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SPECTRAL ASYMPTOTICS OF FOLIATED MANIFOLDS

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Introduction

In this paper we study the spectral distribution functions for the analytic and combinatorial leafwise Laplacians on a foliated manifold admitting a transverse invariant measure. We show (Theorem 4.1) that the dilational equivalence classes near zero of the two spectral distribution functions are the same. An immediate consequence is that this dilational equivalence class is independent of the metric (in the analytic case) and the bounded triangulation (in the combinatorial case) used to define it. Our main result (Theorem 2.6) is that this dilational equivalence class is invariant under a measure preserving leafwise homotopy equivalence. This result is equivalent to the invariance of the dilational equivalence class near infinity of the trace of the leafwise heat kernels.

Our results are motivated by those of [Ef-Sh], [G-Sh] and [Ef] concerning the equivariant homotopy invariance of the asymptotic behaviour of the spectral distribution function for Riemannian manifolds with a free isometric action of a discrete group. We extend many of the ideas of these papers to our situation. The main techniques used in the proof of the homotopy invariance, however, are those developed in [H-L2] (where we prove the leafwise homotopy invariance of the foliation betti numbers), the simplicial techniques of [D] and [W], and those of [H-L1].

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1. Introductory material: von Neumann algebras of foliations, leafwise operators, triangulations and simplicial theory

Let F be a smooth oriented foliation of a compact oriented manifold M. We first recall briefly some facts about transverse measures. For details see [C], [M-S], chapter IV and [H-L1], §2.3. A transversal to F is a Borel subset of M which intersects each leaf in a countable set. A smooth transversal is a proper embedded submanifold of M which is also a transversal. The set of

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transversals forms a σ-ring and a transverse measure ν is a measure on this σ-ring. The transverse measure is called invariant if it is invariant under the holonomy pseudogroup acting on smooth transversals. A Riemannian metric g on M gives rise to a volume form λ_L on each leaf L of F. The family $\lambda = \{\lambda_L\}$ and an invariant transverse measure ν give rise to a measure $\mu = \lambda\, d\nu$ on M ([C], [M-S] chapter IV).

Let E be a smooth vector bundle on M with smooth metric, $E|L$ the bundle restricted to the leaf L, and L_x the leaf through the point x. Let $H_x = L^2(L_x, E|L_x)$, the Hilbert space of L^2 sections of $E|L_x$ over L_x with inner product $(\ ,\ )_x$. A section of the family $\{H_x\}$ is a function $s\colon M \to \bigcup H_x$ with $s(x) \in H_x$. A measurable structure on $\{H_x\}$ is a sequence $\{s_n\}$ of sections such that for each x, $\{s_n(x)\}$ generates H_x as a Hilbert space. See [Dix], p. 161 for a complete discussion. The measurable structure we use is given in the appendix of [H-L2]. A section s of $\{H_x\}$ is called measurable if $(s(x), s_n(x))_x$ is a Borel function on M for all n. An inner product on the measurable sections is given by

$$\langle s, t\rangle = \int_M (s(x), t(x))_x \, d\mu(x).$$

A measurable section s is square integrable if $\|s\|^2 = \langle s, s\rangle < \infty$. Let $\tilde{H}$ be the collection of square integrable sections with the inner product $\langle\ ,\ \rangle$, and identify two sections if they are equal almost everywhere on M (with respect to the measure μ). $\tilde{H}$ is then the direct integral of the family $\{H_x\}$.

Let U and V be foliation charts on M and γ a leafwise path from U to V. Let $U \times \gamma V$ denote the subset of $U \times V$ consisting of those points (x, y) such that $x \in$ domain of h_γ and $y \in h_\gamma(P_x)$. Here h_γ is the holonomy map corresponding to γ, P_x is the placque of x in U and $h_\gamma(P_x)$ is the placque of $h_\gamma(x)$ in V. Let $C_0^\mu(U, V, \gamma)$ denote the set of leafwise smooth, uniformly bounded, measurable sections of the bundle $E \boxtimes E^*$ over $M \times M$ which are compactly supported in $U \times \gamma V$. An element of $C_0^\mu(U, V, \gamma)$ gives rise to a measurable family of bounded operators on $\{H_x\}$ and hence a bounded operator on $\tilde{H}$. $W(F; E)$ is the von Neumann algebra of operators on $\tilde{H}$ generated by the $C_0^\mu(U, V, \gamma)$ where U and V are in a fixed cover of M by foliation charts and γ ranges over the holonomy classes of leafwise paths from U to V. (See [H-L1, 2], [C], [M-S] and [Dix] p. 181.)

An invariant transverse measure ν determines a trace, tr_ν, on $W(F; E)$ ([C], [M-S], [H-L1]. For an element $k \in C_0^\mu(U, V, \gamma)$, $k(x, y) \in E_x \otimes E_y^*$, so $\mathrm{tr}\, k(x, x)$ is well defined and $\mathrm{tr}_\nu\, k$ is given by

$$\mathrm{tr}_\nu\, k = \int_M \mathrm{tr}\, k(x, x) \lambda \, d\nu.$$

We can replace the given family of metrics $\{g_L\}$ on the leaves by any other family $\{w_L\}$ of leafwise smooth transversely measurable leafwise metrics provided that there are constants c_1 and c_2 so that

$$c_1 g_L \leq w_L \leq c_2 g_L$$

for all L. Under such a change of metric, the Hilbert spaces H_x, the square integrable sections $\tilde{H}$ and the von Neumann algebra $W(F; E)$ are unchanged. In particular, they are all independent of the metric g on M.

We will be primarily interested in the bundles $E_k = \wedge^k T^*F \otimes \mathbf{C}$ where T^*F is the cotangent bundle of F. The smooth sections of $E_k|L$ are the smooth k forms on L. Denote by $C_0^\infty(E_k|L)$ the smooth sections of compact support, and by

$$d_k^L \colon C_0^\infty(E_k|L) \to C_0^\infty(E_{k+1}|L)$$

leafwise exterior differentiation. Using the leafwise metrics to construct inner products on $L^2(E_k|L)$, we have the leafwise Laplacians

$$\Delta_k^L = d_{k-1}^L d_{k-1}^{L*} + d_k^{L*} d_k^L.$$

Δ_k^L is an unbounded self adjoint non-negative operator on $L^2(E_k|L)$. We denote by Δ_k, d_k and d_k^* the families $\{\Delta_k^L\}$, $\{d_k^L\}$ and $\{d_k^{L*}\}$.

Now let M' be a second compact oriented Riemannian manifold with oriented foliation F' and invariant transverse measure ν'. Let $f\colon M \to M'$ be a continuous map which takes each leaf of F to a leaf of F'.

DEFINITION 1.1. f is measure preserving if for each transversal T of F and T' of F' for which the restriction $f\colon T \to T'$ is one-to-one and onto, we have $\nu(T) = \nu'(T')$.

We assume that f is measure preserving and that it is a leafwise homotopy equivalence. That is, there is a continuous map $f'\colon M' \to M$ which takes each leaf of F' to a leaf of F and such that $f'f$ and ff' are homotopic to the identity by homotopies which take each leaf of F (respectively F') to itself.

In [H-L2], we construct a von Neumann algebra $W_k(f, f')$ containing the von Neumann algebras $W(F; E_k)$ and $W(F'; E_k')$ where $E_k' = \wedge^k T^*F' \otimes \mathbf{C}$. The assumptions on f and f' imply that there is a trace $\operatorname{tr}_{\nu\otimes\nu'}$ on $W_k(f, f')$ which restricts to the traces tr_ν and $\operatorname{tr}_{\nu'}$ on $W(F; E_k)$ and $W(F'; E_k')$. We briefly describe this construction. For more details see [H-L2], §4.

For $x \in M$, let

$$\begin{aligned} H_x'' &= L^2(L_x; E_k|L_x) \otimes L^2(L'_{f(x)}; E_k'|L'_{f(x)}) \\ &= H_x \otimes H'_{f(x)}. \end{aligned}$$

Given a section s' of $\{H'_y\}$, let $f^*(s')(x) = s'(f(x))$. The measurable structures $\{s_n\}$ and $\{s'_m\}$ on $\{H_x\}$ and $\{H'_y\}$ define a measurable structure $\{s_n\} \cup \{f^*(s'_m)\}$ on $\{H''_x\}$. An inner product on sections $s = (s_1, s_2)$ and $t = (t_1, t_2)$ of $\{H''_x\}$ is given by

$$\langle s, t\rangle = \int_M \big((s_1(x), t_1(x))_x + (s_2(x), t_2(x))_{f(x)}\big)\, d\mu(x).$$

Let $\tilde{H}''$ be the collection of all square integrable sections. Then $\tilde{H}'' = \tilde{H} \oplus f^*\tilde{H}'$ where $f^*\tilde{H}'$ denotes the set of square integrable $f^*(s')$ where s' is a measurable section of $\{H'_y\}$. We now describe generators for $W_k(f, f')$ and how they act on the ambient space $\tilde{H}''$. As elements of $C_0^\mu(U, V, \gamma)$ define bounded operators on $\tilde{H}$, they also give bounded operators on $\tilde{H}'' = \tilde{H} \oplus f^*\tilde{H}'$ by acting as the zero operator on $f^*\tilde{H}'$. Similarly elements of $C_0^\mu(U', V', \gamma')$ define bounded operators on $\tilde{H}'$ so also on $f^*\tilde{H}'$ which extend to bounded operators on $\tilde{H}''$. Now let U be a foliation chart on M, U' and V' foliation charts on M' and γ' a leafwise path from U' to V'. Let $U \times_{\gamma' f} V'$ be the subset of $U \times V'$ consisting of all (x, y) where $x \in U$, $f(x) \in$ domain of $h_{\gamma'}$, $y \in h_{\gamma'}(P'_{f(x)})$ where $P'_{f(x)}$ is the placque of U' containing $f(x)$. Let $C_0^\mu(U, V', \gamma' f)$ consist of all leafwise smooth, transversely measurable, bounded sections $k(u, v)$ of $E_k \boxtimes E_k'^*$ over $M \times M'$ which are compactly supported on $U \times_{\gamma' f} V'$. Such a k defines a bounded operator from $f^*\tilde{H}'$ to $\tilde{H}$ given by

$$(ks')(x)(u) = \int_{h_{\gamma'}(P'_{f(x)})} k(u, v)(s'(f(x))(v))\, d\lambda'(v),$$

and so also a bounded operator on $\tilde{H}''$. Similarly we define $C_0^\mu(U', V, \gamma f')$ where γ is a leafwise path in M. Each $k \in C_0^\mu(U', V, \gamma f')$ defines a bounded operator from $\tilde{H}$ to $f^*\tilde{H}'$ and so also a bounded operator on $\tilde{H}''$.

$W_k(f, f')$ is the von Neumann algebra generated by the sets of bounded operators $C_0^\mu(U, V, \gamma)$, $C_0^\mu(U', V', \gamma')$, $C_0^\mu(U, V', \gamma' f)$ and $C_0^\mu(U', V, \gamma f')$ where U and V are in a fixed cover of M, U' and V' in a fixed cover of M' and γ and γ' range over the holonomy classes of leafwise paths.

We will also need the Sololev spaces associated to $C_0^\infty(E_k|L)$ ([D], §2, [H-L2], §3). Let $A^{s,k}(L)$ be the completion of $C_0^\infty(E_k|L)$ in the norm

$$\|w\|_{s,L}^2 = \left(\left(1 + \Delta_k^L\right)^{s/2} w, \left(1 + \Delta_k^L\right)^{s/2} w\right).$$

The measurable family $\{A_x^{s,k}\}$ where $A_x^{s,k} = A_{L_x}^{s,k}$ has a measurable structure inherited from $\{L^2(E_k|L_x)\}$. Denote by $\tilde{A}^{s,k}$ the direct integral of $\{A_x^{s,k}\}$ and by $\| \quad \|_s$ its norm.

In order to make use of the leafwise simplicial structure, we will need the bounded leafwise triangulations K constructed in [H-L2], §2. K is a family $\{K_L\}$, where K_L is a triangulation of the leaf L. The K_L are leafwise smooth and transversely measurable. They possess the crucial property that the volumes and diameters of the simplices are uniformly (over all leaves) bounded away from zero. We denote by S^rK the rth standard subdivision of a bounded triangulation K. Given a continuous map $f\colon M \to M'$ taking leaves to leaves and bounded leafwise triangulations K and K', there is an integer r and a Borel map $g\colon M \to M'$ such that for each leaf L, $g(L) \subset f(L)$ and g restricts to a simplicial map from $S^r(K_L)$ to $K_{f(L)}$ which is a simplicial approximation to f. For details, see [H-L2], §2.

Given a bounded triangulation K of M, denote by $C_2^k(K_L)$ the completion of the finite simplicial cochains $C^k(K_L)$ in the inner product $(g, h) = \Sigma_\tau g(\tau)h(\tau)$, the sum being over all k simplices in K_L. The simplicial coboundary operator gives rise to a bounded operator

$$\delta\colon C_2^k(K_L) \to C_2^{k+1}(K_L).$$

We form the measurable family $\{C_2^k(K_x)\}$ where $C_2^k(K_x) = C_2^k(K_{L_x})$. A measurable structure on this family is given in the appendix of [H-L2]. We can then form the direct integral $\tilde{C}_2^k(K)$. The map $\int\colon A^{s,k}(K_L) \to C_2^k(K_L)$, ([D], §3) given by $(\int w)(\sigma) = \int_\sigma w$ and the Whitney map $W\colon C_2^k(K_L) \to A^{s,k}(K_L)$, ([D], [W], p. 138), give rise to maps $\int\colon \tilde{A}^{s,k} \to \tilde{C}_2^k(K)$ and $W\colon \tilde{C}_2^k(K) \to \tilde{A}^{s,k}$, with $\int \cdot W = Id$. Details of this construction can be found in [H-L2]. As above, for the foliation F' on M', these maps will be denoted by $\int'$ and W'.

2. Definition of the spectral function

Let M, F and ν be as above. Following [Ef-Sh], we introduce the spectral functions $N_\nu(\lambda)$. Fix an integer k and set $E = E_k$, $\Delta^L = \Delta_k^L$, $\Delta^x = \Delta^{L_x}$ and $\Delta = \{\Delta^x\}$. Δ^x extends to an unbounded operator on $A_x^{s,k}$ for all s and x. Thus if h is a bounded Borel function on $\mathbf{R}$, then $h(\Delta^x)$ is a bounded operator on $A_x^{s,k}$ for all s and x, and so defines a bounded operator $h(\Delta)$ on $\tilde{A}^{s,k}$. In particular $h(\Delta)$ is a bounded operator on $\tilde{H} = \tilde{A}^{0,k}$.

Let χ_λ be the characteristic function of the interval $[0, \lambda]$.

THEOREM 2.1. *$\chi_\lambda(\Delta)$ lies in $W(F; E)$ and* $\mathrm{tr}_\nu(\chi_\lambda(\Delta)) < \infty$.

LEMMA 2.2. *If h is a bounded Borel function with compact support then $h(\Delta^L)$ is a smoothing operator on $A^{0,k}(L) = L^2(L, E|L)$ so the Schwartz kernel $k(x, y)$ of the operator $h(\Delta)$ on $\tilde{H}$ is leafwise smooth. If h is the pointwise limit of uniformly bounded smooth functions, then $k(x, y)$ is measurable on $M \times M$.*

Proof. Let $\|h(\Delta^L)\|_{0,s}$ be the norm of the operator $h(\Delta^L)$: $A^{0,k}(L) \to A^{s,k}(L)$. By the spectral theorem,

$$\|h(\Delta^L)\|_{0,s} \leq \sup(1+\lambda)^{s/2} h(\lambda) < \infty,$$

so $h(\Delta^L)$ is smoothing.

Now if $k_L(x, y)$ is the Schwartz kernel of $h(\Delta^L)$, then the Schwartz kernel $k(x, y)$ of $h(\Delta)$ is

$$k(x,y) = \begin{cases} k_L(x,y) & \text{if } x, y \in L \\ 0 & x, y \text{ not on same leaf.} \end{cases}$$

Let $h(\lambda) = \lim_{n\to\infty} h_n(\lambda)$ where the limit is pointwise, the h_n are smooth with uniform compact support and uniformly bounded. Let $k_n(x, y)$ be the Schwartz kernel of $h_n(\Delta)$. Theorem (2.3.9) of [H-L1] says that $k_n(x, y)$ is (leafwise smooth and) measurable on $M \times M$.

Next, assume for the moment that E is a one dimensional trivial bundle. Let δ_x^L be the δ function at $x \in L$. By the spectral theorem,

$$k_n(x,y) = \left(h_n(\Delta^L)\delta_x^L, \delta_y^L\right) \to \left(h(\Delta^L)\delta_x^L, \delta_y^L\right) = k(x,y)$$

where the convergence is pointwise. In addition, $|k_n(x, y)|$ is bounded by

$$\left\|(1+\Delta^L)^s h_n(\Delta^L)\right\|_{0,0} \|\delta_x^L\|_{-s} \|\delta_y^L\|_{-s}.$$

As $\|\delta_x^L\|_{-s}$ is (for fixed $s > \dim L/2$) bounded independently of x and L and

$$\left\|(1+\Delta^L)^s h_n(\Delta^L)\right\|_{0,0} \leq \sup(1+\lambda)^s h_n(\lambda)$$

which is bounded independently of n, it follows that $|k_n(x, y)|$ is uniformly bounded. Thus $k(x, y)$ is measurable.

For arbitrary E, we merely replace δ_x^L and δ_y^L by the δ sections $\delta_x^{v,X}$ and $\delta_y^{w,Y}$ of the proof of (2.3.9) of [H-L1].

Note that $\int_M \operatorname{tr} k_n(x, x)\, d\mu < \infty$ and that this integral converges to

$$\int_M \operatorname{tr} k(x,x)\, d\mu(x) \quad \text{as } n \to \infty$$

by dominated convergence. As $|\operatorname{tr} k_n(x, x)|$ is bounded, the latter integral is finite and we may interpret it as $\operatorname{tr}_\nu(h(\Delta))$ once we have the following.

LEMMA 2.3. *If h is a bounded Borel function with compact support which is the pointwise limit of uniformly bounded smooth functions, then $h(\Delta) \in W(F; E)$.*

Proof. Suppose that h is smooth. As in [Roe], write h as a limit of smooth uniformly bounded functions h_n whose Fourier transforms are compactly supported. By Theorem 2.3.7 of [H-L1], $h_n(\Delta) \in W(F; E)$. For each $x \in M$, $h_n(\Delta^x) \to h(\Delta^x)$ strongly. If T and the sequence T_n are decomposable operators on $\tilde{H}$ with $\sup\|T_n\| < \infty$ for almost all x, and T_n^x converges strongly to T^x, then T_n converges strongly to T [Dix], p. 183. Thus $h_n(\Delta) \to h(\Delta)$ strongly and $h(\Delta) \in W(F; E)$.

For arbitrary h, write h as a pointwise limit of h_n where each h_n is smooth and compactly supported and the h_n are uniformly bounded. Then for each $x \in M$, $h_n(\Delta^x) \to h(\Delta^x)$ strongly. By Lemma 2.2, $\{h(\Delta^x)\}$ is a measurable family of uniformly bounded operators, hence it defines a bounded operator $h(\Delta)$ on $\tilde{H}$. As $h_n(\Delta) \to h(\Delta)$ strongly, $h(\Delta) \in W(F; E)$.

DEFINITION 2.4. $N_\nu^k(\lambda) = \mathrm{tr}_\nu(\chi_\lambda(\Delta))$. $N_\nu^k(\lambda)$ is the (kth) spectral distribution function of F and ν.

Let $\mathfrak{S}_\lambda$ be the set of orthogonal projections P on $\tilde{H}$ which satisfy the conditions

(i) $P \in W(F; E)$
(ii) Image of $P \subset$ domain of Δ
(iii) $P(\Delta - \lambda)P \leq 0$.

The proof of Theorem (3.1) of [Ef-Sh] in the context of $W(F; E)$ shows

$$N_\nu^k(\lambda) = \sup \mathrm{tr}_\nu(P) \tag{2.5}$$

where the sup is taken over $P \in \mathfrak{S}_\lambda$.

Following [Ef-Sh] and [Ef], we consider non-decreasing functions $F(\lambda)$ on **R** which vanish on $(-\infty, 0)$. We write $F \ll G$ if there is $C > 0$ and $\lambda_0 > 0$ such that $F(\lambda) \leq G(C\lambda)$ for $\lambda \leq \lambda_0$. If $F \ll G$ and $G \ll F$, we say F and G are dilationally equivalent near zero.

Let (M, F, ν) and (M', F', ν') be as in section one. One of our main results is the following foliation version of the main result of [G-Sh].

THEOREM 2.6. *Suppose there is a measure preserving leafwise homotopy equivalence between (M, F, ν) and (M', F', ν'). Then the spectral distribution functions $N_\nu^k(\lambda)$ and $N_{\nu'}^k(\lambda)$ are dilationally equivalent near zero.*

As in [Ef-Sh], we can reformulate this result in terms of the leafwise heat kernels $e^{-t\Delta}$. For non-increasing functions on $\mathbf{R}^+$, we write $f \ll g$ if there is $c > 0$ and $t_0 > 0$ such that $f(t) < g(ct)$ for $t \geq t_0$. If $f \ll g$ and $g \ll f$, we say f and g dilationally equivalent near ∞.

For a non-decreasing function N, let $\Theta(t) = \int_0^\infty e^{-\lambda t}\, dN(\lambda)$. In [Ef-Sh], it is observed that N_1 is dilationally equivalent to N_2 near zero if and only if Θ_1 is dilationally equivalent to Θ_2 near ∞. As above we fix k and write $N_\nu(\lambda)$ for $N_\nu^k(\lambda)$ and $\Theta_\nu(t)$ for

$$\Theta_\nu^k(t) = \int_{\mathbf{R}} e^{-\lambda t}\, dN_\nu(\lambda).$$

Lemma 2.7. $\Theta_\nu(t) = \operatorname{tr}_\nu(e^{-t\Delta})$.

Proof. $N_\nu(\lambda) = \operatorname{tr}_\nu(\chi_\lambda(\Delta)) = \int_M \operatorname{tr} k_\lambda(x, x)\, d\mu(x)$ where k_λ is the Schwartz kernel of $\chi_\lambda(\Delta)$. As in the proof of Lemma 2.2, we have $|\operatorname{tr} k_\lambda(x, x)| \leq C \cdot (1 + \lambda)^s$ for sufficiently large s and C independent of x and λ. Thus

$$N_\nu(\lambda) \leq C \cdot \left(\int_M d\mu\right)(1 + \lambda)^s$$

and we may integrate by parts to obtain

$$\begin{aligned}\Theta_\nu(t) &= t\int_{\mathbf{R}} e^{-\lambda t} N_\nu(\lambda)\, d\lambda \\ &= t\int_{\mathbf{R}}\int_M e^{-\lambda t} \operatorname{tr} k_\lambda(x, x)\, d\mu(x)\, d\lambda.\end{aligned}$$

Let $k_t^L(x, y)$ be the Schwartz kernel of $e^{-t\Delta^L}$. Then $k_t^L(x, x) = \int_{\mathbf{R}} e^{-\lambda t} d(\operatorname{tr} k_\lambda^L(x, x))$ where k_λ^L is the kernel of $\chi_\lambda(\Delta^L)$. Thus

$$\begin{aligned}\operatorname{tr}_\nu(e^{-t\Delta}) &= \int_M \int_{\mathbf{R}} e^{-\lambda t} d\big(\operatorname{tr} k_\lambda^L(x, x)\big)\, d\mu(x) \\ &= t\int_M\int_{\mathbf{R}} e^{-\lambda t} \operatorname{tr}\big(k_\lambda^L(x, x)\big)\, d\lambda\, d\mu(x).\end{aligned}$$

As $k_\lambda^L(x, x) = k_\lambda(x, x)$, Tonelli's theorem gives the result.

Corollary 2.8. *Under the hypothesis of Theorem* (2.6), *we have*

$$\operatorname{tr}_\nu(e^{-t\Delta}) \quad \textit{and} \quad \operatorname{tr}_{\nu'}(e^{-t\Delta'})$$

are dilationally equivalent near ∞ *(here* Δ' *is the leafwise Laplacian on leafwise* k *forms on* M'*). In particular the dilational equivalence class of* $\operatorname{tr}_\nu(e^{-t\Delta})$ *is an invariant of* (M, F, ν) *and does not depend on the metric on* M.

Let $\beta_\nu = \beta_\nu^k$ be the kth foliation betti number of F (see [H-L2], §1). From [H-L1], (2.3.11), we have $\lim_{t\to\infty}\Theta_\nu(t) = \beta_\nu$. A similar argument shows that $\lim_{\lambda\to 0^+} N_\nu(\lambda) = \beta_\nu$. The main theorem of [H-L2] states that β_ν is invariant under measure preserving homotopy equivalence. The argument of [Ef-Sh], Proposition 5.3 shows.

Corollary 2.9. *Let* $\alpha > 0$. $N_\nu(\lambda) - \beta_\nu$ *is dilationally equivalent to* λ^α *near zero if and only if* $\operatorname{tr}_\nu(e^{t\Delta}) - \beta_\nu$ *is dilationally equivalent to* $\lambda^{-\alpha}$ *near* ∞. *The dilational equivalence classes are invariant under measure preserving leafwise homotopy equivalence.*

The Hodge decomposition on each leaf yields

$$L^2(L_x; E_k|L_x) = \ker(\Delta_k^x) \oplus \overline{\operatorname{im}(d_{k-1}^x)} \oplus \overline{\operatorname{im}(d_k^x)^*}$$

where d_{k-1}^x and $(d_k^x)^*$ are the usual operators on the leaf L_x. Denote by d_{k-1} and d_k^* the induced operators on $\tilde{H}$. Now Δ_k is a self adjoint operator on $\tilde{H}$ and we have a Hodge decomposition.

$$\tilde{H} = \ker\Delta_k \oplus \overline{\operatorname{im} d_{k-1}} \oplus \overline{\operatorname{im} d_k^*}\,.$$

As in [G-Sh], (3.4), we define the functions, $F_\nu^k(\lambda)$ and $G_\nu^k(\lambda)$ where $F_\nu^k(\lambda) = \sup\operatorname{tr}_\nu(P)$ where the sup is taken over the subset $\mathfrak{S}_{\lambda,\perp}$ of $\mathfrak{S}_\lambda$ consisting of those P with

$$\text{image } P \subset \overline{\operatorname{im} d_k^*} = (\ker d_k)^\perp\,.$$

Similarly $G_\nu^k(\lambda) = \sup\operatorname{tr}_\nu(P)$ where the sup is over the subset of $\mathfrak{S}_\lambda$ consisting of those P with

$$\text{image } P \subset \overline{\operatorname{im} d_{k-1}} = (\ker d_{k-1}^*)^\perp\,.$$

The proofs of [G-Sh] extend to our situation to give

$$N_\nu^k(\lambda) = G_\nu^k(\lambda) + \beta_k^\nu + F_\nu^k(\lambda)$$

and

$$F_\nu^k(\lambda) = G_\nu^{k+1}(\lambda).$$

As β_k^ν is an invariant under measure preserving leafwise homotopy equivalence, we need only prove theorem (2.6) for $F_\nu^k(\lambda)$.

It is convenient to have a second definition of $F_\nu^k(\lambda)$, as in [Ef]. Namely, let S_λ be the set of orthogonal projections P on $\tilde{H}$ such that

(i) $P \in W(F; E)$
(ii) image $P \subset (\ker d_k)^\perp \cap E_\lambda$

where $E_\lambda = \text{image}\, \chi_\lambda(\Delta_k)$.

PROPOSITION 2.10. $F_\nu^k(\lambda) = \sup \operatorname{tr}_\nu(P)$ *where the sup is over* $P \in S_\lambda$.

Proof. We again drop the k. $\chi_\lambda(\Delta)$ is a bounded smoothing operator on $\tilde{H}$ so $E_\lambda \subset \tilde{A}^{s,k}$ for all $s \geq 0$ and $E_\lambda \subset$ domain Δ. Observe that for $w \in E_\lambda$, $\langle \Delta w, w\rangle \leq \lambda \|w\|^2$. Thus $S_\lambda \subset \mathfrak{S}_{\lambda,\perp}$.

LEMMA 2.11. *Let Q be projection onto* $(\ker d)^\perp$. *Then* $Q \in W(F; E)$.

Proof. Denote the characteristic function of $(0, \infty)$ by χ. Then $Q = Q\chi(\Delta)$. Choose $f_n \in C_0^\infty(\mathbf{R})$ so that

1. support $f_n \subset [1/(n+1), n+1]$,
2. $0 \leq f_n \leq 1$,
3. $f_n|_{[1/n, n]} \equiv 1$.

Then for each x, $f_n(\Delta^x)$ converges strongly to $\chi(\Delta^x)$ and $\|f_n\| \leq 1$. As Q is a uniformly bounded operator on each leaf, $Q^x f_n(\Delta^x)$ converges strongly to $Q^x\chi(\Delta^x) = Q^x$. Thus we need only show that $Qf_n(\Delta) \in W(F; E)$. Let $\chi_{[\lambda,\infty)}$ be the characteristic function of $[\lambda, \infty)$, $\lambda > 0$ and note that on the image of $\chi_{[\lambda,\infty)}$, $Q = d^*d\Delta^{-1}$. Let $h_{n,m}$ be a sequence of functions as in the proof of (2.3) converging to f_n. By (2.3.7) of [H-L1], $h_{n,m}(\Delta)$ is a finite sum of elements of the $C_0^\mu(U, V, \gamma)$. As d^*d is a measurable leafwise differentiable operator, $d^*dh_{n,m}(\Delta)$ is also such a finite sum, i.e., $d^*dh_{n,m}(\Delta) \in W(F; E)$. Set $g_n(x) = (1/x)f_{n+2}(x)$. By (2.3), $g_n(\Delta) \in W(F; E)$. Thus

$$\begin{aligned} g_n(\Delta)d^*\, dh_{n,m}(\Delta) &= d^*\, dg_n(\Delta)h_{n,m}(\Delta) \\ &= d^*\, df_{n+4}(\Delta)g_n(\Delta)h_{n,m}(\Delta) \\ &= d^*\, dg_{n+2}(\Delta)f_{n+2}(\Delta)h_{n,m}(\Delta) \\ &= Qf_{n+2}(\Delta)h_{n,m}(\Delta) \in W(F; E). \end{aligned}$$

For each x, $h_{n,m}(\Delta^x)$ converges strongly to $f_n(\Delta^x)$. $Q^x f_{n+2}(\Delta^x)$ is uniformly bounded so $Q^x f_{n+2}(\Delta^x) h_{n,m}(\Delta^x)$ converges strongly to $Q^x f_{n+2}(\Delta^x) f_n(\Delta^x)$. Thus

$$Qf_{n+2}(\Delta) f_n(\Delta) \in W(F; E).$$

But

$$Qf_{n+2}(\Delta) f_n(\Delta) = Qf_n(\Delta).$$

Let P_λ be projection onto $(\ker d)^\perp \cap E_\lambda$. As $P_\lambda = \chi_\lambda(\Delta) Q$, (see below), $P_\lambda \in W(F; E)$, so $P_\lambda \in S_\lambda$. We claim that for $P \in \mathfrak{S}_{\lambda, \perp}$, $\operatorname{tr}_\nu(P)$ takes its supremium on P_λ. The Proposition then follows. To see the claim, note that $d\Delta = \Delta d$ implies that

$$E_\lambda = (E_\lambda \cap \ker d) \oplus \left(E_\lambda \cap (\ker d)^\perp\right).$$

Thus $\chi_\lambda(\Delta)|(\ker d)^\perp$ is the projection onto $(\ker d)^\perp \cap E_\lambda$. Given $P \in \mathfrak{S}_{\lambda, \perp}$, set $V_P = \text{image } P$. Then $\chi_\lambda(\Delta)|_{V_P}$ is $1-1$. If not, there is a nonzero $w \in V_P$ such that $w \in \text{image } \chi_{(\lambda,\infty)}(\Delta)$, where $\chi_{(\lambda,\infty)}$ is the characteristic function of (λ, ∞). Now Δ preserves this space and its restriction to it is invertible with inverse bounded by $1/\lambda$. Thus

$$\begin{aligned}\langle w, w\rangle &= \langle \Delta^{-1}\Delta w, w\rangle = \langle \Delta^{-1}\Delta^{1/2} w, \Delta^{1/2} w\rangle \\ &\le \|\Delta^{-1}\| \, \|\Delta^{1/2} w\|^2 \le \frac{1}{\lambda}\langle \Delta^{1/2} w, \Delta^{1/2} w\rangle = \frac{1}{\lambda}\langle \Delta w, w\rangle.\end{aligned}$$

So $\langle \Delta w, w\rangle \ge \lambda\langle w, w\rangle$. As $w \in V_P$, $\langle \Delta w, w\rangle = \lambda\langle w, w\rangle$. But this is impossible since on each leaf, $\chi_{(\lambda+1/n,\infty)}(\Delta^L)$ converges strongly to $\chi_{(\lambda,\infty)}(\Delta^L)$. Set $w_n^L = \chi_{(\lambda+1/n,\infty)}(\Delta^L) w^L$ and write

$$w^L = w_n^L + \overline{w}_n^L.$$

Then $\overline{w}_n^L \to 0$ as $n \to \infty$ and $(\Delta w_n^L, \overline{w}_n^L) = 0 = (w_n^L, \overline{w}_n^L)$ as $\overline{w}_n^L \in \text{image } \chi_{(\lambda,\lambda+1/n]}(\Delta^L)$. Then

$$\begin{aligned}(\Delta^L w^L, w^L) &= \left(\Delta^L w_n^L + \Delta \overline{w}_n^L, w_n^L + \overline{w}_n^L\right) \\ &= \left(\Delta^L w_n^L, w_n^L\right) + \left(\Delta^L \overline{w}_n^L, \overline{w}_n^L\right) \\ &\ge \left(\lambda + \frac{1}{n}\right)\left(w_n^L, w_n^L\right) + \lambda\left(\overline{w}_n^L, \overline{w}_n^L\right) \\ &> \lambda\left(w_n^L + \overline{w}_n^L, w_n^L + \overline{w}_n^L\right) = \lambda(w^L, w^L).\end{aligned}$$

The strict inequality follows from the fact that for n large, $(w_n^L, w_n^L) \neq 0$. Thus $\langle \Delta w, w\rangle > \lambda\langle w, w\rangle$ a contradiction to $w \in V_P$.

Since $\chi_\lambda(\Delta)$ and P are in $W(F; E)$, P_1 = projection onto the closure of $\chi_\lambda(\Delta)(V_P)$ is in $W(F; E)$. If we write $\chi_\lambda(\Delta)P = U|\chi_\lambda(\Delta)P|$, the polar decomposition, we have $P = U^*U$ and $P_1 = UU^*$. It follows that $\mathrm{tr}_\nu(P) = \mathrm{tr}_\nu(P_1)$. But image $P_1 \subset (\ker d_k)^\perp \cap E_\lambda$ so $\mathrm{tr}_\nu(P_1) \leq \mathrm{tr}_\nu(P_\lambda)$.

We finish this section with two remarks. First, for $P \in \mathfrak{S}_{\lambda,\perp}$, we may replace condition (iii) of (2.4) by

$$(iii)' \qquad \|dw\| \leq \sqrt{\lambda}\,\|w\| \text{ for all } w \in \text{image } P$$

(see [G-Sh]). Second, note that for $w \in (\ker d)^\perp \cap E_\lambda$, $\langle \Delta w, w\rangle = \|dw\|^2$. Thus for such w,

$$\|dw\| \leq \sqrt{\lambda}\,\|w\|.$$

3. Proof of Theorem 2.4

The main part of the proof consists of constructing maps $\tilde{A}^{s,k}(M) \rightleftarrows \tilde{A}^{s,k}(M')$ which, composed with appropriate projections, give rise to isomorphisms which lie in the von Neumann algebra $W(f, f')$. This is based on the constructions of [H-L2].

Let s be a fixed integer, $s \gg \dim M, \dim M'$.

THEOREM 3.1. *Given bounded triangulations K and K' of M and M', there exist bounded operators*

$$G\colon \tilde{A}^{s,k}(M) \to \tilde{A}^{s,k}(M')$$
$$G'\colon \tilde{A}^{s,k}(M') \to \tilde{A}^{s,k}(M)$$

satisfying $d'G = Gd$, $dG' = G'd'$, and bounded operators

$$T\colon \tilde{A}^{s,k+1}(M) \to \tilde{A}^{s,k}(M)$$
$$T'\colon \tilde{A}^{s,k+1}(M') \to \tilde{A}^{s,k}(M')$$

such that

$$G'G = W\int + dT + Td$$

$$GG' = W'\int' + d'T' + T'd'.$$

We defer the proof of (3.1) to the end of this section.

As above, we denote the image of a projection P onto a closed subspace of $\tilde{H}$ by V_P. If $P \in W(F; E)$, $\dim_\nu(V_P)$ is by definition $\mathrm{tr}_\nu(P)$. $G|V_P$ will be denoted G_P. For $P \in S_\lambda$, we have $V_P \subset E_\lambda \subset \tilde{A}^{s,k}(M)$ for all s. For $w \in V_P$,

$$\begin{aligned}\|w\|_0^2 &\le \|w\|_s^2 = \langle w, w\rangle_s \\ &= \left\langle (1+\Delta)^{s/2} w, (1+\Delta)^{s/2} w\right\rangle \le (1+\lambda)^s \|w\|_0^2.\end{aligned}$$

Hence, $\| \quad \|_0$ and $\| \quad \|_s$ are equivalent norms on V_P. Let $\| \quad \|$ always denote $\| \quad \|_0$ and $\| \quad \|_\perp$ this norm restricted to $(\ker d)^\perp$. Similarly, $\| \quad \|_{s,\perp}$ will denote $\| \quad \|_s$ on $(\ker d)^\perp$. Let Q and Q' denote projection onto $(\ker d)^\perp$ and $(\ker d')^\perp$.

We now prove the analogue of proposition (4.1) of [G-Sh].

THEOREM 3.2. *There exist constants $C > 0$ and $\lambda_0 > 0$ such that for $\alpha \in V_P$, $P \in S_\lambda$,*

$$\left\| d'(1+\Delta')^{s/2} G(\alpha) \right\| \le C\sqrt{\lambda} \left\| (1+\Delta')^{s/2} G(\alpha)\right\|_\perp$$

for $\lambda \in (0, \lambda_0)$. The map

$$\begin{aligned} Q'(1+\Delta')^{s/2} G_P \colon V_P \to (\ker d')^\perp \\ \simeq \tilde{A}^{0,k}(M') \operatorname{mod}(\ker d')\end{aligned}$$

is injective with image a closed subspace.

Proof. First we recall a result from the proof of (3.14) of [H-L2]. Given a bounded triangulation K of M, there is a constant C_1 such that for any standard subdivision $S^r(K)$ and any $\alpha \in \tilde{A}^{s,k}(M)$,

$$\left\| \left(W_0 \int - I \right) \alpha \right\| \le \eta_r C_1 \|\alpha\|_s \tag{3.3}$$

where η_r is the mesh of $S^r(K)$, C_1 is independent of r, and W_0 is the modified Whitney map corresponding to the bounded triangulation $S^r(K)$. This fact is based on (7.19) of [D-P]. With some minor care in the construction, we may use a smooth Whitney map in place of W_0. Let $\lambda_1 > 0$ be fixed. Choose r large enough so that $C_1 \eta_r (1+\lambda_1)^s < 1/4$. We assume $\lambda < \lambda_1$ in the subsequent discussion. Replace K by $S^r K$ and let W be a Whitney map for which (3.3) holds. Let G, G', T, and T' be as in (3.1). Using the facts that G is bounded and d preserves E_λ, we have for any $\alpha \in V_P$

$$\begin{aligned}\left\| d'(1+\Delta')^{s/2} G(\alpha)\right\| &= \|d' G(\alpha)\|_s \\ &= \|G\, d\alpha\|_s \le \|G\|_s \|d\alpha\|_s \\ &= \|G\|_s \left\|(1+\Delta)^{s/2}\, d\alpha\right\| \le \|G\|_s (1+\lambda_1)^s \|d\alpha\| \\ &\le \|G\|_s (1+\lambda_1)^s \sqrt{\lambda}\, \|\alpha\|.\end{aligned} \tag{3.4}$$

Write $G(\alpha) = \alpha_0' + \alpha_1'$ where $\alpha_0' \in \ker d'$, $\alpha_1' \in (\ker d')^\perp$ in the $\langle \ , \ \rangle_s$ metric. From (3.1) we have

$$G'G(\alpha) = \alpha + \left(W\int - I\right)\alpha + dT\alpha + Td\alpha$$

and as $G'G(\alpha) = G'(\alpha_1') \bmod(\ker d)$ we have

$$\alpha = G'(\alpha_1') - \left(W\int - I\right)\alpha - Td\alpha \quad \bmod(\ker d).$$

Now $\alpha \in (\ker d)^\perp$ so

$$\begin{aligned}\|\alpha\| &= \left\| Q\left(G'(\alpha_1') - \left(W\int - I\right)\alpha - Td\alpha\right)\right\|_\perp \\ &\leq \left\| G'(\alpha_1') - \left(W\int - I\right)\alpha - Td\alpha \right\| \\ &\leq \|G'(\alpha_1')\| + \left\| \left(W\int - I\right)\alpha \right\| + \|Td\alpha\|.\end{aligned}$$

Also

$$\left\| \left(W\int - I\right)\alpha \right\| \leq \eta_r C_1 \|\alpha\|_s \leq \eta_r C_1 (1 + \lambda_1)^s \|\alpha\| \leq \tfrac{1}{4}\|\alpha\|,$$

and

$$\|Td\alpha\|_s \leq \|T\|_s \|d\alpha\|_s \leq \|T\|_s (1 + \lambda)^s \|d\alpha\| \leq \|T\|_s (1 + \lambda)^s \sqrt{\lambda}\, \|\alpha\|.$$

Choose $\lambda_0 < \lambda_1$ so that

$$\|T\|_s (1 + \lambda)^s \sqrt{\lambda} \leq \tfrac{1}{4} \text{ for } 0 < \lambda < \lambda_0.$$

As $\|G'(\alpha_1')\| \leq \|G'(\alpha_1')\|_s \leq \|G'\|_s \|\alpha_1'\|_s$ we have for $\lambda < \lambda_0$

$$\|\alpha\| \leq \|G'\|_s \|\alpha_1'\|_s + \tfrac{1}{2}\|\alpha\|,$$

or

$$\|\alpha\| \leq 2\|G'\|_s \|\alpha_1'\|_s = 2\|G'\|_s \|G(\alpha)\|_{s,\perp}.$$

It is easy to see that $\|G(\alpha)\|_{s,\perp} = \|(1 + \Delta')^{s/2} G(\alpha)\|_\perp$. Thus

$$\|\alpha\| \leq 2\|G'\|_s \left\|(1 + \Delta')^{s/2} G(\alpha)\right\|_\perp. \tag{3.5}$$

This combined with (3.4) gives the first part of the theorem. As

$$\left\|(1+\Delta')^{s/2}G(\alpha)\right\|_{\perp}=\left\|Q'(1+\Delta')^{s/2}G(\alpha)\right\|,$$

(3.5) gives the second part.

THEOREM 3.6. *Let $P \in S_\lambda$ and P' be the projection onto the image of*

$$Q'(1+\Delta')^{s/2}G_P \subset (\ker d')^{\perp}.$$

Then

$$P' \in W(F'; E') \quad \text{and} \quad \dim_\nu V_P = \dim_{\nu'} V_{P'}.$$

A similar statement holds for $P' \in S'_\lambda$.

We defer the proof of this as it involves the construction in the proof of (3.1).

THEOREM 3.7. *For $\lambda \in (0, \lambda_0)$, $F_\nu^k(\lambda) \leq F_{\nu'}^k(C^2\lambda)$ where C is the constant in Theorem* 3.2.

Proof. Let $P \in S_\lambda$ and P' as in (3.6). By (3.6), $P' \in W(F'; E')$ with

$$\text{image } P' \subset (\ker d')^{\perp} = \overline{\operatorname{im} d'^*} \subset \operatorname{dom} d'^*.$$

Let $\alpha' \in$ image P' and set

$$\alpha' = Q'(1+\Delta')^{s/2}G(\alpha) \quad \text{for } \alpha \in V_P.$$

As $\alpha \in \operatorname{dom} d$, $(1+\Delta')^{s/2}G(\alpha) \in \operatorname{dom} d'$. Now

$$(1-Q')(1+\Delta')^{s/2}G(\alpha) \in \ker d' \subset \operatorname{dom} d'.$$

Thus $\alpha' \in \operatorname{dom} d'$, so image $P' \subset \operatorname{dom} \Delta'$.

As Q' is projection onto $(\ker d')^{\perp}$ we have

$$d'\alpha' = d'\big(Q'(1+\Delta')^{s/2}G(\alpha)\big) = d'(1+\Delta')^{s/2}G(\alpha).$$

Now $\|(1+\Delta')^{s/2}G(\alpha)\|_{\perp} = \|Q'(1+\Delta')^{s/2}G(\alpha)\|$ so (3.2) implies that

$$\|d'\alpha'\| \leq \sqrt{C^2\lambda}\,\|\alpha'\|.$$

Thus $P' \in \mathfrak{S}'_{C^2\lambda,\perp}$ and as $\dim_\nu V_P = \dim_{\nu'} V_{P'}$ we are done.

By symmetry, $F^k_{v'}(\lambda) \le F^k_v(C'^2\lambda)$ for some constant C' so we have proven Theorem (2.4).

Proof of 3.1. We recall the constructions of §3 of [H-L2]. Choose j large enough so that for each simplex $\sigma \in S^j(K_L)$, and each leaf L of F, $\overline{st(\sigma)}$ is contained in the first regular neighborhood of a simplex of K_L. Next, choose l large enough so that each simplex $\sigma' \in S^l(K'_L)$ lies in the first regular neighborhood of a simplex of K'_L and so that there is a measurable leafwise simplicial approximation g' of f', g': $S^k(K') \to S^j(K)$. Choose k large enough so that there is a measurable leafwise simplicial approximation g of f, g: $S^k(K) \to S^l(K')$. Then G is the composition

$$\begin{aligned}\tilde{A}^{s,*}(M) \xrightarrow{\int} \tilde{C}_2^*(K) \xrightarrow{\tilde{\pi}_1} \tilde{C}_2^*(S^jK) \xrightarrow{\tilde{g}'} \tilde{C}_2^*(S^lK')\\ \xrightarrow{\tilde{S}^l} C_2^*(K') \xrightarrow{W'} \tilde{A}^{s,*}(M')\end{aligned}$$

and G' is the composition

$$\begin{aligned}\tilde{A}^{s,*}(M') \xrightarrow{\int'} \tilde{C}_2^*(K') \xrightarrow{\tilde{\pi}_2} \tilde{C}_2^*(S^lK') \xrightarrow{\tilde{g}} \tilde{C}_2^*(S^kK)\\ \xrightarrow{\tilde{S}^k} \tilde{C}_2^*(K) \xrightarrow{W} \tilde{A}^{s,*}(M).\end{aligned}$$

Here π_1 and π_2 are the induced leafwise simplicial maps from canonical maps $S^jK \to K$ and $S^lK' \to K'$. S^l and S^k are induced from the natural chain maps. From [H-L2], §3 we have the formula, at the chain level on each leaf,

$$\begin{aligned}(3.8)\qquad \pi^j_{\#}S^m g'_{\#} g_{\#} S^k(\tau) - \pi^k_{\#}S^{m+k}(\tau) &= \partial(\bar{\gamma}S^m rDS^k)(|\tau|)\\ &\quad + (\bar{\gamma}S^m rDS^k)\partial(|\tau|)\end{aligned}$$

where π^j: $S^{m+j}K \to K$ and π^k: $S^{m+k}K \to K$ are canonical simplicial maps, D takes singular i simplices to singular $i+1$ prisms, r takes singular $i+1$ prisms to singular $i+1$ chains, $\bar{\gamma}$ takes singular simplices supported in the first regular neighborhood of K to simplices of K, and π^j and π^k are chosen so that $\pi^j_{\#}(\rho) = \bar{\gamma}(|\rho|)$ for each simplex ρ, and similarly for π^k. The $\#$ indicates the induced map on chains. On S^lK, $S^l\pi_{2,\#}$ is chain homotopic to the identity, so we may write $S^l\pi_{2,\#} = 1 + D'\partial + \partial D'$ for some D'. Thus

$$(3.9)\quad \pi_{1,\#}g'_{\#}S^l\pi_{2,\#}g_{\#}S^k - \pi_{1,\#}g'_{\#}g_{\#}S^k = \pi_{1,\#}g'_{\#}(D'\partial + \partial D')g_{\#}S^k.$$

In (3.8), use $\pi^j_{\#}S^m = \pi_{1,\#}$ and $\pi^k_{\#}S^{m+k} = I$ and add (3.8) to (3.9) to obtain

$$(3.10)\qquad \left(\pi_{1,\#}g'_{\#}S^l\right)\left(\pi_{2,\#}g_{\#}S^k\right) - I = \partial T_0 + T_0\partial$$

where $T_0 = \bar{\gamma} S^m r D S^k + \pi_{1,\#} g'_\# D' g_\# S^k$. Now combine the cochain version of (3.10) with W, W', $\int$ and $\int'$. Since $\int' W' = I$, we have

$$(3.11)\quad \left(WS^k g^\# \pi_2^\# \int'\right)\left(W' S^l g'^\# \pi_1^\# \int\right) = W\int + d\left(WT_0 \int\right) + \left(WT_0\int\right)d.$$

We observe that from the constructions in §3 of [H-L2], the operator T_0 and hence $T = WT_0\int$ are bounded operators whose operator norms are uniformly bounded over all leaves. Passing to the direct integral we have $G'G = W\int + dT + Td$. By an identical construction we have $GG' = W'\int' + d'T' + T'd'$.

Proof of 3.6. We recall from [H-L2], (4.2) that tr_ν, $\mathrm{tr}_{\nu'}$ and f (since f is measure preserving) give rise to a trace, $\mathrm{trace}_{\nu\oplus\nu'}$ on $W(f, f')$ which restricts to tr_ν and $\mathrm{tr}_{\nu'}$ on the appropriate subalgebras. For a projection $P \in W(F; E)$, it follows from [Dix], p. 187 that P is decomposable, $P = \{P_x\}$ for $x \in M$. Similarly for $P' \in W(F', E')$. Thus any projection $P \in W(F; E)$ determines a projection $P \in W(f, f')$ and $\mathrm{tr}_{\nu\oplus\nu'}(P) = \mathrm{tr}_\nu(P)$. Similarly $P' \in W(F'; E')$ determines $P' \in W(f, f')$, since f is a leafwise homotopy equivalence, and $\mathrm{tr}_{\nu\oplus\nu'}(P') = \mathrm{tr}_{\nu'}(P')$.

Now let $P \in S_\lambda$. Then $G(V_P) \subset \tilde{A}^{s,k}(M')$, and $W_P = Q'(1 + \Delta')^{s/2} G(V_P)$ is a closed subspace of $\tilde{A}^{0,k} = \tilde{H}'$. W_P determines a closed subspace, also denoted W_P, of $f^*(\tilde{H}')$. Let P' be the projection onto W_P in $\tilde{H}'$. We have a decomposition

$$\tilde{H}'' = V_P \oplus V_P{}^\perp \oplus W_P \oplus W_P{}^\perp$$

where $V_P{}^\perp$ is the orthogonal complement of V_P in $\tilde{H}$ and $W_P{}^\perp$ that of W_P in $f^*(\tilde{H}')$. Let P_1, P_2, P_3, P_4 be the projections onto these subspaces. $P_1 = P \in W(F; E)$ so P_1 and also P_2 lie in $W(f, f')$. Denote $Q'(1 + \Delta')^{s/2} G P_1$ by A_P. From (3.2) we have $\|\alpha\| \le C\|A_P(\alpha)\|$ for $\alpha \in V_P$. Hence $A_P^{-1} = (A_P|_{VP})^{-1} P_3$ is well defined and bounded. We define

$$T_P\colon \tilde{H}'' \to \tilde{H}''$$

by

$$T_P(x_1, x_2, x_3, x_4) = \left(A_P^{-1}(x_3), x_2, A_P(x_1), x_4\right).$$

We will show that P_3, P_4, $A_P P_1$ and $A_P^{-1} P_3$, and hence T_P, lie in $W(f, f')$. As $P_3 \in W(f, f')$, $P' = P_3 \in W(F'; E')$. Since $T_P^{-1} = T_P$, it follows (see [H-L2], (4.6)) that

$$\begin{aligned}\mathrm{tr}_\nu(P) &= \mathrm{tr}_{\nu\oplus\nu'}(P_1) = \mathrm{tr}_{\nu\oplus\nu'}\left(T_P P_1 T_P^{-1}\right) \\ &= \mathrm{tr}_{\nu\oplus\nu'}(P_3) = \mathrm{tr}_{\nu'}(P').\end{aligned}$$

Now $\mathrm{tr}_\nu(P) = \dim_\nu(V_P)$ and $\mathrm{tr}_{\nu'}(P') = \dim_{\nu'}(V_{P'})$ so the theorem follows.

To prove $A_P \in W(f, f')$, we appeal to (4.5) of [H-L2]. There it is shown that for $k \in C_0^\mu(U_0, V_0, \gamma_0)$, Gk is a finite sum of elements of the $C_0^\mu(U', V, \gamma f')$. For s even, $(1 + \Delta')^{s/2}$ is a measurable leafwise differential operator, so $(1 + \Delta')^{s/2} Gk$ is also such a finite sum. It follows that $(1 + \Delta')^{s/2} GP_1 \in W(f, f')$. By Lemma (2.11), Q' is in $W(F'; E')$, so also in $W(f, f')$, and so A_P and $A_P P_1$ are in $W(f, f')$.

To prove that P_3 and $P_4 \in W(f, f')$, write $A_P = U|A_P|$, the polar decomposition. Let $f_n(x) = 1/x$ for $x \geq 1/n$ and $f(x) = 1/n$ otherwise. Then for each $x \in M$, $A_P^x f_n(|A_P^x|)$ has U^x as strong limit [R-S, p. 247, no. 20]. As $\|A_P f_n(|A_P|)\| \leq 1$, it follows that U, U^* and $UU^* = P_3$ and so also P_4 lie in $W(f, f')$.

Finally to show A_P^{-1} and so also $A_P^{-1} P_3$ lies in $W(f, f')$, first note that restricted to $V_P \oplus V_P^\perp \oplus W_P^\perp$, $P_1 f_n(|A_P|) U^*$ and A_P^{-1} are zero. For $\beta \in W_P$, $A_P f_n(|A_P|) U^*(\beta)$ converges in norm to $\beta = A_P A_P^{-1}(\beta)$. Let

$$\beta_n = f_n(|A_P|) U^*(\beta) - A_P^{-1}(\beta).$$

Since $A_P(\beta_n) = A_P P_1(\beta_n) \to 0$ in norm, and $\|A_P P_1(\beta_n)\| \geq \text{const} \|P_1(\beta_n)\|$ we have $\|P_1(\beta_n)\| \to 0$. Hence $\|P_1 f_n(|A_P|) U^*(\beta) - A_P^{-1}(\beta)\| \to 0$ and $P_1 f_n(|A_P|) U^*$ converges strongly to A_P^{-1}, so A_P^{-1} and $A_P^{-1} P_3$ lie in $W(f, f')$.

4. The combinatorial spectral distribution functions

In this section we introduce the combinatorial spectral distribution functions $N_\nu^{c,k}(\lambda)$ and prove our second main result, namely:

THEOREM 4.1. *For all k, the analytic and combinatorial spectral distribution functions, $N_\nu^k(\lambda)$ and $N_\nu^{c,k}(\lambda)$ are dialationally equivalent near zero.*

To begin, we construct the combinatorial analog $W^c(F; E)$ of the von Neumann algebra $W(F; E)$. Let K be a bounded triangulation for the foliated manifold M, F. Then $W^c(F, E)$ is the von Neumann algebra of operators on the direct integral space $\tilde{C}_2^k(K)$ generated by the sets $C_0^\mu(U, V, \gamma)$ described below. Here U and V are sets in the open cover of M which is associated to K [H-L2, p. 324] and γ is a leafwise path from U to V. Let $A = A(\Sigma_0, \ldots, \Sigma_k)$ and $B = B(\Sigma_0', \ldots, \Sigma_k')$ be measurable families of simplices in K as in [H-L2], p. 326, 330, with $A \subset U$, $B \subset V$, and let h be a measurable function on the placques of U. Then h, A and B determine an element of $C_0^\mu(U, V, \gamma)$ denoted $hA \otimes B$ as follows. Given $x \in U$, denote the unique (if its exists) simplex in A contained in P_x, the placque of x, by A_x. Then for $\sigma \in \tilde{C}_2^k(K)$,

$$((hA \otimes B)\sigma)(x) = h(x)(A_x, \sigma_x) B_{\gamma(x)}.$$

Here we identify A_x and $B_{\gamma(x)}$ with their dual simplicial cochains and $(\ ,\ )$ is the inner product given in Section 1.

The invariant transverse measure ν defines a trace on $W^c(F; E)$ which is given on generators by

$$\operatorname{tr}_\nu(hA \otimes B) = \int_{T_U} h(x)(A_x, B_{\gamma(x)})\, d\mu(x)$$

where T_U is a transversal in U.

Note that the simplicial coboundary δ: $\tilde{C}_2^k(K) \to \tilde{C}_2^{k+1}(K)$ and the inner product give rise to the combinatorial Laplacian Δ^c: $\tilde{C}_2^k(K) \to \tilde{C}_2^k(K)$ which is a bounded self adjoint element of $W^c(F; E)$.

Definition 4.2. $N_\nu^{c,k}(\lambda) = \operatorname{tr}_\nu(\chi_\lambda(\Delta^c))$. $N_\nu^{c,k}(\lambda)$ is the kth combinatorial spectral distribution function of F and ν.

Let χ_0 be the characteristic function of $0 \in \mathbf{R}$.

Definition 4.3. $\beta_\nu^{c,k} = \operatorname{tr}_\nu(\chi_0(\Delta^c))$ is the kth combinatorial foliation betti number of F.

We shall show below that for all k, $\beta_\nu^{c,k} = \beta_\nu^k$.

Let $\mathfrak{S}_{\lambda,\perp}^c$ be the set of orthogonal projections P on $\tilde{C}_2^k(K)$ which satisfy:

(i) $P \in W^c(F; E)$,

(ii) image $P \subset (\ker \delta)^\perp$,

(iii) for all $\alpha \in$ image P, $\|\delta\alpha\| \leq \sqrt{\lambda}\, \|\alpha\|$.

Set $F_\nu^{c,k}(\lambda) = \sup \operatorname{tr}_\nu(P)$ where the sup is over $\mathfrak{S}_{\lambda,\perp}^c$. As in the analytic case, we similarly define $G_\nu^{c,k}(\lambda)$ and we have

$$N_\nu^{c,k}(\lambda) = G_\nu^{c,k}(\lambda) + \beta_\nu^{c,k} + F_\nu^{c,k}(\lambda),$$

and

$$F_\nu^{c,k}(\lambda) = G_\nu^{c,k+1}(\lambda).$$

In fact the proofs are easier in this case since the simplicial coboundary δ, its adjoint δ^* and the Laplacian Δ^c are all bounded operators. Thus we need only prove that $F_\nu^{c,k}(\lambda)$ is dialationally equivalent to $F_\nu^k(\lambda)$ near zero.

First we show that we may replace the bounded triangulation K by any standard subdivision (and $W^c(F, E)$ by its analog) in computing $F_\nu^{c,k}(\lambda)$. Let S^rK be the rth standard subdivision of K and denote the associated simplicial complex and von Neumann algebra by $\tilde{C}_2^k(S^rK)$ and $W_r^c(F; E)$

respectively. Note that the natural cochain maps

$$\pi^r\colon \tilde{C}_2^*(K) \to \tilde{C}_2^*(S^r k)$$
$$S^r\colon \tilde{C}_2^*(S^r K) \to \tilde{C}_2^*(K)$$

are bounded maps satisfying

$$S^r \pi^r = I$$
$$\pi^r S^r = I + D\delta + \delta D$$

where D is a bounded cochain map

$$D\colon \tilde{C}_2^*(S^r K) \to \tilde{C}_2^{*-1}(S^r K).$$

See [HL-2]. The fact that we may replace K by $S^r K$ then follows from the proof of Prop. 4.1 of [G-Sh] and the material below.

Let $\hat{W}$ be the von Neumann algebra acting on $\tilde{C}_2^k(K) \oplus \tilde{C}_2^k(S^r K)$ generated by W^c (acting by zero on the second factor), W_r^c (acting by zero on the first factor), and the sets $C_0^\mu(U, V', \gamma)$ and $C_0^\mu(U', V, \gamma)$ where U and V are as above for K, U' and V' are associated to $S^r K$, and γ is a leafwise path from U to V' (U' to V respectively). An element $hA \otimes B' \in C_0^\mu(U, V', \gamma)$ is defined just as before, except that B' is a measurable family of simplices in $S^r K$. Similarly, $hA' \otimes B$ defines an element in $C_0^\mu(U', V, \gamma)$ where A' is a measurable family of simplices in $S^r K$. Each $hA \otimes B'$ defines a bounded map from $\tilde{C}_2^k(K)$ to $\tilde{C}_2^k(S^r K)$ and similarly for $hA' \otimes B$. Note that if π_i is projection of $\tilde{C}_2^k(K) \otimes \tilde{C}_2^k(S^r K)$ onto the ith factor, $\pi_1 \hat{W} \pi_1 = W^c$ and $\pi_2 \hat{W} \pi_2 = W_r^c$. Also note that π^r and S^r are in $\hat{W}$.

Let Q^c be orthogonal projection onto $(\ker \delta)^\perp$ in $\tilde{C}_2^k(K)$ and Q_r^c the projection onto $(\ker \delta)^\perp$ in $\tilde{C}_2^k(S^r K)$. $(\ker \delta)^\perp = (\ker \delta^*\delta)^\perp$ is a spectral space of the bounded self adjoint operator $\delta^*\delta$. As $\delta^*\delta \in W^c$, so is Q^c. Similarly $Q_r^c \in W_r^c$. Given an orthogonal projection $P^c \in W^c$, let P_r^c be orthogonal projection onto the closure of the image of $Q_r^c \pi^r P^c$. Similarly, given an orthogonal projection $P_r^c \in W_r^c$, let P^c be orthogonal projection onto the closure of the image of $Q^c S^r P_r^c$. Given P^c, $Q_r^c \pi^r P^c \in \hat{W}$, so $P_r^c \in \hat{W}$, so also in W_r^c and image $P_r^c \subset (\ker \delta)^\perp$. Similarly, given P_r^c, $P^c \in \hat{W}$, so also in W^c, and image $P^c \subset (\ker \delta)^\perp$. Now suppose λ is sufficiently small. Then the proof of Prop. 4.1 of [G-Sh] implies that given $P^c \in \mathfrak{S}_{\lambda,\perp}^c$, $P_r^c \in \mathfrak{S}_{\lambda,\perp}^c$ (for W_r^c) and given $P_r^c \in \mathfrak{S}_{\lambda,\perp}^c$, $P^c \in \mathfrak{S}_{\lambda,\perp}^c$.

The invariant transverse measure ν defines a trace, $\mathrm{tr}_{\nu\oplus\nu}$ on $\hat{W}$, extending the traces on W^c and W_r^c. We need to prove $\mathrm{tr}_\nu(P^c) = \mathrm{tr}_\nu(P_r^c)$. To do so, we proceed as in the proof of 3.6 and show that $\mathrm{tr}_{\nu\oplus\nu}(P^c) = \mathrm{tr}_{\nu\oplus\nu}(P_r^c)$. Given P^c, what is required is a bounded isomorphism A: image $P^c \to$ image P_r^c with bounded inverse A^{-1} and AP^c and $A^{-1}P_r^c \in \hat{W}$. It follows from [G-Sh] and the proof of 3.6 that $A = Q_r^c \pi^r | \text{image } P^c$ provides just such a map. Similarly, given P_r^c, $A = Q^c S^r | \text{image } P_r^c$: image $P_r^c \to$ image P^c is a bounded isomorphism with bounded inverse and AP_r^c and $A^{-1}P^c \in \hat{W}$. Thus we may replace K by $S^r K$ for any r.

Next, we remark that we may replace $F_\nu^k(\lambda)$ by an equivalent function, also denoted $F_\nu^k(\lambda)$, computed from the complex of Hilbert spaces

$$\cdots \longrightarrow \tilde{A}^{s,k} \xrightarrow{d_k} \tilde{A}^{s,k+1} \longrightarrow \cdots$$

Let $W^s = W^s(F; E)$ be the von Neumann algebra $W(F; E)$ acting on $\tilde{A}^{s,k}$ generated by the sets $C_0^\mu(U, V, \gamma)$. There is a trace, denoted tr_ν, defined on W^s just as before. The results of section 2 remain valid if we replace $\tilde{H}^k$ by $\tilde{A}^{s,k}$, W by W^s, and $\langle \ , \ \rangle$ by $\langle \ , \ \rangle_s$. In what follows we will work with $\tilde{A}^{s,k}$ where s is fixed and sufficiently large (e.g., $s > 2 \dim L$).

We shall show that for λ sufficiently small and r sufficiently large, there are constants $c_1, c_2 > 0$ so that each $P \in S_\lambda$ determines an element $P^c \in \mathfrak{S}_{c_1\lambda,\perp}^c$ and that each $P^c \in \mathfrak{S}_{\lambda,\perp}^c$ determines $P \in \mathfrak{S}_{c_2\lambda,\perp}$. Finally, we shall show that for two such associated projections, $\text{tr}_\nu(P) = \text{tr}_\nu(P^c)$. This will give Theorem 4.1.

Recall the maps

$$\tilde{A}^{s,k} \underset{W}{\overset{\int}{\rightleftarrows}} \tilde{C}_2^k(S^r K).$$

Then $dW = W\delta$, $\delta\int = \int d$, $\int W = I$, and W and $\int$ (for s large) are bounded maps. Given $P \in S_\lambda$, let P^c be orthogonal projection on the closure of the image of $Q^c \int P$. Given $P^c \in \mathfrak{S}_{\lambda,\perp}^c$, let P be orthogonal projection onto the closure of the image of QWP^c.

Proposition 4.4. *There exists constants $c_1, c_2 > 0$ such that, for λ sufficiently small and r sufficiently large, if $P^c \in \mathfrak{S}_{\lambda,\perp}^c$, then $P \in \mathfrak{S}_{c_1\lambda,\perp}$ and if $P \in S_\lambda$, then $P^c \in \mathfrak{S}_{c_2\lambda,\perp}$.*

Proof. Given P^c, we need to show that for $w \in$ image P,

$$\|dw\|_s \leq \sqrt{c_1\lambda}\, \|w\|_s$$

and that $P \in W(F, E)$. We defer the latter proof to the end of this section. Following [G-Sh], we let $\alpha \in$ image $P^c \subset \ker(\delta)^\perp$ be such that $w = QW\alpha$. Then

$$\|dw\|_s = \|dQW\alpha\|_s = \|dW\alpha\|_s = \|W\delta\alpha\|_s \leq \|W\|\, \|\delta\alpha\| \leq \|W\| \sqrt{\lambda}\, \|\alpha\|.$$

In $\tilde{A}^{s,k}$ we write

$$W\alpha = w_1 + w_2, w_1 \in (\ker d)^{\perp}, w_2 \in \overline{\ker d}.$$

Then $\|QW\alpha\|_s = \|w_1\|_s$ and as $\delta \int w_2 = \int dw_2 = 0$ and $\alpha \in (\ker \delta)^{\perp}$

$$\alpha = Q^c\alpha = Q^c \int W\alpha = Q^c \int w_1.$$

Thus

$$\begin{aligned} \|\alpha\| &= \left\| Q^c \int w_1 \right\| \leq \left\| \int w_1 \right\| \\ &\leq \left\| \int \right\| \|w_1\|_s = \left\| \int \right\| \|QW\alpha\|_s. \end{aligned}$$

Setting $c_1 = \|W\|^2 \|\int\|^2$ we have

$$\|dw\|_s \leq \sqrt{c_1 \lambda}\, \|w\|_s.$$

COROLLARY 4.5. *If $P^c \in \mathfrak{S}^c_{\lambda, \perp}$, then QW is injective on* image P^c *with closed image and bounded inverse.*

To prove the second half of Prop. 4.4, we use the inequality

$$\text{(3.3)} \qquad \left\| w - W \int w \right\| \leq c\eta_r \|w\|_s$$

where $w \in \tilde{A}^{s,k}$ (s sufficiently large), c is independent of r, and η_r is the mesh of S^rK. We assume in what follows that r is so large and λ is so small that

$$\eta = c\eta_r(1+\lambda)^s < 1 \quad \text{and} \quad (1+\lambda)^s < 2.$$

Recall that for $w \in E_\lambda$,

$$\|w\|^2 \leq \|w\|_s^2 \leq (1+\lambda)^s \|w\|.$$

Now we have the analog of Lemma 3.3 of [Ef].

LEMMA 4.6. *If $w \in E_\lambda$, then*

$$\|Qw\| \leq \frac{\|W\|}{1-\eta} \left\| Q^c \int w \right\|.$$

Proof. By (3.3) and the above remarks, the operator

$$B = \left(I - W\int\right)\chi_\lambda(\Delta)\colon \tilde{H} \to \tilde{H}$$

is bounded by η. Thus $\|QB\| \le \eta$, and we have

$$\begin{aligned}\|Qw\| &= \left\|Q\left(W\int w + Bw\right)\right\| \\ &\le \left\|QW\int w\right\| + \|QBw\| \\ &\le \left\|QW\int w\right\| + \|QB\|\,\|w\| \\ &\le \left\|QW\int w\right\| + \|QB\|\,\|Qw\|.\end{aligned}$$

Since $W(\ker\delta) \subset \ker d$,

$$\begin{aligned}\left\|QW\int w\right\| &= \left\|QW\left(Q^c\int w\right)\right\| \\ &\le \left\|WQ^c\int w\right\| \le \|W\|\left\|Q^c\int w\right\|.\end{aligned}$$

COROLLARY 4.7. *If $P \in S_\lambda$ and $\eta < 1$, then $Q^c\int$ is injective on* image P *with closed image and bounded inverse.*

Now for $P \in S_\lambda$ we need to show that for $\alpha \in$ image P^c,

$$\|\delta\alpha\| \le \sqrt{c_2\lambda}\,\|\alpha\|.$$

Let $w \in$ image $P \subset (\ker d)^\perp \cap E_\lambda$ with $\alpha = Q^c\int w$. Then

$$\begin{aligned}\|\delta\alpha\| &= \left\|\delta Q^c\int w\right\| = \left\|\delta\int w\right\| \\ &= \left\|\int dw\right\| \le \left\|\int\right\|\|dw\|_s \\ &\le \left\|\int\right\|\sqrt{\lambda}\,\|w\|_s \le \left\|\int\right\|\sqrt{\lambda}\,(1+\lambda)^s\|w\| \\ &= \left\|\int\right\|\sqrt{\lambda}\,(1+\lambda)^s\|Qw\| \\ &\le \left\|\int\right\|\|W\|\frac{(1+\lambda)^s}{1-\eta}\sqrt{\lambda}\left\|Q^c\int w\right\|.\end{aligned}$$

Set $\sqrt{c_2} = 2\|\int\|\,\|W\|/(1-\eta)$.

Finally, we show that given $P \in S_\lambda$, then $P^c \in W_r^c(F, E)$ and given $P^c \in \mathfrak{S}^c_{\lambda, \perp}$, then $P \in W(F, E)$ and $\operatorname{tr}_\nu(P) = \operatorname{tr}(P^c)$. To do so, we construct as before a von Neumann algebra $\hat{W}$ containing W and W_r^c and a trace on $\hat{W}$ extending those on W and W_r^c.

Let $\hat{W}$ be the von Neumann algebra acting on $\tilde{A}^{s,k} \oplus \tilde{C}_2^k(S^rK)$ generated by W (acting by zero on the second factor), W_r^c (acting by zero on the first factor), and the sets $C_0^\mu(U, V', \gamma)$ and $C_0^\mu(U', V, \gamma)$ described below. U and V are foliation charts on M, U' and V' are foliation charts on M associated to S^rK, and γ is a leafwise path from U to V' or U' to V. An element $h \otimes B$ of $C_0^\mu(U, V', \gamma)$ is given by a leafwise smooth, measurable, bounded dual k form h with support in U, and a measurable family of simplicies B in S^rK as above with $B \subset V'$. This defines a bounded map from $\tilde{A}^{s,k}$ to $\tilde{C}_2^k(S^rK)$ which on a k form ϕ is given by

$$(h \otimes B)(\phi)(x) = \left[\int_{P_x} h(z)(\phi(z))\, dz\right] B_{\gamma(x)}$$

where P_x is the placque of x in U.

Similarly, an element $A \otimes g \in C_0^\mu(U', V, \gamma)$ is given by a measurable family of simplices $A \subset U'$, and a leafwise smooth, measurable, bounded k form g with support in V. This defines a bounded map from $\tilde{C}_2^k(S^rK)$ to $\tilde{A}^{s,k}$ is an obvious way. Note that if π_i is projection of $\tilde{A}^{s,k} \oplus \tilde{C}_2^k(S^rK)$ onto the ith factor, then $W = \pi_1 \hat{W} \pi_1$ and $W_r^c = \pi_2 \hat{W} \pi_2$ and that the maps $\int$ and W are in $\hat{W}$.

We define a linear functional $\operatorname{tr}_{\nu\oplus\nu}$ on generators of $\hat{W}$ as follows. For $k \in W$ or W_r^c, $\operatorname{tr}_{\nu\oplus\nu}(k) = \operatorname{tr}_\nu(k)$. For $k \in C_0^\mu(U, V', \gamma)$ or $C_0^\mu(U', V, \gamma)$, $\operatorname{tr}_{\nu\oplus\nu}(k) = 0$. An argument similar to the proof of Theorem 4.2 of [H-L2] shows that $\operatorname{tr}_{\nu\oplus\nu}$ extends to a trace on $\hat{W}$.

To finish the proof of Theorem 4.1, we once again proceed as in the proof of 3.6. Let λ be sufficiently small and r sufficiently large. Given $P^c \in \mathfrak{S}^c_{\lambda, \perp}$, $QWP^c \in \hat{W}$ so $P \in W$ and so also in $\mathfrak{S}_{c_1\lambda, \perp}$. By Corollary 4.5, QW (which is in $\hat{W}$) provides a bounded isomorphism between the image of P^c and the image of P with bounded inverse. Thus

$$\begin{aligned} \operatorname{tr}_\nu(P^c) &= \operatorname{tr}_{\nu\oplus\nu}(P^c) = \operatorname{tr}_{\nu\oplus\nu}(P) \\ &= \operatorname{tr}_\nu(P). \end{aligned}$$

Similarly, given $P \in S_\lambda$, $P^c \in \mathfrak{S}_{c_2\lambda, \perp}$ and $Q^c \int \in \hat{W}$ provides the requisite isomorphism between image P and image P^c, and we are done.

Finally we prove the following.

THEOREM 4.8. *The analytic and combinatorial betti numbers β_ν^k and $\beta_\nu^{c,k}$ are the same.*

Proof. Let Q_0: $A^{s,k} \to \ker \Delta$ and Q_0^c: $\tilde{C}_2^k(S^r K) \to \ker \Delta^c$ be the projections. It is not difficult to show that $\operatorname{tr}_\nu Q_0^c$ is independent of r and $\ker \Delta$ is obviously independent of s. Now Q_0 and $Q_0^c \in \hat{W}$ so $Q_0 W$: $\ker \Delta^c \to \ker \Delta$ and $Q_0^c \int$: $\ker \Delta \to \ker \Delta^c$ are also in W. Both $\ker \Delta$ and $\ker \Delta^c$ are closed subspaces of $A^{s,k}$ and $\tilde{C}_2^k(S^r K)$ respectively. By Theorem 3.9 of [H-L2], the inclusion $\ker \Delta \to \tilde{H}^k$ into the kth de Rham cohomology group is a Hilbert space isometry. A similar proof shows that the inclusion $\ker \Delta^c \to \tilde{H}_c^k$ into the kth simplicial cohomology group is also a Hilbert space isometry. By Theorem 3.15 of [H-L2], the induced maps W: $\tilde{H}_c^k \to \tilde{H}^k$ and $\int$: $\tilde{H}^k \to \tilde{H}_c^k$ are bounded Hilbert space isomorphisms, which are inverses of each other. But $Q_0 W$ is the composition

$$\ker \Delta^c \simeq \tilde{H}_c^k \xrightarrow{W} \tilde{H}^k \simeq \ker \Delta$$

and $Q_0^c \int$ is the composition

$$\ker \Delta \simeq \tilde{H}^k \xrightarrow{\int} \tilde{H}_c^k \simeq \ker \Delta^c.$$

Thus these maps provide the required isomorphisms between $\ker \Delta$ and $\ker \Delta^c$ and

$$\beta_\nu^k = \operatorname{tr}_\nu Q_0 = \operatorname{tr}_\nu Q_0^c = \beta_\nu^{c,k}.$$

Remarks. Note that the results of this paper as well as those of [H-L2] remain valid if we substitute $L^2(\tilde{L}_x, r^*(E|L_x))$ for $L^2(L_x, E|L_x)$ where $\tilde{L}_x$ is the holonomy cover of L_x and r: $\tilde{L}_x \to L_x$ the covering map. The proofs given here and in [H-L2] carry over mutatis mutandis with the role of $M \times M$ being played by the holonomy groupoid. Note also that the spectral invariants and the betti numbers in the second case (i.e., using $\tilde{L}_x$) are not necessarily the same as those in the first (i.e. using L_x), contrary to our remark on p. 323 of [H-L2], as was pointed out to us by John Roe.

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