ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 4, Winter 1987

ON EXTREME INFINITE DOUBLY STOCHASTIC MATRICES

BY

RYSZARD GRZĄŚLEWICZ

Introduction

Let $\bar{r} = (r_1, r_2, \dots)$ and $\bar{s} = (s_1, s_2, \dots)$ be sequences of non-negative reals. A matrix $P = (p_{ij})$, $i, j = 1, 2, \dots$, is called doubly substochastic with respect to $(\bar{r}, \bar{s})$ if $p_{ij} \geq 0$, $\sum_{j=1}^{\infty} p_{ij} \leq r_i$ and $\sum_{i=1}^{\infty} p_{ij} \leq s_j$ for all $i, j = 1, 2, \dots$. We denote by $\mathscr{D}(\leq \bar{r}, \leq \bar{s})$ the set of all doubly substochastic matrices with respect to $(\bar{r}, \bar{s})$.

Let $\sum_{i=1}^{\infty} r_i = \sum_{j=1}^{\infty} s_j$. We admit the case $\sum_{i=1}^{\infty} r_i = \sum_{j=1}^{\infty} s_j = \infty$. We say that a matrix $P = (p_{ij})$ is doubly stochastic with respect to $(\bar{r}, \bar{s})$ if $p_{ij} \geq 0$, $\sum_{j=1}^{\infty} p_{ij} = r_i$ and $\sum_{i=1}^{\infty} p_{ij} = s_j$, $i, j = 1, 2, \dots$. We denote by $\mathscr{D}(\bar{r}, \bar{s})$ the set of all matrices which are doubly stochastic with respect to $(\bar{r}, \bar{s})$. The sets $\mathscr{D}(\bar{r}, \bar{s})$ and $\mathscr{D}(\leq \bar{r}, \leq \bar{s})$ are convex.

Let $\operatorname{ext} \mathscr{D}(\bar{r}, \bar{s})$ ($\operatorname{ext} \mathscr{D}(\leq \bar{r}, \leq \bar{s})$) denote the set of extreme points of $\mathscr{D}(\bar{r}, \bar{s})$ ($\mathscr{D}(\leq \bar{r}, \leq \bar{s})$). It is not difficult to see that if

$$p_{ij} = \min\left(\left(r_i - \sum_{k=1}^{j-1} p_{ik}\right), \left(s_j - \sum_{k=1}^{i-1} p_{kj}\right)\right)$$

then, $P = (p_{ij}) \in \mathscr{D}(\bar{r}, \bar{s}) \subset \mathscr{D}(\leq \bar{r}, \leq \bar{s})$, hence $\mathscr{D}(\bar{r}, \bar{s})$ and $\mathscr{D}(\leq \bar{r}, \leq \bar{s})$ are non-empty.

In 1946 Birkhoff [2] proved that if

$$r_i = s_i = \begin{cases} 1 & \text{for } i \leq n \\ 0 & \text{for } i > n, \end{cases} \qquad n \in N$$

then the set $\operatorname{ext} \mathscr{D}(\bar{r}, \bar{s})$ coincides with the set of all permutation matrices. Kendal [11] and Isbel [9] generalized this result to the case of infinite doubly stochastic matrices (i.e., $s_i = r_i = 1$, $i = 1, 2, \dots$). Other characterization of extreme points was discovered independently by Douglas [7] and by Lindenstrauss [13]. We see that in the above mentioned cases extremality of doubly stochastic matrices (measures with discrete supports) depends on their

Received October 15, 1985.

supports (graphs). Generally it is not true. Let $p_{11} = p_{i,i+1} = p_{i+1,i} = 1/2$, $i \geq 1$, and $p_{ij} = 0$, otherwise. And let $q_{11} = 1$, $q_{i,i+1} = q_{i+1,i} = 1/i$, $i \geq 1$, and $q_{ij} = 0$, otherwise. The matrix $P = (p_{ij})$ ($Q = (q_{ij})$) is doubly stochastic with respect to $(\bar{r}, \bar{s})$ where $r_i = s_i = 1$ ($r_i = s_i = 1/i + 1/(i+1)$). Obviously supports of P and Q are the same. But is to difficult to check that P is not extreme and Q is extreme. Therefore, in the general case, to characterize extreme doubly stochastic matrix measures with discrete coutable supports in terms of their supports (graphs) we need more subtle description, e.g., ε-summing families, ε-bitrees (see Section 1 for the definitions).

Mirsky [14] showed that ext $\mathscr{D}(\leq \bar{r}, \leq \bar{s})$ coincides with the set of subpermutation matrices for $\bar{r} = \bar{s}$ with

$$r_i = s_i = \begin{cases} 1 & \text{for } i \leq n \\ 0 & \text{for } i > n, \end{cases} \quad n \in N.$$

This result was generalized to the finite-dimensional case by Brualdi [4] (i.e., when $\bar{r} = (r_1, r_2, \ldots, r_n)$, $\bar{s} = (s_1, s_2, \ldots, s_m)$, $n, m \in N$, are arbitrary non-negative vectors).

The purpose of this paper is to describe ext $\mathscr{D}(\bar{r}, \bar{s})$ (Section 1) and ext $\mathscr{D}(\leq \bar{r}, \leq \bar{s})$ (Section 2) for arbitrary infinite non-negative vectors $\bar{r}, \bar{s}$.

With each matrix $P = (p_{ij}) \in \mathscr{D}(\bar{r}, \bar{s})$ or $\mathscr{D}(\leq \bar{r}, \leq \bar{s})$ is associated a graph $G(P)$ defined by the following formula. To the i-th row there corresponds a (row) node x_i ($i = 1, 2, \ldots$) and to the j-th column there corresponds a (column) node y_j. There is an edge joining x_i and y_j if and only if $p_{ij} > 0$; there are to be no other edges. Therefore to each edge $x_i y_j$ in $G(P)$ there corresponds a positive entry p_{ij}. Note that the sum of all entries p_{ij} which correspond to edges joined with fixed node $x_i(y_j)$ is equal $r_i(s_j)$. For a matrix

$$P \in \mathscr{D}((r_1, r_2, \ldots, r_m)(s_1, s_2, \ldots, s_n)),$$

P is extreme if and only if the connected components of $G(P)$ are trees, i.e., the graph $G(P)$ has no cycle (for example, see [4]). Note that this result was extended by Bartoszek [1] to the case of infinite sequences $\{r_i\}, \{s_j\}$ such that $\sum_{i=1}^{\infty} r_i = \sum_{j=1}^{\infty} s_j < \infty$ (cf. Corollary 1). For other expositions of this result in finite dimensional case see [10], [3] and [15]. The problem of description of extreme doubly stochastic measures with the discrete coutable supports (or equivalently infinite doubly stochastic matrices with given marginals) was also considered by Letac [12], Denny [6] and Mukerjee [16].

1. Extreme infinite doubly stochastic matrices

A set $\{\varepsilon_{k_1, k_2, \ldots, k_n} : n \in N\}$ of non-negative numbers is said to be an ε-summing family if

$$\sum_{k_{n+1} \in A_{k_1, \ldots, k_n}} \varepsilon_{k_1 k_2 \cdots k_n k_{n+1}} = \varepsilon_{k_1 k_2 \cdots k_n}, \quad n \in N$$

and

$$\sum_{k_1 \in A} \varepsilon_{k_1} = \varepsilon > 0,$$

where A, A_{k_1} $(k_1 \in A)$, $A_{k_1 k_2}$ $(k_1 \in A,\ k_2 \in A_{k_1}), \dots$ are disjoint subsets of N.

We say that the graph $G(P)$, $P \in \mathscr{D}(\bar{r}, \bar{s})$, has an ε-bitree, if there exists a subgraph H of the graph $G(P)$ which for certain ε-summing families $\{\varepsilon_{k_1, k_2, \dots, k_n}\}, \{\varepsilon'_{k_1, k_2, \dots, k_n}\}$ satisfies the following conditions:

(a$_1$) The graph H includes an edge $x_{i_0} y_{j_0}$ with $0 < \varepsilon \le p_{i_0 j_0}$.

(a$_2$) The graph H includes edges $y_{j_0} x_{i_1}$, $i_1 \in A$ with $0 < \varepsilon_{i_1} \le p_{i_1 j_0}$ and edges $x_0 y_{j_1}$, $j_1 \in A'$ with $0 < \varepsilon'_{j_1} \le p_{i_0 j_1}$ and $i_0 \notin A$, $j_0 \in A'$. (Obviously $\sum_{i_1 \in A} \varepsilon_{i_1} = \sum_{j_1 \in A'} \varepsilon_{j_1} = \varepsilon > 0$.)

(a$_3$) the graph H includes edges $x_{i_1} y_{j_1}$, $i_1 \in A$, $j_1 \in A_{i_1}$ with $0 < \varepsilon_{i_1 j_1} \le p_{i_1 j_1}$ and edges $y_{j_1} x_{i_1}$, $j_1 \in A'$, $i_1 \in A'_{j_1}$ with $0 < \varepsilon'_{j_1 i_1} \le p_{i_1 j_1}$ and $A' \cap A'_{j_1} = \varnothing$, $A \cap A_{i_1} = \varnothing$. (Obviously $\sum_{j_1 \in A_{i_1}} \varepsilon_{i_1 j_1} = \varepsilon_{i_1}$, $\sum_{i_1 \in A_{j_1}} \varepsilon'_{j_1 i_1} = \varepsilon'_{j_1}$).

(a$_4$) The graph H includes edges $y_{j_1} x_{i_1}$, $j_1 \in A_{i_1}$, $i_2 \in A_{i_1 j_1}$, $i_1 \in A$ with $0 < \varepsilon_{i_1 j_1 i_2} \le p_{i_2 j_1}$ and edges $x_{i_1} y_{j_2}$, $i_1 \in A'_{j_1}$, $j_2 \in A'_{j_1 i_1}$, $j_1 \in A'$ with $0 < \varepsilon'_{j_1 i_1 j_2} \le p_{i_1 j_2}$ and the sets A, A'_{j_1} $(j_1 \in A')$, $A_{i_1 j_1}$ $(i_1 \in A,\ j_1 \in A_{i_1})$ are disjoint and the sets A', A_{i_1} $(i_1 \in A)$, $A'_{j_1 i_1}$ $(j_1 \in A',\ i_1 \in A'_{j_1})$ are disjoint.

⋮

The graph H includes only edges described in (a$_1$), (a$_2$), (a$_3$), ... (Fig. 1).

If in a graph $G(P)$ there exists a subgraph H which is an ε-bitree, it is not difficult to see that there exists also an ε-bitree H_1 such that every node of H_1 is joined with only a finite number of edges and H_1 is a subgraph of H.

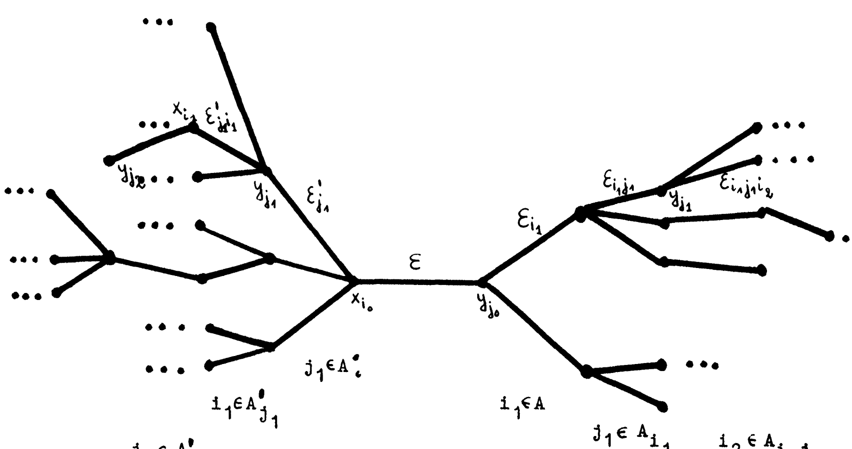

FIG. 1.

LEMMA 1. *If $P \in \mathcal{D}(\bar{r}, \bar{s})$ has an ε-bitree, then $P \notin \operatorname{ext} \mathcal{D}(\bar{r}, \bar{s})$.*

Proof. Let H be a subgraph of $G(P)$ such that H is an ε-bitree. We define a matrix $T = (t_{ij})$ (using the notation from the definition of an ε-bitree):

(b$_1$) $t_{i_0 j_0} = \varepsilon > 0$.

(b$_2$) $t_{i_1 j_0} = -\varepsilon_{i_1}$, $i_1 \in A$; $t_{i_0 j_1} = -\varepsilon'_{j_1}$, $j_1 \in A'$ ($i_0 \notin A$, $j_0 \in A'$)

(b$_3$) $t_{i_1 j_1} = \varepsilon_{i_1 j_1}$, $i_1 \in A$, $j_1 \in A_{i_1}$; $t_{i_1 j_1} = \varepsilon'_{j_1 i_1}$ $j_1 \in A'$, $i_1 \in A'_{j_1}$. (The sets $\{i_0\}, A, A'_{j_1}$ ($j_1 \in A'$) are disjoint and the sets $\{j_0\}, A', A_i$ ($i_1 \in A$) are disjoint.

(b$_4$) $t_{i_2 j_1} = -\varepsilon_{i_1 j_2 i_2}$, $i_1 \in A$, $j_1 \in A_{i_1}$ $i_2 \in A_{i_1 j_2}$; $t_{i_1 j_2} = -\varepsilon'_{j_1 i_1 j_2}$, $j_1 \in A'$, $i_1 \in A'_{j_1}$, $j_2 \in A'_{j_1 i_1}$. (The sets $\{i_0\}, A, A'_j$ ($j_1 \in A'$), $A_{i_1 j_1}$ ($i_1 \in A$, $j_1 \in A_{i_1}$) are disjoint and the sets $\{j_0\}, A', A_{i_1}$ ($i_1 \in A$), $A'_{j_1 i_1}$ ($j_1 \in A'$, $i_1 \in A'_{j_1}$) are disjoint.

⋮

If an edge $x_i y_j$ is not in the graph H, then we let $t_{ij} = 0$. It is easy to see that $\Sigma_i t_{ij} = \Sigma_j t_{ij} = 0$ and $p_{ij} \geq |t_{ij}|$. Thus $P \pm T \in \mathcal{D}(\bar{r}, \bar{s})$, so P is not extreme.

THEOREM 1. *Let $P \in \mathcal{D}(\bar{r}, \bar{s})$. Then $P \in \operatorname{ext} \mathcal{D}(\bar{r}, \bar{s})$ if and only if the graph $G(P)$ has no cycle and $G(P)$ has no ε-bitree.*

Proof. Suppose that the graph $G(P)$ has a cycle. Let the sequence $x_{i_1}, y_{j_1}, x_{i_2}, y_{j_2}, \ldots, x_{i_n}, y_{j_n}, x_{i_1}$ describe this cycle. We may and do assume that our cycle is simple. We have $p_{i_1 j_1}, p_{i_2 j_1}, p_{i_2 j_1}, \ldots, p_{i_n j_n}, p_{i_1 j_n} > 0$. Let

$$\varepsilon = \min\{p_{i_1 j_1}, p_{i_2 j_1}, \ldots, p_{i_n j_n}, p_{i_1 j_n}\}$$

Obviously $\varepsilon > 0$. Let us define a matrix $T = (t_{ij})$ by

$$t_{ij} = \begin{cases} \varepsilon & \text{if } (i, j) = (i_k, j_k),\ k = 1, 2, \ldots, n, \\ -\varepsilon & \text{if } (i, j) = (i_1, j_n) \\ -\varepsilon & \text{if } (i, j) = (i_{k+1}, j_k),\ k = 1, 2, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $p_{ij} \pm t_{ij} \geq 0$ for all $i, j = 1, 2, \ldots$ and $\Sigma_i t_{ij} = \Sigma_j t_{ij} = 0$. Thus $P \pm T \in \mathcal{D}(\bar{r}, \bar{s})$.

Suppose that the graph $G(P)$ has an ε-bitree. By Lemma 1, P is not extreme. Therefore if $P \in \operatorname{ext} \mathcal{D}(\bar{r}, \bar{s})$, then the graph $G(P)$ has no cycle and no ε-bitree.

Now suppose that $P \in \mathcal{D}(\bar{r}, \bar{s})$ is not extreme. Then there exists a non-zero matrix $T = (t_{ij})$ with $P \pm T \in \mathcal{D}(\bar{r}, \bar{s})$. Obviously $|t_{ij}| \leq p_{ij}$ for all $i, j =$

$1, 2, \ldots$ Moreover, there exists (i_0, j_0) with $t_{i_0 j_0} \neq 0$. It is also easy to see that $\Sigma_i t_{ij} = \Sigma_j t_{ij} = 0$. Let $|T| = (|t_{ij}|)$. A graph $G(|T|)$ is a subgraph of $G(P)$. If $G(|T|)$ has a cycle, then $G(P)$ has also a cycle. Now assume that $G(|T|)$ has no cycle. It is sufficient to show that there exist ε-summing families $\{\varepsilon_{k_1, \ldots, k_n}\}, \{\varepsilon'_{k_1, \ldots, k_n}\}$ and a subgraph H of the graph $G(|T|)$ which satisfy the conditions $(\mathrm{a}_1), (\mathrm{a}_2), (\mathrm{a}_3), \ldots$. Let $t_{i_0 j_0} \neq 0$. Let $\varepsilon = |t_{i_0 j_0}|$. Now we define a graph H as follows:

(c_1) H includes the edge $x_{i_0} y_{j_0}$.

(c_2) Let $A = \{i \neq i_0\colon t_{ij_0} \neq 0\}$ and $A' = \{j \neq j_0\colon t_{i_0 j} \neq 0\}$. H includes edges $y_{j_0} x_{i_1}$, $i_1 \in A$, and edges $x_{i_0} y_{j_1}$, $j_1 \in A'$. We choose positive numbers $\varepsilon_{i_1}, \varepsilon'_{j_1}$, $i_1 \in A$, $j_1 \in A'$ in such way that $\Sigma_{i_1 \in A} \varepsilon_{i_1} = \varepsilon$, $\Sigma_{j_1 \in A'} \varepsilon'_{j_1} = \varepsilon$ and $0 < \varepsilon_{i_1} \leq |t_{i_1 j_0}|$, $0 < \varepsilon'_{j_1} \leq |t_{i_0 j_1}|$. We are able to choose $\varepsilon_{i_1}, \varepsilon'_{j_1}$ by the above formula because

$$\sum_{i_1 \in A} t_{i_1 j_0} + t_{i_0 j_0} = 0 \quad \text{and} \quad \sum_{j_1 \in A'} t_{i_0 j_1} + t_{i_0 j_0} = 0.$$

$\vdots$

H includes only edges described in (c_1), (c_2), ... Note that in the above construction of the graph H the sets $\{i_0\}, A, A'_{j_1}$ $(j_1 \in A'), \ldots$ are disjoint and the sets $\{j_0\}, A, A_{i_1}$ $(i_1 \in A), \ldots$ are disjoint, since $G(|T|)$ has no cycle. Therefore the graph H (H is a subgraph of $G(|T|)$) is an ε-bitree. Since $|t_{i_j}| \leq p_{ij}$ $i, j = 1, 2, \ldots$, the graph H is an ε-bitree in the graph $G(P)$. This completes the proof.

Suppose that the graph $G(P)$, $P \in \mathscr{D}(\bar{r}, \bar{s})$, has an ε-bitree. Then $\Sigma_{i,j} p_{ij} \geq \Sigma \varepsilon_{k_1 k_2, \ldots, k_n} = \infty$, i.e., $\Sigma_i r_i = \Sigma_j s_j = \Sigma_{ij} p_{ij} = \infty$. Therefore we can write the following corollary. A similar result was presented by Bartoszek in [1] (cf. [12], [6], [16]).

COROLLARY 1. *Let* $\Sigma_{i=1}^{\infty} r_i = \Sigma_{j=1}^{\infty} s_j < \infty$ *and let* $P \in \mathscr{D}(\bar{r}, \bar{s})$. *Then* $P \in \operatorname{ext} \mathscr{D}(\bar{r}, \bar{s})$ *if and only if the connected components of the graph* $G(P)$ *are trees.*

Example 1. Let

$$\bar{r} = (19 + a, 2, 1, 7, 5, 11, 2, 7, 5, 6, 6, 6, \ldots)$$

and

$$\bar{s} = (5 + a, 3, 3, 9, 7, 8, 9, 10, 2, 6, 6, 6, \ldots).$$

The matrix

$$P_a = \begin{bmatrix} a & & & 6 & 8 & 5 & & & & & & & & \\ 2 & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & \\ 3 & & & & & & 4 & & & & & & & 0 \\ & 2 & 3 & & & & & & & & & & & \\ & & & 3 & & & 5 & 1 & 2 & & & & & \\ & & & & & & & 2 & & & & & & \\ & & & & & & & 7 & & & & & & \\ & & & & & 2 & & & & 3 & & & & \\ & & & & & & & & & 3 & 3 & & & \\ & & & & & & & & & & 3 & 3 & & \\ & & & & & & & & & & & 3 & 3 & \\ & & & & & 0 & & & & & & & & \ddots \end{bmatrix}$$

has graph $G(P_a)$ given by Fig. 2. If we let $a = 0$, then in view of Theorem 1, P_a is extreme. If we let $a > 0$ then the graph $G(P_a)$ has a cycle, so P_a is not an extreme doubly stochastic matrix.

Example 2. Let

$$\bar{r} = \left(2\tfrac{5}{6}, 2, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1, 1, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \dots\right)$$

and

$$\bar{s} = \left(3, 1, 1, 1, \tfrac{5}{6}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 1, 1, \dots\right).$$

The matrix

$$P = \begin{bmatrix} 2 & & & \frac{1}{2} & \frac{1}{3} & & & & & & & & & & & \\ 1 & \frac{1}{2} & \frac{1}{2} & & & & & & & & & & & & & \\ & \frac{1}{4} & & & & \frac{1}{8} & \frac{1}{8} & & & & & & & & & \\ & \frac{1}{4} & & & & & & \frac{1}{8} & \frac{1}{8} & & & & & & & 0 \\ & & \frac{1}{4} & & & & & & & \frac{1}{8} & \frac{1}{8} & & & & & \\ & & \frac{1}{4} & & & & & & & & & \frac{1}{8} & \frac{1}{8} & & & \\ & & & \frac{1}{2} & & & & & & & & & & \frac{1}{2} & & \\ & & & & \frac{1}{2} & & & & & & & & & & \frac{1}{2} & \\ & & & & & \frac{1}{16} & & & & & & & & & & \\ & & & & & \frac{1}{16} & & & & & & & & & & \\ & & & & & & \frac{1}{16} & & & & & & & & & \\ & & & & & & \frac{1}{16} & & & & & & & & & \\ & & & & & & & \frac{1}{16} & & & & & & & & \\ & 0 & & & & & & \frac{1}{16} & & & & & & & & \\ & & & & & & & & \ddots & & & & & & & \end{bmatrix}$$

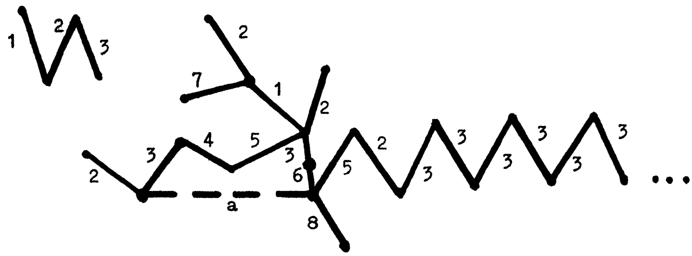

FIG. 2.

has graph $G(P)$ given by Fig. 3. Note that Fig. 3a and Fig. 3b presents subgraphs of the graph $G(P)$ which are ε-bitree with ε equal to $\frac{5}{6}$ and $\frac{1}{4}$, respectively.

Example 3. Let a doubly stochastic matrix P have the graph $G(P)$ given by Fig. 4. It is easy to see that $G(P)$ has no cycle and $G(P)$ has no ε-bitree. Thus P is extreme.

If for $P, P' \in \mathscr{D}(\bar{r}, \bar{s})$, the condition that $G(P')$ is a subgraph of $G(P)$ implies $P = P'$, then we say that a matrix $P \in \mathscr{D}(\bar{r}, \bar{s})$ is uniquely determined in $\mathscr{D}(\bar{r}, \bar{s})$ by its graph. The elements of ext $\mathscr{D}(\bar{r}, \bar{s})$ in the finite-dimensional case can also be characterized as those matrices in $\mathscr{D}(\bar{r}, \bar{s})$ which are uniquely determined in $\mathscr{D}(\bar{r}, \bar{s})$ by their graph (see Brualdi [4], Theorem 2.1.). This result can be extended. Indeed, if $P \notin \operatorname{ext} \mathscr{D}(\bar{r}, \bar{s})$ then $P = (P_1 + P_2)/2$, $P_1, P_2 \in \mathscr{D}(\bar{r}, \bar{s})$, $P_1 \neq P_2$. Obviously, $G(P_1)$ is a subgraph of $G(P)$, so P is not uniquely determined in $\mathscr{D}(\bar{r}, \bar{s})$ by its graph. Now assume that $P, P' \in \mathscr{D}(\bar{r}, \bar{s})$ are distinct such that the graph $G(P')$ is a subgraph of $G(P)$. Put $T = P - P'$. Obviously $\sum_i t_{ij} = \sum_j t_{ij} = 0$ and $t_{i_0 j_0} \neq 0$ for some (i_0, j_0). If $G(|T|)$ has a cycle then $G(P)$ has also a cycle and $P \notin \operatorname{ext} \mathscr{D}(\bar{r}, \bar{s})$. Suppose now that $|T|$ has no cycle. We may and do assume that $t_{i_0 j_0} > 0$. Define the family of sets

$$
\begin{aligned}
A &= \{i \neq i_0 \colon t_{ij_0} < 0\}, \\
A' &= \{j \neq j_0 \colon t_{i_0 j} < 0\}, \\
A_{i_1} &= \{j \neq j_0 \colon t_{i_1 j} > 0\}, \quad i_1 \in A, \\
A'_{j_1} &= \{i \neq i_0 \colon t_{ij_1} > 0\}, \quad j_1 \in A', \\
A_{i_1 j_2} &= \{i \neq i_1 \colon t_{ij_1} < 0\}, \quad i_1 \in A,\ j_2 \in A_{i_1}, \\
A'_{j_1 i_2} &= \{j \neq j_1 \colon t_{i_2 j} < 0\}, \quad j_1 \in A',\ i_2 \in A'_{j_1},
\end{aligned}
$$

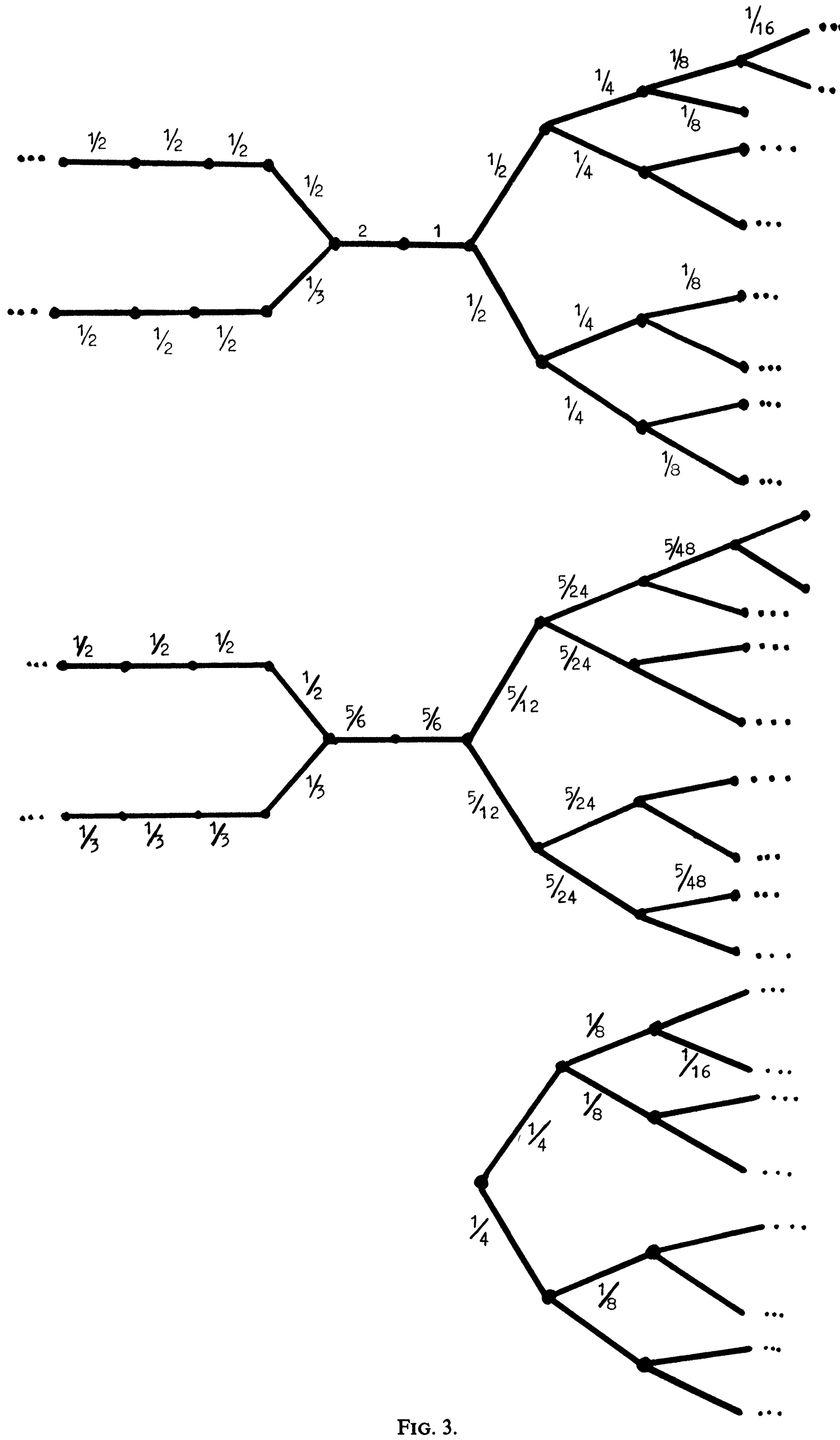
1/16
1/4
1/8
1/2
1/2
1/2
1/8
1/2
1/4
1/2
2
1
1/3
1/8
1/2
1/4
1/2
1/2
1/2
1/4
1/8
5/48
5/24
1/2
1/2
1/2
5/24
1/2
5/12
5/6
5/6
1/3
5/24
5/12
1/3
1/3
1/3
5/48
5/24
1/8
1/16
1/8
1/4
1/4
1/8

FIG. 3.

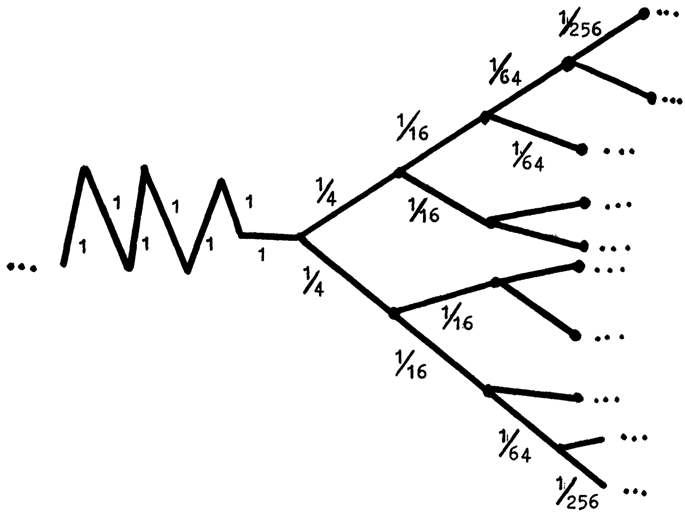

FIG. 4.

It is not difficult to see that using the sets A, A', A_{i_1}, A'_{j_1}, $A_{i_1 j_2}, \ldots$ we can find an ε-bitree in $G(P)$ (with $\varepsilon = t_{i_0 j_0}$), i.e., $P \notin \operatorname{ext} \mathscr{D}(\bar{r}, \bar{s})$. Thus we proved the following fact.

PROPOSITION 1. *The extreme points of $\mathscr{D}(\bar{r}, \bar{s})$ are those matrices in $\mathscr{D}(\bar{r}, \bar{s})$ which are uniquely determined in $\mathscr{D}(\bar{r}, \bar{s})$ by their graphs.*

We recall that a point q_0 in a convex set Q is exposed if there exists a functional ξ such that $\xi(q_0) \geq \xi(q)$ for all $q \in Q \setminus \{q_0\}$.

PROPOSITION 2. *The set of all extreme points of $\mathscr{D}(\bar{r}, \bar{s})$ coincides with the set of all exposed points of $\mathscr{D}(\bar{r}, \bar{s})$.*

Proof. Obviously each exposed point is extreme. Now let $\alpha_i > 0$ be such that $\sum_i \alpha_i = 1$. Let $T = (t_{ij}) \in \operatorname{ext} \mathscr{D}(\bar{r}, \bar{s})$. We define a function ξ on $\mathscr{D}(\bar{r}, \bar{s})$ by

$$\xi(P) = \sum_i \sum_j \left[2(\operatorname{sgn} t_{ij}) - 1\right] \alpha_i p_{ij}/r_i,$$

$P = (p_{ij}) \in \mathscr{D}(\bar{r}, \bar{s})$. It is easy to see that $\xi(P) \leq 1 = \xi(T)$ for all $P \in \mathscr{D}(\bar{r}, \bar{s})$. Suppose that $\xi(P) = 1$ for some $P \in \mathscr{D}(\bar{r}, \bar{s})$. Because $t_{ij} = 0$ for fixed (i, j) implies that $p_{ij} = 0$, the graph $G(P)$ is a subgraph of $G(T)$. By Proposition 1, $P = T$, i.e., T is exposed by ξ.

2. Extreme infinite doubly substochastic matrices

The extreme points of $\mathscr{D}(\leq \bar{r}, \leq \bar{s})$ were shown by Mirsky [14] to be the $n \times n$ subpermutation matrices (i.e., matrices of 0's and 1's with at most one 1 in each row and column) when $\bar{r} = \bar{s} = (1, 1, 1, \ldots, 1)$, n 1's.

We say that the i-th row sum (j-th column sum) of a matrix $P \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$ is unattained if $\sum_j p_{ij} < r_i (\sum_i p_{ij} < s_j)$. Brualdi [4] generalized Mirsky's result: A matrix $P \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$ is extreme if and only if the connected components of the graph $G(P)$ are trees and at most one node of each tree corresponds to a row or a column of P whose sum in P is unattained [4, Theorem 2.2]

Let $\bar{r} = (r_1, r_2, \ldots)$, $\bar{s} = (s_1, s_2, \ldots)$ be arbitrary non-negative vectors. We say that the graph $G(P)$, $P = (p_{ij}) \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$, has an infinite ε-path if there exist sequences $\{i_k\}_{k=1}^{\infty}$, $\{j_k\}_{k=1}^{\infty}$ with $i_k \neq i_e$, $j_k \neq j_e$ if $k \neq e$ such that

$$\inf\{p_{i_1 j_1}, p_{i_2 j_1}, p_{i_2 j_2}, p_{i_3 j_2}, \ldots\} \geq \varepsilon > 0.$$

Let $P \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$. We say that the connected component H of the graph $G(P)$ is an extreme tree if H is a tree satisfying the following conditions:

(*) H has no ε-bitree.

(**) H has at most one node corresponding to a row or a column of P whose sum in P is unattained.

(***) If H has one node corresponding to a row or a column of P whose sum in P is unattained, then H has no infinite ε-path, or equivalently, if H has an infinite ε-path, then H has no node corresponding to a row or a column of P whose sum in P is unattained.

THEOREM 2. *Let $P \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$. Then $P \in \operatorname{ext} \mathscr{D}(\leq \bar{r}, \leq \bar{s})$ if and only if the connected components of $G(P)$ are extreme trees.*

Proof. Assume first that there exists a connected component H of the graph $G(P)$ such that H is not an extreme tree. Obviously if there is a cycle in H then P is not extreme, so we may and do assume that H is a tree, but not an extreme tree.

If H does not satisfy (*), i.e., H has an ε-bitree, then by arguments similar to those in the proof of Lemma 1 we obtain $P \notin \operatorname{ext} \mathscr{D}(\leq \bar{r}, \leq \bar{s})$.

Suppose that H does not satisfy (**), i.e. there exist two (or more) nodes each corresponding to a row or a column whose sum in P is unattained. Let z_1, z_2 be two distinct nodes of H having this property. Then either these nodes are both row nodes, or both column nodes, or one of them is a row node and the other is a column node. We can and do assume that $z_1 = x_{i_0}$ is a row node and $z_2 = y_{j_0}$ is a column node. In the remaining cases the reasoning is analogous. Since H is a tree and in a connected graph there is a path $x_{i_0}, y_{i_2}, y_{j_1}, \dots, x_{i_k}, y_{j_0}$ between x_{i_0} and y_{j_0} in the graph H, the entries $p_{i_0 j_1}, p_{i_1 j_1}, \dots, p_{i_k j_0}$ of the matrix P are positive and $\sum_j p_{i_0 j} < r_{i_0}, \sum_i p_{i j_0} < s_{j_0}$. Let

$$\varepsilon = \min\left\{ p_{i_0 j_1}, \dots, p_{i_k j_0}, \left(r_{i_0} - \sum_j p_{i_0 j} \right), \left(s_j - \sum_i p_{i j_0} \right) \right\}.$$

We define $T = (t_{ij})$ by setting $t_{ij} = 0$ except for $t_{i_0 j_1} = t_{i_1 j_2} = \cdots = \varepsilon$ and $t_{i_1 j_1} = t_{i_2 j_2} = \cdots = -\varepsilon$. Then $P \pm T \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$, i.e., P is not extreme.

Now suppose that H does not satisfy $(***)$, i.e., H has a node corresponding to a row or a column of P whose sum in P is unattained and H has an infinite ε-path. Let x_{i_0} be a row node of H with this property (analogously we can consider a column node y_{j_0}). Let the infinite ε-path be determined by sequences $\{i_k\}, \{j_k\}$. Since H is a tree and a connected graph we can and do assume that $i_0 = i_1$. We define $T = (t_{ij})$ by

$$t_{ij} = \begin{cases} \varepsilon & \text{if } (i, j) = (i_k, j_k),\ k = 1, 2, \dots, \\ -\varepsilon & \text{if } (i, j) = (i_{k+1}, j_k),\ k = 1, 2, \dots, \\ 0 & \text{otherwise} \end{cases}$$

where $\varepsilon = \inf\{ p_{i_1 j_1}, p_{i_2 j_2}, p_{i_2 j_2}, \dots, (r_{i_1} - \sum_j p_{i_1 j}) \} > 0$. It is easy to see that $P \pm T \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$, so P is not extreme. Therefore if $P \in \operatorname{ext} \mathscr{D}(\leq \bar{r}, \leq \bar{s})$, then the connected components of the graph $G(P)$ are extreme trees.

Now let $T = (t_{ij})$ be such that $P \pm T \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$ and the connected components of the graph $G(P)$ are extreme trees. The graph $G(|T|)$ is a subgraph of $G(P)$ and $|t_{ij}| \leq p_{ij}$ Let H be a connected component of the graph $G(|T|)$. Since $G(P)$ has no ε-bitree, the graph H has no ε-bitree.

We claim that at most one node z_0 of the graph H has the property $\sum_j t_{i_0 j} \neq 0$ if $z_0 = x_{i_0}$ or $\sum_i t_{i j_0} \neq 0$ if $z_0 = y_{j_0}$. Indeed, suppose that there exist two distinct nodes z_1, z_2 with this property. For example, suppose $z_1 = x_{i_1}$ is a row node and $z_2 = y_{j_2}$ is a column node (analogously we can consider the case when z_1, z_2 are both row nodes or both column nodes). Since $P \pm T \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$ we have

$$\sum_j p_{i_1 j} \pm \sum_j t_{i_1 j} \leq r_{i_1} \quad \text{and} \quad \sum_i p_{i j_2} \pm \sum_i t_{i j_2} \leq s_{j_2},$$

so $\sum_j p_{i_1 j} < r_{i_1}$ and $\sum_i p_{i j_2} < s_{j_2}$. Clearly nodes x_{i_1}, y_{j_2} belong to the same connected component of the graph $G(P)$, but this is impossible in view of $(**)$. This contradiction proves our claim.

Now let us consider two possibilities:

(1^0) $\sum_k t_{kj} = \sum_k t_{ik} = 0$ for all i, j corresponding to nodes in the graph H. If some $t_{i_0 j_0} \neq 0$, then using arguments similar to those of the second part of the proof of Theorem 1 we conclude that H has an ε-bitree. This contradiction proves that $t_{ij} = 0$ for all i, j.

(2^0) There is exactly one node z_0 in the graph H corresponding to a non-zero row or column sum of the matrix (t_{ij}). Assume, that $z_0 = x_{i_0}$ is a row node of H (analogously we can assume that $z_0 = y_{j_0}$ is a column node). We have $\Sigma_k t_{kj} = \Sigma_k t_{ik} = 0$ for all $i \neq i_0$ and j corresponding to nodes in the graph H and $\Sigma_k t_{i_0 k} \neq 0$. Suppose $t_{i_0 j_0} \neq 0$. Now we define the sets

$$\begin{aligned}
&A = \{i_1 \neq i_0 \colon t_{i_1 j_0} \neq 0\}\\
&A_{i_1} = \{j_1 \neq j_0 \colon t_{i_1 j_1} \neq 0\}, \quad i_1 \in A,\\
&A_{i_1 j_1} = \{i_2 \neq i_1 \colon t_{i_2 j_1} \neq 0\}, \quad i_1 \in A,\ j_1 \in A_{i_1},\\
&A_{i_1 j_1 i_2} = \{j_2 \neq j_1 \colon t_{i_2 j_2} \neq 0\}, \quad i_1 \in A,\ j_1 \in A_{i_1},\ i_2 \in A_{i_1 j_1},\\
&\vdots
\end{aligned}$$

In view of the conditions $\Sigma_k t_{ik} = \Sigma_k t_{kj} = 0$ the sets $A, A_{i_1}, A_{i_1 j_1}, A_{i_1 j_1 i_2}, \ldots$ are non-empty.

We claim that each of these sets has exactly one element. Indeed, suppose that there are two distinct elements i', i'' in $A_{i_1 j_1 i_2 \cdots i_k j_k}$ for some k. Let $|t_{i' j_k}| \geq |t_{i'' j_k}| > 0$. Let $\delta = -t_{i'' j_k}/t_{i' j_k}$. We define $s = s_{ij}$ by

$$\begin{aligned}
&s_{i'' j_{k+1}} = t_{i'' j_{k+1}}, \quad j_{k+1} \in A_{i_1, \ldots, j_k i''},\\
&s_{i_{k+1} j_{k+1}} = t_{i_{k+1} j_{k+1}}, \quad i_{k+1} \in A_{i_1, \ldots, j_k i'' j_{k+1}},\ j_{k+1} \in A_{i_1, \ldots, j_k i''},\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
&s_{i' j_{k+1}} = \delta t_{i j_{k+1}}, \quad j_{k+1} \in A_{i_1, \ldots, j_k i''},\\
&s_{i_{k+1} j_{k+1}} = \delta t_{i_{k+1} j_{k+1}}, \quad i_{k+1} \in A_{i_1, \ldots, i'', j_{k+1}},\ j_{k+1} \in A_{i_1, \ldots, j_k, i'}.\\
&\vdots
\end{aligned}$$

and for other i, j we set $s_{ij} = 0$.

We have $\Sigma_i s_{ij} = \Sigma_j s_{ij} = 0$. It is easy to see that the graph corresponding to the matrix $|S|$ contains an ε-bitree, so the graph $G(P)$ has also an ε-bitree. This contradicts the condition $(*)$ and ends the proof of our claim.

Therefore $A = \{i_1\}$, $A_{i_1} = \{j_1\}$, $A_{i_1 j_1} = \{i_2\}, \ldots$. Thus $t_{i_0 j_0} - t_{i_1 j_0} = t_{i_1 j_1} = -t_{i_2 j_1} = \cdots$. Since $G(P)$ has no infinite ε-path (condition $(* * *)$) and $|t_{ij}| \leq p_{ij}$ we obtain $(t_{i_0 j_0}| \leq \inf\{p_{i_0 j_0}, p_{i_1 j_0}, \ldots\} = 0$. Thus $t_{ij} = 0$ for all i, j. By (1^0) and (2^0) it follows that $P \pm T \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$ implies $T = 0$, i.e., $P \in \operatorname{ext} \mathscr{D}(\leq \bar{r}, \leq \bar{s})$, and the proof of the theorem is completed.

COROLLARY 2. *Let* $\Sigma_i r_i < \infty$ *or* $\Sigma_j s_j < \infty$ *and let* $P \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$. *Then* $P \in \operatorname{ext} \mathscr{D}(\leq \bar{r}, \leq \bar{s})$ *if and only if the connected components of the graph* $G(P)$ *are trees and at most one node of each of those trees corresponds to a row or a column of* P *whose sum in* P *is unattained.*

Proof. $\Sigma_{(i,j)} p_{ij} \leq \Sigma_i r_i$ and $\Sigma_{(i,j)} p_{ij} \leq \Sigma_j s_j$. Thus $\Sigma_{(i,j)} p_{ij} < \infty$. In such a case P has no ε-bitree. P has no infinite ε-path, because $\Sigma_{(i,j)} p_{ij} \geq \Sigma_k p_{i_k j_k} = \varepsilon + \varepsilon + \varepsilon + \varepsilon \cdots = \infty$, where $\{i_k\}, \{j_k\}$ are sequences from the definition of an infinite ε-path. Now, we use Theorem 2 and the proof is complete.

3. The facial structure in the finite-dimensional case

Let $\bar{r} = (r_1, r_2, \dots, r_m)$ and $\bar{s} = (s_1, s_2, \dots, s_n)$. We define the dimension of the face generated by P in $\mathscr{D}(\leq \bar{r}, \leq \bar{s})$ by

$$\dim_{\mathscr{D}(\leq r, \leq s)} = \dim \operatorname{lin}\{R \colon P \pm R \in \mathscr{D}(\bar{r}, \bar{s})\}.$$

Obviously P is extreme if and only if dimension of the face generated by P is equal to 0. Brualdi and Gibson [5] have given the dimension of the face of $\mathscr{D}(\bar{r}, \bar{s})$ (see also [8], Property 2),

$$\dim_{\mathscr{D}(r,s)} P = \sigma(P) - n - m + k_0$$

where $P = (p_{ij})$, $\sigma(P) = \Sigma_{i=1}^{m} \Sigma_{j=1}^{n} \operatorname{sign} p_{ij}$ and k_0 denotes the number of connected components of the graph $G(P)$. In this section we present analogous result for $\mathscr{D}(\leq r, \leq s)$. We say that a matrix P is elementary if the graph $G(P)$ is connected. Let the graph $G(P)$ have k_0 connected components. Then we can represent P as the direct sum of k_0 elementary matrices P_k. In such case $\dim_{\mathscr{D}(\leq \bar{r}, \leq \bar{s})} P = \Sigma_{k=1}^{k_0} \dim_{\mathscr{D}(\leq \bar{r}, \leq \bar{s})} P_k$.

PROPOSITION 3. *Let* $\bar{r} = (r_1, r_2, \dots, r_m)$ *and* $\bar{s} = (s_1, s_2, \dots, s_n)$ $(r_i > 0$, $i \leq m$, $s_j > 0$, $j \leq n$, *m and n finite*). *For an elementary matrix* $P = (p_{ij}) \in \mathscr{D}(\leq \bar{r}, \leq \bar{s})$, *if all nodes of* $G(P)$ *correspond to rows and columns whose sum in* P *is attained* $(P \in \mathscr{D}(\bar{r}, \bar{s}))$,

$$\dim_{\mathscr{D}(\leq \bar{r}, \leq \bar{s})} = \sigma(P) - m - n + 1.$$

Otherwise

$$\dim_{\mathscr{D}(\leq \bar{r}, \leq \bar{s})} = \sigma(P) - m_0 - n_0$$

where $m_0 = \operatorname{card}\{i \colon \Sigma_{j=1}^{m} p_{ij} = r_i\}$, $n_0 = \operatorname{card}\{j \colon \Sigma_{i=1}^{m} p_{ij} = s_j\}$.

Proof. We define functionals

$$\varphi_i\big((t_{ij})\big) = \sum_{j=1}^{n} t_{ij}, \quad i \le m,$$

$$\psi_j\big((t_{ij})\big) = \sum_{i=1}^{m} t_{ij}, \quad j \le n.$$

In [8, pp. 685–686] it is proved that

$$\dim \operatorname{lin}\big(\{\varphi_i \colon i \le m\} \cup \{\psi_j \colon j \le n\}\big) = m + n - 1.$$

Moreover we have $\sum_{i=1}^{m}\varphi_i = \sum_{j=1}^{n}\psi_j$. Let

$$F = \{\varphi_i \colon i \in A\} \cup \{\psi_j \colon j \in B\}$$

where

$$A = \left\{i \colon \sum_{j=1}^{n} p_{ij} = r_i\right\}, \quad B = \left\{j \colon \sum_{i=1}^{m} p_{ij} = s_j\right\}.$$

(card $A = m_0$, card $B = n_0$). Thus

$$\dim \operatorname{lin} F = \begin{cases} n + m - 1, & \text{if } m + n = m_0 + n_0, \\ m_0 + n_0, & \text{otherwise.} \end{cases}$$

We have

$$\dim_{\mathscr{D}(\le \bar r,\, \le \bar s)} P = \dim\{T \in X \colon \varphi_i(T) = \psi_j(T) = 0,\ i \in A,\ j \in B\},$$

where

$$X = \{T = (t_{ij}) \colon t_{ij} = 0 \text{ for all } (i, j) \text{ such that } p_{ij} = 0\}$$

(dim $X = \sigma(P) = mn - z$, z denotes the number of zero entries of P). Hence $\dim_{\mathscr{D}(\le \bar r,\, \le \bar s)} P$ is equal to dim X minus the number of linearly independent in the set F.

THEOREM 3. *Let* $\bar r = (r_1, r_2, \ldots, r_m)$ *and* $\bar s = (s_1, s_2, \ldots, s_n)$ *(n, m are finite). For* $P = (p_{ij}) \in \mathscr{D}(\le \bar r, \le \bar s)$ *we have*

$$\dim_{\mathscr{D}(\le \bar r,\, \le \bar s)} P = \sigma(P) - m_0 - n_0 + k_0$$

where

$$\sigma(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} \operatorname{sign} p_{ij},$$

$$m_0 = \operatorname{card}\left\{ i \colon \sum_{j=1}^{n} p_{ij} = r_i > 0 \right\}, \quad n_0 = \operatorname{card}\left\{ j \colon \sum_{i=1}^{m} p_{ij} = s_j > 0 \right\},$$

k_0 is the number of connected components of the graph $G(P)$ all of whose nodes correspond to rows or columns of P whose sum in P is attained.

References

1. W. Bartoszek, *Geometric properties of stochastic operators*, Master Thesis, Technical University, Wrocław, 1979 (in Polish).
2. G.D. Birkhoff, *Tres obsercaviones sobre el algebra lineal*, Univ. Nac. Tucumán Rev. Ser. A, vol. 5 (1946), pp. 147–151;
3. R.A. Brualdi, *Convex sets of non-negative matrices*, Canadian J. Math., vol. 20 (1968), pp. 144–157.
4. ______, *Combinatorical properties of symmetric non-negative matrices*, Colloquio Internazionale sulle Theorie Combinatorie Roma, 3-15 settembre 1973, Tomo II, Roma, Accademia Nazionale Dei Lincei, 1976, pp. 99–120;
5. R.A. Brualdi and P.M. Gibson, *Convex polyhedra of doubly stochastic matrices I: Applications of the permanent function*, J. Combinatorial Theory, vol. 22 (1977), pp. 194–230.
6. J.L. Denny, *The support of discrete measures with given marginals*, Michigan Math. J., vol. 27 (1980), pp. 59–64.
7. R.G. Douglas, *On extremal measures and subspace density*, Michigan Math. J., vol. 11 (1964), pp. 243–246.
8. R. Grząślewicz, *Extreme positive contractions on finite dimensional l^p-spaces*, Canadian J. Math., vol. 37 (1985), pp. 682–699.
9. J.R. Isbel, *Infinite doubly stochastic matrices*, Canadian Math. Bull., vol. 5 (1962), pp. 1–4.
10. W.B. Jurkat and H.J. Ryser, *Term ranks and permanents of non-negative matrices*, J. Algebra, vol. 5 (1967), pp. 342–357.
11. M.G. Kendal, *On infinite doubly stochastic matrices and Birkhoff's problem* 111, J. London Math. Soc., vol. 35 (1960), pp. 81–84.
12. G. Letac, *Reprèsentation des measures de probabilitè sur le produit de deux espaces dènombérables, de marges donnèes*, Illinois J. Math., vol. 10 (1966), pp. 497–507.
13. J. Lindenstrauss, *A remark on extreme doubly stochastic matrix measures*, Amer. Math. Monthly, vol. 72 (1965), pp. 379–382.
14. L. Mirsky, *On a convex set of matrices*, Arch. math., vol. 10 (1959), pp. 88–92.
15. ______, *Results and problems in the theory of doubly stochastic matrices*, Z. Wahrscheinlichkeitstheorie, vol. 1 (1963), pp. 319–334.
16. H.G. Mukerjee, *Supports of extremal measures with given marginals*, Illinois J. Math., vol. 29 (1985), pp. 248–260.

Technical University of Wroclaw
Wroclaw, Poland.