad (h_1, h_2 \in \mathscr{L}(X^{**})).$$

Therefore,

$$F \times G = F^{\flat *} \cdot G \quad \text{where } F^{\flat *} := (\pi^*)^{-1}(\pi^* F)^{\flat *} \quad (F, G \in K_0(X)^{**}).$$

$E = (\pi^*)^{-1}(\mathrm{id}_{X^{**}})$ is determined uniquely as mixed unit and satisfies

$$E \cdot F = F = F \cdot E; \; E \times F = F, \; F \times E = F^{\flat *} \quad (F \in K_0(X)^{**}).$$

This provides a short proof of part (ii) of the above proposition: Let $(S, T) = \delta_F \in M(A)$, i.e. $F \in \delta^{-1}(M(A)) = (\pi^*)^{-1}(\mathscr{L}(X))$. Then

$$S^{**}(E) = F \times E = F^{\flat *} = F = F \cdot E = T^{**}(E).$$

The situation where π^* is an isomorphism has been studied in [9] and [16].

For a brief discussion of the respective converses of (i) and (ii), we anticipate some results which will be proved later on.

The converse of (i) is false: Take X to be the James Tree Space [14]. Then $X^{*(2k)}$ has the RNp for every $k = 1, 2, \ldots,$ while $X^{*(2k-1)}$ does not. Hence $I(X^{**}) = N(X^{**})$ which implies $I(X^*) = N(X^*)$. Since X^{**} has the a.p. and the RNp, X^{**} and, consequently, also X^* have the b.a.p. (cf. [16]).

Whether the converse of (ii) is valid is not known to the author. If X were a counterexample then X^* would have to have the b.a.p. (so that it makes sense to say that $K_0(X^{**})$ is semi-regular). $I(X^*) \neq N(X^*)$ would imply $I(X^{*(k)}) \neq N(X^{*(k)})$ for all $k = 1, 2, 3, \ldots$. Therefore, X^* and X^{**} (hence all duals $X^{*(k)}$) would have to fail the RNp. By 5.5., X would have to be non-Cartesian.

For the class of Cartesian Banach spaces, the converse of part (ii) of Proposition 4.3. is true. A Banach space X is said to be *Cartesian* if it contains a complemented subspace Y isomorphic (not necessarily isometrically) to $X \oplus X$ [17, 2.1.4].

5.5. THEOREM. *Let X be a Cartesian Banach space such that X^* has the b.a.p. Then $K_0(X)$ is semi-regular if and only if $I(X^*) = N(X^*)$.*

Proof. The sufficiency is (ii) in 4.3.; the necessity follows by the method of [9, p. 50]. □

Important examples of (non-reflexive) Cartesian spaces are $C[0,1]$, $L^1[0,1]$, c_0, l^1, l^∞; moreover, if X is Cartesian then $\mathscr{L}(X)$, $K_0(X)$, $N(X)$, $I(X)$

are also Cartesian, as well as the algebras of compact (resp. weakly compact) operators on X.

Let us now consider the question which of the three properties mentioned in Proposition 5.4 are inherited by X from its dual X^*.

The presence of the RNp in X and X^* is independent: c_0 and $l^\infty = c_0^{**}$ do not have the RNp, while $l^1 = c_0^*$ does [6, p. 218]; all the even duals of the James Tree space have the RNp while all the odd duals fail to have it [6, p. 214].

On the other hand, the property $I(Y) = N(Y)$ is inherited from X^* by X provided X^* has the a.p, or if X is a dual space: By [6, VIII.2.11. and VIII.3.7], $f \in I(X) \setminus N(X)$ implies $f^* \in I(X^*) \setminus N(X^*)$ if X has the a.p.. If $X = Z^*$, the result follows from

$$f = \iota_Z^* \circ f^{**} \circ \iota_{Z^*}$$

The following theorem shows that semi-regularity is also preserved when passing from X^* to X (if X^{**} has the b.a.p.).

5.6. THEOREM. *Let X^{**} have the b.a.p.. If $K_0(X^*)$ is semi-regular then so is $K_0(X)$.*

Proof. The proof is similar to that of Theorem 5.3. Let $A = K_0(X)$, $B = K_0(X^*)$. Let E be a mixed unit in A^{**}, E_1 a mixed unit in B^{**}. Let φ denote the map from A to B which assigns to every $a \in A$ its adjoint a^*. Then φ^*: $B^* \to A^*$ assigns to $f \in I(X^{**})$ the operator $f^\flat \in I(X^*)$:

$$\begin{aligned}
\langle y \otimes y', \varphi^*(f)\rangle &= \langle \varphi(y \otimes y'), f\rangle \\
&= \langle y' \otimes \iota y, f\rangle \\
&= \langle y', f\iota y\rangle \\
&= \langle f\iota y, \iota y'\rangle \\
&= \langle y, \iota^* f^* \iota y'\rangle \\
&= \langle y \otimes y', f^\flat\rangle \quad (y \in X,\ y' \in X^*,\ f \in I(X^{**}))
\end{aligned}$$

Let φ_1: $A^* \to B^*$ denote the map $f \mapsto f^*$. Then A^* is a (topological) direct summand in B^* with φ_1 as embedding and φ^* as projection. Consequently, A^{**} is a direct summand in B^{**} with φ^{**} as embedding and φ_1^* as projection. We claim that $E_2 = E_1 - \varphi^{**}\varphi_1^*(E_1) + \varphi^{**}(E)$ is also a mixed unit in B^{**}. Let $b \in B$, $f \in B^*$. It will suffice to show that $\varphi^{**}\varphi_1^*(E_1) - \varphi^{**}(E)$ annihilates fb and bf. We shall treat only the case of fb, the case of bf being similar.

Observe that $fb \in B^*B = N(X^{**})$. Therefore, $\varphi^*(fb) = (fb)^\flat$ and $\varphi_1\varphi^*(fb) = (fb)^{\flat *}$ are also nuclear. By Corollary 3.5, we have

$$\begin{aligned}\left\langle fb, \varphi^{**}\varphi_1^*(E_1) - \varphi^{**}(E)\right\rangle &= \left\langle \varphi^*(fb), \varphi_1^*(E_1) - E\right\rangle \\ &= \left\langle \varphi_1\varphi^*(fb), E_1\right\rangle - \left\langle \varphi^*(fb), E\right\rangle \\ &= \operatorname{trace}\left((fb)^{\flat *}\right) - \operatorname{trace}\left((fb)^\flat\right) \\ &= 0.\end{aligned}$$

Since B is semi-regular, for every $b_1 \in \mathscr{L}(X^*)$, $f_1 \in I(X^{**})$ we have

$$\langle b_1^* \circ f_1 - f_1 \circ b_1^*, E_2\rangle = 0$$

by Lemma 5.1. Now let $b_1 = b^*(b \in \mathscr{L}(X))$, $f_1 = f^*(f \in I(X^*))$. Then

$$\begin{aligned}0 &= \langle b^{**} \circ f^* - f^* \circ b^{**}, E_2\rangle \\ &= \left\langle \varphi_1(f \circ b^* - b^* \circ f), E_1 - \varphi^{**}\varphi_1^*(E_1) + \varphi^{**}(E)\right\rangle \\ &= \left\langle f \circ b^* - b^* \circ f, \varphi_1^*(E_1) - \varphi_1^*\varphi^{**}\varphi_1^*(E_1) + \varphi_1^*\varphi^{**}(E)\right\rangle \\ &= \langle f \circ b^* - b^* \circ f, E\rangle\end{aligned}$$

since $\varphi_1^*\varphi^{**}$ is the identity map on A^{**}. Again by Lemma 5.1, this establishes the result. □

For the study of semi-regularity of $K_0(X)$, the following terminology is useful:

5.7. DEFINITION. Let A be a Banach algebra possessing a b.a.i.. A mixed unit E in A^{**} is called *good* if $S^{**}(E) = T^{**}(E)$ for all $(S, T) \in M(A)$. E is called *bad* if it is not good.

By Lemma 5.1, $E \in K_0(X)^{**}$ is good if and only if $\langle b^* \circ f, E\rangle = \langle f \circ b^*, E\rangle$ for all $b \in \mathscr{L}(X)$, $f \in I(X^*)$. Obviously, $K_0(X)$ is semi-regular if and only if each mixed unit in its bidual is good.

V. Losert has constructed an example such that $K_0(X)^{**}$ has only bad mixed units. It will be presented in Section 6.

5.8. LEMMA. (i) *If $b^* \circ f - f \circ b^*$ is nuclear for every $b \in \mathscr{L}(X)$, $f \in I(X^*)$, then either all mixed units in $K_0(X)^{**}$ are good or all are bad.*

(ii) *If there exist $b \in \mathscr{L}(X)$, $f \in I(X^*)$ such that $b^* \circ f - f \circ b^*$ is not nuclear then $K_0(X)^*$ is not semi-regular, i.e., there exists a bad mixed unit in $K_0(X)^{**}$ (provided X^* has the b.a.p.).*

Proof. (i) If E were a good and E_1 a bad mixed unit then $G = E_1 - E$ would annihilate $A^*A + AA^*$ $(A = K_0(X))$ which equals $N(X^*)$ in the present case. However, there exist $b \in \mathscr{L}(X)$, $f \in I(X^*)$ such that

$$\langle b^* \circ f - f \circ b^*, G \rangle = \langle b^* \circ f - f \circ b^*, E_1 \rangle \neq 0.$$

This contradicts the assumption made in (i).

(ii) Choose $G \in N(X^*)^{\perp}$ as to satisfy $\langle b^* \circ f - f \circ b^*, G \rangle \neq 0$; then at least one of E and $E_1 = E + G$ has to be bad. □

Slight modifications of the proofs of Theorems 5.3 and 5.6. (let $E = \iota_0^*(E_1)$ and $E = \varphi_1^*(E_1)$ respectively, giving $E_2 = E_1$ in both cases) yield the following:

5.9. PROPOSITION. (i) *Let X be a Banach space such that X^* has the b.a.p. and let Y be a complemented subspace of X. If $K_0(X)^{**}$ contains a good mixed unit, then so does $K_0(Y)^{**}$.*

(ii) *Let X be a Banach space such that X^{**} has the b.a.p. If $K_0(X^*)^{**}$ contains a good mixed unit, then so does $K_0(X)^{**}$.*

6. Semi-regularity of $K_0(X)$ for $X = C(K)$ and $X = L^1(\mu)$

In this section, we give a complete answer to the question of semi-regularity of $K_0(X)$ if X is the space $C(K)$ of continuous complex-valued functions on a compact Hausdorff space K or if X is the space $L^1(\mu) = L^1(\Omega, \mathfrak{A}, \mu)$ of (equivalence classes of) integrable functions on a measure space $(\Omega, \mathfrak{A}, \mu)$: In the former case, $K_0(X)$ is semi-regular if and only if K contains no (non-empty) subset being dense-in-itself (i.e., if K is "scattered"); in the latter case, $K_0(X)$ is never semi-regular (provided $\dim X = \infty$, of course).

Let us consider the case $X = C(K)$. To prove the main result, we shall use the following three lemmas from which (at least) the first is well known.

6.1. LEMMA. *Let T denote an infinite completely regular topological space and $C(T)$ the Banach space of bounded, complex-valued, continuous functions on T, equipped with the supremum norm. Let T_1 be a closed subset of T such that $T \setminus T_1$ is infinite. Then $C(T)$ contains a subspace isometrically isomorphic to $c_0(\mathbf{N})$, consisting of functions equal to zero on T_1.*

Proof. If $U_0 := T \setminus T_1$ contains an infinite set $\{y_1, y_2, \dots\}$ such that all $\{y_i\}$ are open, define $W_i := \{y_i\}$. Otherwise, let $x_0 \in U_0$ such that $\{x_0\}$ is not open; let $y_0 \in U_0 \setminus \{x_0\}$. By induction, choose open neighbourhoods $U_1, U_2, \dots$. of x_0 and elements $y_1, y_2, \dots$ such that

$$\overline{U_i} \subseteq U_{i-1} \setminus \{y_{i-1}\}, \quad y_i \in U_i \setminus \{x_0\}.$$

Let $W_i := U_i \setminus \overline{U_{i+1}}$. In any case, we get non-empty, open, pairwise disjoint subsets W_i of $T \setminus T_1$ such that $y_i \in W_i$. There exist $f_1, f_2, \ldots \in C(T)$ satisfying $0 \leq f_i \leq 1$, $f_i(y_i) = 1$, $f_i \equiv 0$ on $T \setminus W_i$. For a complex sequence $\lambda = (\lambda_n)$ tending to zero, define $f_\lambda \colon T \to \mathbf{C}$ to be equal to $\lambda_i f_i$ on W_i and $f_\lambda \equiv 0$ outside of $\bigcup_i W_i$. It is routine to check that $f_\lambda \in C(T)$, $f_\lambda \equiv 0$ on T, and that $\lambda \mapsto f_\lambda$ is an isometric embedding of c_0 into $C(T)$. □

If $(\Omega, \mathfrak{A}, \mu)$ is a finite measure space and μ is atomless then $L^1(\mu)$ contains a complemented subspace Y isomorphic to $L^1[0,1]$: Construct a separable sub-σ-algebra $\mathfrak{A}_\infty$ of $\mathfrak{A}$ and consider the space Y of all $f \in L^1(\mu)$ which are measurable with respect to $\mathfrak{A}_\infty$ (cf. [12, 41.C]). For the sake of completeness, we include a proof of this (well-known) fact, using standard techniques from martingale theory.

Let

$$I_{n,k} := \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right), \quad n \in \mathbf{N},\ k = 0, 1, \ldots, 2^n - 1;$$

Let $\mathfrak{B}_n$ denote the σ-algebra generated by

$$\{I_{n,k} | k = 0, \ldots, 2^{n-1}\}.$$

$\bigcup_{n=1}^\infty \mathfrak{B}_n$ generates the σ-algebra of Borel subsets of $[0,1]$. For $f \in L^1[0,1]$, let $f_n = E(f|\mathfrak{B}_n)$ be the conditional expectation of f with respect to $\mathfrak{B}_n$, i.e. (denoting the characteristic function of a set A by c_A),

$$f_n := \sum_{k=0}^{2^{n-1}} c_{I_{n,k}} 2^n \left(\int_{I_{n,k}} f(t)\, dt \right).$$

A routine computation shows that $\|f_n\|_1 \leq \|f\|_1$ and $\|f - f_n\|_\infty \to 0$ for continuous f. This implies $\|f - f_n\|_1 \to 0$ for $f \in C[0,1]$ and, the latter being dense in $L^1[0,1]$, also for $f \in L^1[0,1]$.

In the following, we write $L^1(\mathfrak{A}, \mu)$ instead of $L^1(\mu)$ to emphasize that we consider (equivalence classes of) integrable functions measurable with respect to a particular σ-algebra $\mathfrak{A}$.

6.2. LEMMA. *Let $(\Omega, \mathfrak{A}, \mu)$ be a finite measure space with μ nonnegative and atomless. Then there exists a sub-σ-algebra $\mathfrak{A}_\infty$ of $\mathfrak{A}$ such that the complemented subspace $L^1(\mathfrak{A}_\infty, \mu)$ of $L^1(\mathfrak{A}, \mu)$ is isometrically isomorphic to $L^1[0,1]$ and $\bigcirc L^\infty(\mathfrak{A}_\infty, \mu)$ corresponds to $\mu(\Omega) \cdot \bigcirc L^\infty[0,1]$ under this isomorphism.*

Proof. By [12, 41(2)] it is possible to select sets

$$A_{n,k} \in \mathfrak{A} \quad (n \in \mathbf{N}, 0 \leq k \leq 2^n - 1)$$

satisfying

$$A_{n+1,2k} \cup A_{n+1,2k+1} = A_{n,k} \quad (A_{00} := \Omega),$$
$$A_{n+1,2k} \cap A_{n+1,2k+1} = \varnothing,$$
$$\mu(A_{n,k}) = 2^{-n} \cdot \mu(\Omega).$$

Let $\mathfrak{A}_n$ denote the σ-algebra generated by $\{A_n | 0 \leq k \leq 2^n - 1\}$ and $\mathfrak{A}_\infty$ the σ-algebra generated by $\bigcup_{n=1}^\infty \mathfrak{A}_n$. Then

$$f = \sum_k \alpha_k c_{I_{n,k}} \mapsto \frac{1}{\mu(\Omega)} \sum_k \alpha_k c_{A_{n,k}} \quad \left(f \in L^1([0,1], \mathfrak{B}_n)\right)$$

gives rise to a well-defined isometric map from

$$X := \bigcup_{n=1}^{\infty} L^1([0,1], \mathfrak{B}_n) \quad \text{onto} \quad \bigcup_{n=1}^{\infty} L^1(\mathfrak{A}_n, \mu).$$

Since X is dense in $L^1[0,1]$, this map can be extended to an isometric embedding j: $L^1[0,1] \to L^1(\Omega, \mathfrak{A}_\infty, \mu)$. Since each $g \in L^1(\mathfrak{A}_\infty, \mu)$ is the $\|\cdot\|_1$-limit of the martingale $(E(g|\mathfrak{A}_n), \mathfrak{A}_n)$ and $E(g|\mathfrak{A}_n) \in j(X)$ (for example, see [4, 60.4–6]), j is surjective. The assertion concerning $\bigcirc L^\infty$ is evident from the construction of j. □

6.3. LEMMA. *Let K denote a compact Hausdorff space, μ a nonnegative atomless Radon measure on K. Let F_1, F_2 be disjoint compact subsets of K, $\mu(F_1) > 0$ and F_2 infinite. Then there exist disjoint open neighbourhoods U_1 and U_2 of F_1 and F_2 respectively, and a bounded linear operator T: $C(K) \to C(K)$ with the following properties*:

(i) *If $f \in C(K)$ vanishes on F_1 then $T(f) = 0$.*
(ii) *$T(f)$ vanishes on $K \setminus U_2$ for any $f \in C(K)$.*
(iii) *T is integral, but not nuclear.*

Proof. We shall construct T as the composition of four operators. Since K is a normal space, there exist disjoint open neighbourhoods U_1 and U_2 of F_1 and F_2 respectively. Let μ_1 denote $\mu/\mu(F_1)$ restricted to F_1, i.e.,

$$\mu_1(E) := \mu(E \cap F_1)/\mu(F_1)$$

for every Borel subset E of K. μ_1 is atomless and concentrated on F_1. Let T_1: $C(K) \to L^1(\mu_1)$ be the canonical "embedding". T_1 is integral [6, VIII.2.9], $T_1(\bigcirc C(K))$ is dense in $\bigcirc L^\infty(\mu_1)$ with respect to the norm of $L^1(\mu_1)$.

$L^1(\mu_1)$ contains a complemented subspace isomorphic to $L^1[0,1]$ by Lemma 6.2. Let T_2: $L^1(\mu_1) \to L^1[0,1]$ denote the corresponding projection. Let T_3:

$L^1[0,1] \to c_0$ be the usual Fourier transform. Finally, by Lemma 6.1, there is an isometric embedding T_4 of c_0 into the subspace of $C(K)$ consisting of all $f \in C(K)$ vanishing on $K \setminus U_2$. The operator T: $C(K) \to C(K)$, defined by

$$T := T_4 \circ T_3 \circ T_2 \circ T_1$$

is integral since T_1 is integral. However, $T_2 \circ T_1$ maps $\bigcirc C(K)$ on a subset which is $\|\cdot\|_1$-dense in $\bigcirc L^\infty[0,1]$; the latter set contains all functions $t \mapsto \exp(2\pi ikt)$, $k \in \mathbf{Z}$. Therefore, the closure of the image of $\bigcirc C(K)$ under $T_3 \circ T_2 \circ T_1$, containing the set of all vectors $(0,\dots,0,1,0,\dots) \in c_0$, is not compact. This shows that T itself is not compact and, hence, not nuclear. To conclude the proof, assume that $f \in C(K)$ vanishes on F_1. Then f and the function $g \equiv 0$ differ only on $K \setminus F_1$ which has measure 0 with respect to μ_1. Consequently, $T_1(f) = 0$ in $L^1(\mu_1)$. □

Without conditions (i) and (ii), Lemma 6.3. also results from [6, VI.4.6].

6.4. THEOREM. *Let K be a compact Hausdorff space, $X = C(K)$. The following conditions are equivalent*:

(i) *K is scattered, i.e., K has no subset $A \neq \varnothing$ dense-in-tself.*
(ii) *Every Radon measure on K is atomic.*
(iii) *$X^* = M(K)$ has the RNp.*
(iv) *Every integral operator T: $X^* \to X^*$ is nuclear.*
(v) *Every integral operator T: $X \to X$ is nuclear.*
(vi) *$K_0(X)$ is semi-regular.*

Proof. (i) ↔ (ii). This is [20, 19.7.6].
(ii) → (iii) By [20, 19.7.7], X^* is isomorphic to $l^1(K)$, which has the RNp (for example, see [6, p. 64]).
(iii) → (iv) Proposition 5.4.
(iv) → (v) [6, VIII.2.11 and VIII.3.7].
(iv) → (vi) Proposition 5.4.
We conclude the proof by showing that for K not being scattered, neither (v) nor (vi) can be satisfied, i.e., (v) as well as (vi) imply (i).

If K is not scattered, there exists a (closed, maximal) dense-in-itself subset A of K carrying an atomless non-negative Borel measure μ [20, 8.5.2. and 19.7.6., 19.7.3]. By [12, 41(2)], there exist disjoint Borel subsets A_1, A_2 of A such that $\mu(A_1) = \mu(A_2) = \mu(A)/2$. Consequently, there exist disjoint compact subsets F_1, F_2 of A_1 and A_2 respectively such that $\mu(F_1) > 0$ and F_2 is infinite (with $\mu(F_2)$ positive as well). Invoking Lemma 6.3., we obtain open neighbourhoods U_1, U_2 of F_1 and F_2 respectively and a bounded linear operator T: $C(K) \to C(K)$ satisfying the properties stated in the lemma; the third property gives the implication (v) → (i). Now let $f \in C(K)$, $f \equiv 1$ on F_1 and $f \equiv 0$ outside of U_1. Denote by M: $C(K) \to C(K)$ the operation of multiplication by f. Then for any $g \in C(K)$, $g - M(g) \equiv 0$ on F_1. By property (i) in the lemma, $T(g) = (T \circ M)(g)$. On the other hand, $M \circ T \equiv 0$ by property (ii) in the lemma. Therefore, $T \circ M - M \circ T = T$ is not nuclear. Taking adjoints, Lemma 5.8(ii) shows that $K_0(X)$ is not semi-regular. Thus also (vi) implies (i). □

6.5. COROLLARY. *Let K be a locally compact (non-compact) Hausdorff space. Then theorem 6.4. is valid for X being the space $C_0(K)$ of all continuous complex-valued functions on K tending to zero at infinity.*

Proof. Let αK denote the one-point compactification of K. Obviously, K contains a nonempty dense-in-itself subset if and only if αK does. Moreover, $C_0(K)$ is a complemented subspace of $C(\alpha K)$ having codimension 1. Therefore, each of the conditions (i)–(v) in 6.4 with respect to αK and $C(\alpha K)$ is equivalent to the corresponding condition for K and $C_0(K)$ respectively. Finally, if $K_0(C(\alpha K))$ is semi-regular then so is $K_0(C_0(K))$ (by Theorem 5.3.). For the converse, we show that $K_0(C_0(K))$ is not semi-regular if K is not scattered: Starting as in the proof of (vi) $\to$ (i) in Theorem 6.4 with K replaced by αK, we eliminate ∞ from A if necessary and choose U_1, U_2 such that $\infty \notin U_1 \cup U_2$. Then it is obvious that the images of T and M are contained in $C_0(K)$. Thus we have $T_1 \circ M_1 - M_1 \circ T_1 = T_1$ for the restrictions T_1, M_1 of T and M respectively to $C_0(K)$. Since T_1 is integral but not nuclear, an application of Lemma 5.8 (ii) concludes the proof. □

Now we turn to $K_0(L^1(\mu))$. In the following lemma, we consider the special case of Lebesgue measure on $[0,1]$.

6.6. LEMMA. *There exists an integral operator*

$$T\colon L^1[0,1] \to L^1[0,1]$$

which is not nuclear.

Proof. Let T_1 be a quotient map from $L^1[0,1]$ onto $l^1(\mathbf{N})$, for example,

$$T_1(f) := \left(\int_{1/(n+1)}^{1/n} f(t)\,dt\right)_{n=1}^{\infty} \quad (f \in L^1[0,1]).$$

Let T_2 be a quotient map from $l^1(\mathbf{N})$ onto $C[0,1]$ (for example, see [5, I.1.11]). Let T_3 denote the canonical embedding of $C[0,1]$ into $L^1[0,1]$. Let $T := T_3 \circ T_2 \circ T_1$. T is integral since T_3 is [6, VIII.2.9]. However, the image of $\bigcirc L^1[0,1]$ under $T_2 \circ T_1$ is dense in $\bigcirc C[0,1]$. Therefore, the closure of $T(\bigcirc L^1[0,1])$, containing the functions $t \mapsto \exp(2\pi i k t)$, $k \in \mathbf{Z}$, is not compact. Consequently, T is not nuclear. □

The integration operator $(Tf)(x) := \int_0^x f(t)\,dt$ could as well serve to prove Lemma 6.6.

According to Theorem 5.5, $K_0(L^1[0,1])$ is not semi-regular.

Now let X be the space $L^1(\mu)$ of (equivalence classes of) integrable real-valued functions on a measure space $(\Omega, \mathfrak{A}, \mu)$ where Ω is a set, $\mathfrak{A}$ is a σ-field of subsets of Ω and μ is a nonnegative measure on $\mathfrak{A}$ (not necessarily finite). By a theorem of Kakutani [20, 26.3.3], X can be assumed to be the direct sum of a space $l^1(\Gamma)$ and of spaces $L^1(\mu_i)$ where Γ is a set and the μ_i's are nonnegative atomless Radon measures on respective compact spaces K_i (i assumes values in an index set I).

6.7. THEOREM. *Let X be a space $L^1(\mu)$ as described above having infinite dimension. Then $K_0(X)$ is not semi-regular.*

Proof. By the preceding remarks, X contains a complemented subspace Y isomorphic to $l^1(\mathbf{N})$ or isomorphic to $L^1(\mu)$ where μ is a nonnegative atomless Radon measure on a compact space K. By Theorem 5.3, it is sufficient to show that $K_0(Y)$ is not semi-regular.

Case 1. $Y = l^1(\mathbf{N})$. Y^* is isomorphic to $C(\beta\mathbf{N})$ where $\beta\mathbf{N}$ denotes the Stone-Čech compactification of $\mathbf{N}$. $\beta\mathbf{N} \setminus \mathbf{N}$ is dense-in-itself. Therefore, by Theorem 6.4, there exists an integral operator T: $Y^* \to Y^*$ which is not nuclear. Since $Y = l^1(\mathbf{N})$ is Cartesian, Theorem 5.5 shows that $K_0(Y)$ is not semi-regular.

Case 2. $Y = L^1(\mu)$, μ a non-negative atomless Radon measure on some compact space K. By Lemma 6.2, Y has a complemented subspace isomorphic to $L^1[0,1]$. By Lemma 6.6. and Theorem 5.5, $K_0(L^1[0,1])$ is not semi-regular. By Theorem 5.3, this is sufficient to show that $K_0(L^1(\mu))$ is not semi-regular. □

Now replace "real-valued" in the definition of $L^1(\mu) = L^1(\mu, \mathbf{R})$ by "complex-valued" to obtain $L^1(\mu, \mathbf{C})$. This space is isomorphic (though not isometrically) to the complexification $L^1(\mu) \oplus iL^1(\mu)$.

6.8. COROLLARY. *If $X = L^1(\mu, \mathbf{C})$ is of infinite dimension then $K_0(X)$ is not semi-regular.*

Proof. A review of the proofs involved on our way to Theorem 6.7 shows that there exist T, $M \in \mathscr{L}(L^1(\mu))$ such that T is integral and $T \circ M - M \circ T$ is not nuclear. This yields corresponding $\mathbf{C}$-linear operators T_1, $M_1 \in \mathscr{L}(L^1(\mu, \mathbf{C}))$ having the same properties. Again, Lemma 5.8 (ii) gives the desired result. □

Let us point out that for $X = l^1(\mathbf{N})$, $I(X) = N(X)$ (since $l^1(\mathbf{N})$ has the RNp; cf. [6, p. 64, VI.4.8 and VIII.2.10]), but $K_0(X)$ is not semi-regular—contrary to $X = C(K)$ where these properties are equivalent by

Theorem 6.4. For $X = L^1(\mu)$, μ not purely atomic, we always have $I(X) \neq N(X)$, and $K_0(X)$ is never semi-regular for such μ.

If G denotes an infinite compact Hausdorff group, Theorem 6.4 and 6.7 tell us that $K_0(C(G))$ and $K_0(L^1(G))$ are not semi-regular. However, it seems worth noting that the operators $T \in I(X)$ and $M \in \mathscr{L}(X)$ having the property that $T \circ M - M \circ T$ is not nuclear, can be chosen as convolution by an element of $L^\infty(G)$ and as multiplication by a continuous function on G, respectively. This is what the next (and last) theorem states.

For a locally compact Hausdorff group G with modular function Δ, we define left and right translation operators by

$$(L_x f)(y) := f(x^{-1}y), \quad (R_x f)(y) := f(yx^{-1})\Delta(x^{-1}) \quad (x, y) \in G,$$

where f is a function (resp. an equivalence class of measurable functions) on G. It is easy to check that $(R_x f) * g = f * (L_x g)$ for $f, g \in L^1(G)$ and $x \in G$.

6.9. Lemma. *Let G be a compact Hausdorff group. Then there exists a partition of unity $(\tau_1, \ldots, \tau_n)$ with the following property: for each pair (i, j) of indices, there is an $x \in G$ satisfying $\tau_i \cdot (L_x \tau_j) = 0$.*

Proof. Fix $z \in G$, $z \neq e$. Separate e and z by open neighbourhoods U_1, U_2 and let $U := U_1 \cap z^{-1}U_2$. By compactness, there are $x_1, \ldots, x_n \in G$ such that G is covered by the union of $x_1U, \ldots, x_nU$. Now let $(\tau_1, \ldots, \tau_n)$ be a partition of unity subordinate to $x_1U, \ldots, x_nU$ (for example, see [13, I.3.1]). Then $x := x_i z x_j^{-1}$ satisfies $\tau_i \cdot (L_x \tau_j) = 0$. □

6.10. Theorem. *Let G be an infinite compact Hausdorff group. Then for every $f \in L^\infty(G)$ which is not equal a.e. to any continuous function, there exist a continuous function τ: $G \to [0,1]$ and $x \in G$ such that the following holds:*

If S denotes left convolution by $R_x f$ and M denotes multiplication by τ, then S is integral and $M \circ S - S \circ M$ is not nuclear, on $L^1(G)$ as well as on $C(G)$.

By Lemma 5.8 (ii), neither $K_0(L^1(G))$ nor $K_0(C(G))$ nor $K_0(X)$ where X is a dual space of any order of $L^1(G)$ or $C(G)$, is semi-regular.

Proof. Let $(\tau_1, \ldots, \tau_n)$ be a partition of unity as in Lemma 6.9. Let $f \in L^\infty(G)$ such that there is no continuous function on G which equals f almost everywhere [10, lemma]. Then the convolution operator

$$S_f \colon \varphi \mapsto f * \varphi$$

is integral, but not nuclear on $L^1(G)$ (resp. $C(G)$) [19, Remark 3.9]. Let M_i denote the operator on X defined by multiplication by τ_i. Since $(\sum_i M_i) \circ S_f \circ (\sum_i M_i) = S_f$ is not nuclear, there are i, j such that $M_i \circ S_f \circ M_j$ is not nuclear. Now choose $x \in G$ satisfying $\tau_i(L_{x^{-1}}\tau_j) = 0$. Let M_0 denote multiplication by $L_{x^{-1}}\tau_j$ and let $S := S_{R_x f}$. Then $M_i \circ S \circ M_0$ (which is equal to $M_i \circ S_f \circ M_j \circ L_x$ by the remark preceding Lemma 6.9.) also fails to be nuclear. Let $\tau = \tau_i$, $M = M_i$. If $M \circ S - S \circ M$ were nuclear, then $(M \circ S - S \circ M) \circ M_0 = M_i \circ S \circ M_0 - S \circ M_i \circ M_0 = M_i \circ S \circ M_0$ would be as well which is a contradiction. □

The case of compact topological groups even shows that $K_0(C(K))$ can be very far from being semi-regular: There exist compact groups G such that all mixed units in $K_0(C(G))^{**}$ are bad. The following example has been pointed out to me by V. Losert.

Let $G_0 = \mathbf{T}^2 \times_s SL(2, \mathbf{Z})$ where $\mathbf{T}^2 = (\mathbf{R}/\mathbf{Z})^2$ is the two-dimensional torus and SL$(2, \mathbf{Z})$ acts by matrix multiplication modulo 1. For $f \in L^\infty(G_0)$, let f_I denote the restriction of f to $\mathbf{T}^2 \times \{I\} \cong \mathbf{T}^2$ where I is the identity matrix. Clearly, $f_I \in L^\infty(\mathbf{T}^2)$. If the equivalence class of f_I does not contain a continuous function then S_{f_I}: $\varphi \to f_I * \varphi$ is an integral operator on $C(\mathbf{T}^2)$ which is not nuclear by [19, 3.7]. To show that each mixed unit in $K_0(C(\mathbf{T}^2))^{**}$ is bad we consider the following bounded linear operator on $C(\mathbf{T}^2)$: For $A \in SL(2, \mathbf{Z})$ and $\varphi \in C(\mathbf{T}^2)$, define

$$(T_A\varphi)(t) = \varphi(At) \quad (t \in \mathbf{T}^2).$$

We claim that $T_A \circ S_g \circ T_A^{-1} = S_{g_A}$ for every $A \in SL(2, \mathbf{Z})$ and every $g \in L^\infty(\mathbf{T}^2)$ where $g_A(t) := g(At)$:

$$\begin{aligned}
\left(T_A \circ S_g \circ T_A^{-1}\right)(\varphi)(t) &= \left(S_g \circ T_A^{-1}\right)(\varphi)(At) \\
&= \left(g * \left(T_A^{-1}\varphi\right)\right)(At) \\
&= \int g(y)\varphi\left(A^{-1}(-y + At)\right) dy \\
&= \int g(y)\varphi\left(-A^{-1}y + t\right) dy \\
&= \int g(Az)\varphi(-z + t)\, dz \\
&= (g_A * \varphi)(t) \\
&= S_{g_A}(\varphi)(t).
\end{aligned}$$

Assuming that the trace functional on $N(C(\mathbf{T}^2))$ has a continuous extension E_0 to $I(C(\mathbf{T}^2))$ satisfying

$$(*) \qquad \langle S \circ T - T \circ S, E_0 \rangle = 0 \quad \left(T \in \mathscr{L}\left(C(\mathbf{T}^2)\right),\ S \in I\left(C(\mathbf{T}^2)\right)\right),$$

we consider the map M: $L^\infty(G_0) \to \mathbf{C}$ defined by $M(f) := \langle S_{f_I}, E_0\rangle$. Then M is a conjugation-invariant mean in the sense of [15] extending the Dirac measure δ_e, e the unit element of G_0. For the proof, define

$$(\tau_x^* f)(y) := f(xyx^{-1}) \quad (x, y \in G_0);$$

if $x = (s, A) \in G_0$ and $t \in \mathbf{T}^2$, it follows that

$$\begin{aligned}(\tau_x^* f)_I(t) &= (\tau_x^* f)(t, I)\\ &= f((s, A)(t, I)(-A^{-1}s, A^{-1}))\\ &= f(s + At - s, I)\\ &= f_I(At)\\ &= (f_I)_A(t).\end{aligned}$$

Therefore,

$$\begin{aligned}M(\tau_x^* f) &= \langle S_{(\tau_x^* f)_I}, E_0\rangle\\ &= \langle S_{(f_I)_A}, E_0\rangle\\ &= \langle T_A \circ S_{f_I} \circ T_A^{-1}, E_0\rangle\\ &= \langle S_{f_I} \circ T_A^{-1} \circ T_A, E_0\rangle\\ &= \langle S_{f_I}, E_0\rangle\\ &= M(f).\end{aligned}$$

Moreover, E_0 extends δ_e since, for $f_I \in C(\mathbf{T}^2)$, S_f is nuclear with trace $f_I(0)$ [19, 3.4, 3.7] which implies

$$M(f) = \langle S_{f_I}, E_0\rangle = \operatorname{trace}(S_{f_I}) = f_I(0) = \delta_e(f) \quad (f \in L^\infty(G_0)).$$

According to [15], Example 1 and Theorem 2, this is impossible. Therefore, $K_0(C(\mathbf{T}^2))^{**}$ cannot possess a good mixed unit since its restriction E_0 to $I(C(\mathbf{T}^2))(\subseteq I(C(\mathbf{T}^2)^*))$ would satisfy $(*)$.

By Proposition 5.9 (ii), $K_0(X)^{**}$ cannot possess a good mixed unit where X is a dual of any order of $C(\mathbf{T}^2)$. Finally, let us point out that $\mathbf{T}^2$ can be replaced by any $\mathbf{T}^k$ $(k > 2)$.

7. Open problems

1. Does there exist a (necessarily non-Cartesian—cf. Theorem 5.5.) Banach space X (with X^* possessing the b.a.p.) such that $K_0(X)$ is semi-regular but $N(X^*) \neq I(X^*)$?

2. Characterize the Banach spaces X for which $K_0(X)$ is semi-regular.

3. Characterize the Banach spaces X satisfying $I(X^*) = N(X^*)$.

4. Characterize the Banach spaces X satisfying $I(X) = N(X)$.

5. For which of the classical Banach spaces is $K_0(X)$ semi-regular (provided X^* has the b.a.p.) resp. does $I(X^*) = N(X^*)$ resp. $I(X) = N(X)$ hold (apart from $X = C(K)$ or $X = L^1(\mu)$)?

6. For which Banach spaces does $K_0(X)^{**}$ contain a good mixed unit?

7. If X is a Banach space such that X^* possesses the a.p. (or the b.a.p.), does trace$(b^* \circ f - f \circ b^*) = 0$ for any $b \in \mathscr{L}(X)$, $f \in I(X^*)$, provided only that $b^* \circ f - f \circ b^* \in N(X^*)$ (resp. $b^* \circ f$ and $f \circ b^*$ both nuclear)? (Compare 5.8 and [11]).

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