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ORIENTATION REVERSING AUTOMORPHISMS OF RIEMANN SURFACES

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It was shown by Jakob Nielsen [N] that the fixed point data determines an orientation preserving automorphism of prime order on a given compact Riemann surface up to topological conjugacy. In this paper we classify up to topological conjugation the orientation reversing automorphisms of order $2p$, for p prime, on compact Riemann surfaces of genus $g_0 \geq 2$. In 1979, Robert Zarrow studied this classification (see [Z1] and [Z2]). However we have found some errors in his works.

We separate our study in two cases: when the automorphisms have order 4 and when the automorphisms have order $2p$, with p an odd prime. In the first case we have proved the following theorem:

THEOREM 1. *Let X be a Riemann surface, suppose that ϕ_1 and ϕ_2 are two orientation reversing automorphisms of X such that ϕ_1^2 and ϕ_2^2 have order* 2 *and they have fixed points. Then ϕ_1 and ϕ_2 are conjugate if and only if ϕ_1^2 and ϕ_2^2 have the same number of fixed points.*

The above theorem agrees with Theorem 1.1 of [Z1] but if the considered automorphisms have fixed point free squares and $g_0 \equiv 1 \pmod 4$ then we find two conjugacy classes instead of one as Zarrow claimed (see Theorem 2).

For the automorphisms of order $2p$ with p an odd prime we have established the following result:

THEOREM 3. *Let X be a Riemann surface and suppose that ϕ_1 and ϕ_2 are two orientation reversing automorphisms of order $2p$ where p is an odd prime. Then ϕ_1 and ϕ_2 are conjugate if and only if* (1) *$X/\langle\phi_1\rangle$ and $X/\langle\phi_2\rangle$ are homeomorphic,* (2) *ϕ_1^2 and ϕ_2^2 are conjugate and* (3) *the action of ϕ_1^2 on* Fix ϕ_1^p *(fixed point set of ϕ_1^p) is conjugate to the action of ϕ_2^2 on* Fix ϕ_2^p.

The conditions of this theorem are different to those proposed in [Z2]. However in the example in Section 3 we show that the conditions of Zarrow's statement are not sufficient.

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1. NEC groups and automorphisms of NEC groups

A non-euclidean crystallographic group (NEC group) is a discrete subgroup Γ of the group of isometries of the hyperbolic plane H^2 (including the orientation reversing isometries, namely reflections and glide reflections) with compact quotient space.

NEC groups may be classified according to their signatures [M]. The signature of an NEC group Γ is a symbol of the form

$$\left(g, \pm, [m_1, \ldots, m_t], \left\{(n_{i_1}, \ldots, n_{is_i}), i = 1, \ldots, k\right\}\right)$$

where g is the genus of the surface H^2/Γ, the sign $+$ or $-$ indicates whether the surface is orientable or non-orientable, the $m_i \geq 0$ (proper periods) represent the branching indices over the interior points of H^2/Γ by the projection p: $H^2 \to H^2/\Gamma$, the $n_{ij} \geq 2$ (linked periods) represent the branching indices over the points of the boundary of the surface under the projection p, and k is the number of boundary components of H^2/Γ. If $s_i = 0$ then the ith bracket is called empty and denoted by ( ).

The groups Γ with sign $+$ in the signature have a canonical presentation given by generators (canonical system of generators):

$x_i, i = 1, \ldots, t$	(elliptic generators)
$e_i, i = 1, \ldots, k$	(boundary generators)
$c_{ij}, i = 1, \ldots, k, j = 0, \ldots, s_i$	(reflection generators)
$a_j, b_j = 1, \ldots, g$	(hyperbolic generators)

and relations

$$x_i^{m_i} = 1, i = 1, \ldots, t$$

$$c_{is_i} = e_i^{-1} c_{i0} e_i, \qquad i = 1, \ldots, k$$

$$c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1} c_{ij})^{n_{ij}} = 1, \qquad i = 1, \ldots, k, \quad j = 1, \ldots, s_i$$

$$e_1 \ldots e_k x_1 \ldots x_t a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 \quad \text{(long relation)}$$

If the group Γ has sign $-$ in its signature it has the same presentation replacing the hyperbolic generators by the glide-reflections d_j, $j = 1 \ldots g$ and the long relation by

$$e_1 \ldots e_k x_1 \ldots x_t d_1^2 \ldots d_g^2 = 1.$$

From the results of Singerman [S] if ϕ is an automorphism of the surface H^2/Γ then there exists an NEC group Γ' such that $\Gamma \lhd \Gamma'$, and $\Gamma'/\Gamma \approx \langle\phi\rangle$.

In our study, the knowledge of some special types of automorphisms of NEC groups will be important.

Let Γ be an NEC group with sign $+$ in its signature and $x_i, e_i, c_{ij}, a_j, b_j$ be a canonical system of generators then the automorphisms to be used are:

ω defined by $\omega(a_1) = a_1 b_1$ and $\omega(y) = y$ for every canonical generator y different from a_1.

ξ defined by $\xi(a_1) = a_1 b_1$, $\xi(b_1) = a_1^{-1}$ and $\xi(y) = y$ for every canonical generator different from a_1 and b_1.

ν_j defined by $\nu_j(a_j) = a_{j+1}$, $\nu_j(b_j) = b_{j+1}$, $\nu_j(a_{j+1}) = c_{j+1}^{-1} a_j c_{j+1}$, $\nu_j(b_{j+1}) = c_{j+1}^{-1} b_j c_{j+1}$, where $c_{j+1} = [a_{j+1}, b_{j+1}]$ and $\nu_j(y) = y$ for every canonical generator y different from a_j, b_j, a_{j+1} and b_{j+1}.

μ defined by $\mu(a_1) = a_2 a_1$, $\mu(a_2) = b_1 a_2 b_1^{-1}$, $\mu(b_1) = b_1$, $\mu(b_2) = a_2 b_2 a_2^{-1} b_1^{-1}$, $\mu(y) = a_2 y a_2^{-1}$ where y is an elliptic, reflection, boundary or hyperbolic generator different from a_1, b_1, a_2, b_2.

σ defined by $\sigma(x_t) = a_1^{-1} x_t a_1$, $\sigma(a_1) = [a_1^{-1}, x_t^{-1}] a_1$, $\sigma(b_1) = b_1 a_1^{-1} x_t a_1$, $\sigma(y) = y$ for every canonical generator different from x_t, a_1 and b_1.

If $s_i = s_{i+1} = 0$, then we define λ_i by $\lambda_i(e_i) = e_i e_{i+1} e_i^{-1}$, $\lambda_i(e_{i+1}) = e_i$, $\lambda_i(c_{i0}) = e_i c_{(i+1)0} e_i^{-1}$, $\lambda_i(c_{(i+1)0}) = c_{i0}$, $\lambda_i(y) = y$ for every canonical generator y different from c_{i0}, $c_{(i+1)0}$, e_i and e_{i+1}.

Assume that $t = 0$; then we can define the automorphism π, by

$$\pi(e_k) = a_1^{-1} e_k a_1, \qquad \pi(c_{ki}) = a_1^{-1} c_{ki} a_1,$$
$$\pi(a_1) = \left[a_1^{-1}, e_k^{-1}\right] a_1, \qquad \pi(b_1) = b_1 a_1^{-1} e_k, a_1,$$

$\pi(y) = y$ for every canonical generator y different from e_k, c_{ki}, a_1 and b_1.

If the sign in the signature of Γ is $-$ and x_i, e_i, c_{ij}, d_j is a canonical system of generators then the automorphisms to be used are:

α_j defined by $\alpha_j(d_j) = d_j^2 d_{j+1} d_j^{-2}$, $\alpha_j(d_{j+1}) = d_j$, $\alpha_j(y) = y$ for every canonical generator y different from d_j and d_{j+1}.

β_j defined by $\beta_j(d_j) = d_j d_{j+1}^{-1} d_j^{-1}$, $\beta_j(d_{j+1}) = d_j d_{j+1}^2$, $\beta_j(y) = y$ for every canonical generator y different from d_j and d_{j+1}.

γ defined by $\gamma(d_1) = x_t d_1$, $\gamma(x_t) = x_t d_1 x_t^{-1} d_1^{-1} x_t^{-1}$, $\gamma(y) = y$ for every canonical generator y different from d_1 and x_t.

Assume that $s_i = s_{i+1} = 0$, then we can define the automorphism δ_i by

$$\delta_i(e_i) = e_i e_{i+1} e_i^{-1}, \quad \delta_i(e_{i+1}) = e_i, \quad \delta_i(c_{i0}) = e_i c_{(i+1)0} e_i^{-1},$$
$$\delta_i(c_{(i+1)0}) = c_{i0},$$

$\delta_i(y) = y$ for every canonical generator y different from $c_{i0}, c_{(i+1)0}$, e_i and e_{i+1}.

Assume that $t = 0$ and $s_k = 0$; then we can consider the automorphism ε defined by

$$\varepsilon(d_1) = e_k d_1, \quad \varepsilon(e_k) = e_k d_1 e_k^{-1} d_1^{-1} e_k^{-1}, \quad \varepsilon(c_{k0}) = e_k d_1 c_{k0} d_1^{-1} e_k^{-1},$$

$\varepsilon(y) = y$ for every canonical generator y different from d_1, c_{k0} and e_k.

2. Orientation reversing automorphisms of order 4

First let us prove Theorem 1 of the introduction.

Proof of Theorem 1. It is clear that if ϕ_1 is conjugate to ϕ_2 then ϕ_1^2 and ϕ_2^2 have the same number of fixed points.

Suppose now that $\#\text{Fix}\,\phi_1^2 = \#\text{Fix}\,\phi_2^2$. Assume $X = H^2/\Gamma$, and that Γ_1, Γ_2 are NEC groups such that $\Gamma_1/\Gamma \approx \langle\phi_1\rangle$, $\Gamma_2/\Gamma \approx \langle\phi_2\rangle$ then the signatures of Γ_1 and Γ_2 are

$$\left(g_i, \pm, \left[(2)^{r_i}(4)^{q_i}\right], \left\{(\ )^{t_{ij}}\right\}\right), \qquad j = 1\dots t, \qquad i = 1,2$$

(see Chapter II of [BEGG]).

Let θ_i: $\Gamma_i \to Z_4 \approx \langle\phi_i\rangle \approx \Gamma_i/\Gamma$ be the natural epimorphism and let x_i, e_i, c_j, a_i, b_i (or d_i according to the sign in the signature) be a canonical system of generators of Γ_i.

Let us prove that t_{ij} must be 0. If $t_{ij} \neq 0$ then the reflection generators c_{j0} satisfy $\theta_i(c_{j0}) = \bar{2}$. If $r_i \neq 0$, $\theta_i(x_j) = \bar{2}$ for x_j some elliptic canonical generator and this contradicts the orientability character of X (see [HS]) and if $r_i = 0$ then some generator d_i, a_i, b_i or x_i must be mapped on $\bar{1}$ by θ_i (because θ_i is an epimorphism) and this also contradicts the fact that X is orientable. Then $t_{1j} = t_{2j} = 0$ $j = 1\dots t$. By the orientability of X, since ϕ_2 is orientation reversing, the sign in the signature must be $-$ and each $q_i = 0$. Since ϕ_1^2 and ϕ_2^2 have the same number of fixed points, $r_1 = r_2$ and by Riemann-Hurwitz formulae $g_1 = g_2$. Then the signature of Γ_1 and Γ_2 is

$$\left(g, -, \left[(2)^r\right]\right) \quad \text{with } r \neq 0.$$

The epimorphisms θ_i necessarily satisfy $\theta_i(d_j) = \bar{1}$ or $\bar{3}$ for every glide reflection generator of Γ_i and $\theta_i(x_j) = \bar{2}$ for the elliptic generators. Using the α_j automorphisms of Γ_i defined in §1, we can order the generators d_j in such a way that: $\theta_i(d_j) = \bar{3}$, $j = 1, \dots, m_i$, $\theta_i(d_j) = \bar{1}$, $j = m_i + 1, \dots, g$, $\theta_i(x_j) = \bar{2}$, $j = 1, \dots, r$.

If $m_i = 0$ or g, then by an automorphism of Z_4, $\theta_i(d_j) = \bar{3}$ for every j.

If $m_i \neq 0$, g then, since $r \neq 0$, using automorphisms γ, α_j of §1 we can construct a new system of generators of Γ_i such that $m_i = g$. Thus we can find an isomorphism $\psi\colon \Gamma_1 \to \Gamma_2$ such that $\theta_1 = \theta_2\psi$ and ϕ_1 is conjugated to ϕ_2 (see Theorem 3 of [M]).

THEOREM 2. *Let X be a Riemann surface of genus $g_0 \geq 2$. If $g_0 \equiv 1$ (mod 4) there are two conjugacy classes of orientation reversing automorphisms of order 4 having squares that are fixed point free automorphisms of X; if g_0 is not congruent to 1 modulo 4 then there are no such automorphisms.*

Proof. Let X be H^2/Γ and ϕ an orientation reversing automorphism of order 4 of X and such that ϕ^2 does not have fixed points. If $\Gamma'/\Gamma \approx \langle\phi\rangle$ then by the same reason as in the proof of Theorem 1 the signature of Γ' is $(g, -, [-], \{-\})$. Let $\theta\colon \Gamma' \to Z_4 \approx \langle\phi\rangle \approx \Gamma'/\Gamma$ be the natural epimorphism. Since $\theta(d_i) = \bar{1}$ or $\bar{3}$ for $i = 1, \ldots, g$ then g must be even and so g_0 is congruent to 1 modulo 4.

Assume that $\theta(d_j) = \bar{1}$ and $\theta(d_{j+1}) = \bar{1}$, with $0 < j < g - 1$. Then using the automorphism β_j of Γ' (see §1) we have $\theta(\beta_j(d_j)) = \theta(\beta_j(d_{j+1})) = \bar{3}$. If there is an even number of generators d_i sent by θ to $\bar{1}$ then using the automorphisms α_j of §1 and the β_j we can obtain a new system of generators of Γ' such that $\theta(d'_j) = \bar{3}$ for $j = 1, \ldots, g$. If there is an odd number of d_i sent to $\bar{1}$ by the same method we can obtain d'_j, $j = 1, \ldots, g$ such that $\theta(d'_j) = \bar{3}$ $j = 1, \ldots, g-1$ and $\theta(d_g) = \bar{1}$.

Then there are at most two conjugacy classes of automorphisms satisfying the conditions of the theorem.

In order to finish the proof let us take two automorphisms ϕ_1, ϕ_2 satisfying the conditions of the theorem. Let θ_1 and θ_2 be the epimorphisms defined by ϕ_1 and ϕ_2 and assume that $\theta_1(d_j) = \bar{3}$ for $j = 1, \ldots, g$ and $\theta_2(d_j) = \bar{3}$ for $j = 1, \ldots, g-1$, $\theta_2(d_g) = \bar{1}$.

Then θ_2 can be defined by $\theta_2(y) = \bar{3}\langle d_1 \ldots d_g, y\rangle + \bar{2}\langle d_g, y\rangle$; $y \in \Gamma_1$, where $\langle\ ,\ \rangle$ is the intersection number modulo 2. If there is an isomorphism $\psi\colon \Gamma_1 \to \Gamma_2$ such that $\theta_1 = \theta_2\psi$ then $\psi(d_1), \ldots, \psi(d_g)$ will be a system of generators of Γ_2 and $\theta_2(\psi(d_j)) = \bar{3}$ for $j = 1, \ldots, g$. Then $\langle d_g, \psi(d_j)\rangle = 0$ for every j, which is impossible. Therefore ϕ_1 and ϕ_2 are not conjugate.

3. Orientation reversing automorphisms of order $2p$, for p an odd prime

Let X be a Riemann surface and ϕ an orientation reversing automorphism of order $2p$ with p an odd prime. The set of fixed points of ϕ^p, Fix ϕ^p, consists of finitely many disjoint closed curves and the orientation preserving automorphism ϕ^2 of order p acts on Fix ϕ^p.

Proof of Theorem 3. If ϕ_1 is a conjugate to ϕ_2 then it is clear that $X/\langle\phi_1\rangle$ is homeomorphic to $X/\langle\phi_2\rangle$, ϕ_1^2 and ϕ_2^2 are conjugate and $\phi_1^2|_{\text{Fix}\,\phi_1^p}$ is also conjugate to $\phi_2^2|_{\text{Fix}\,\phi_2^p}$.

Let ϕ be an automorphism of X of order $2p$. If $X = H^2/\Gamma$, let Γ_1 be an NEC group such that $\langle\phi\rangle \approx \Gamma_1/\Gamma$ and θ: $\Gamma_1 \to Z_{2p} \approx \langle\phi\rangle \approx \Gamma_1/\Gamma$ be the natural projection. To prove the converse of Theorem 3 it is enough to show that θ is completely determined by the topological type of $X/\langle\phi\rangle$, the conjugation class of ϕ^2 and the action of ϕ^2 on Fix ϕ^p.

By the results in Chapter 2 of [BEGG] the signature of Γ_1 is

$$\left(g, \pm, \left[(2)^r, (p)^s, (2p)^q\right], \{(\ )^v\}\right)$$

Since ϕ is orientation reversing there exists an orientation reversing element in Γ_1 and the image under θ of such an element must be a generator of Z_{2p} or $\bar{p} \in Z_{2p}$ because X is orientable. Using the above orientation reversing element, the results of [HS] and the orientability of X it is easy to obtain that $r = q = 0$.

Case 1. $s > 0$. In this case the signature of Γ_1 is $(g, \pm, [(p)^s], \{(\ )^v\})$.

Subcase 1. The sign in the signature of Γ_1 is $-$. In other words the signature of Γ_1 is $(g, -, [(p)^s], (\ )^v\})$. Let $d_i, i = 1, \ldots, g$, $x_i, i = 1, \ldots, s$, $e_i, i = 1, \ldots, v$ be a canonical system of generators of the NEC group Γ_1. Then

$$\theta(d_i) = r_i \in Z_{2p}, \text{with } r_i \text{ odd}, i = 1, \ldots, g,$$

$$\theta(x_i) = \bar{1}_i \in Z_{2p}, \text{with } \bar{1}_i \text{ even}, i = 1, \ldots, s,$$

$$\theta(e_i) = \bar{k}_i \in Z_{2p}, \text{with } \bar{k}_i \text{ even}, i = 1, \ldots, v,$$

$$\theta(c_i) = \bar{p} \in Z_{2p}, i = 1, \ldots, v.$$

The conjugacy class of ϕ^2 completely determines $\theta(x_i)$ for $i = 1, \ldots, s$ (up to order) and the action of ϕ^2 on Fix ϕ^p determines $\theta(e_i)$ up the order of $e_1, \ldots, e_v$ but the automorphism δ_i tells us that such order is not important. In order to finish this case we will find a new set of glide reflection generators for Γ_1 such that the image under θ is completely determined by the data. Using the automorphisms α_i of §1 we can change the order of the d_i's to obtain $\theta(d_j) = \bar{1}$, $j = m, \ldots, g$ and $\theta(d_j) \neq \bar{1}$ for each j from 1 to $m - 1$. Assume $m \neq 1$. Since $s > 0$ there is an $e > 0$ such that $\theta(x_s)^e\theta(d_1) = \bar{1}$. With the automorphism $(\gamma\alpha_1\gamma\alpha_1)^e$ we obtain a new system of generators $d'_1, \ldots, d'_g$ such that $\theta(d'_j) = \theta(d_j)$, $j = m, \ldots, g$ and $\theta(d'_1) = \bar{1}$. Reordering the d'_j we have a new system of generators such that $\theta(d'_{m-1}) = \cdots = \theta(d'_g) = \bar{1}$. Repeating this process we can arrive at a new system $d_1, \ldots, d_g$

such that $\theta(d_2) = \cdots = \theta(d_g) = \bar{1}$ and $\theta(d_1)$ is determined by the relation

$$e_1 \ldots e_v x_1 \ldots x_s d_1^2 \ldots d_g^2 = 1$$

and by the fact that $\theta(d_1) = \bar{f}$ with f odd.

Subcase 2. Signature with sign $+$. In this case the signature is $(g; +; [(p)^s]; \{(\ )^v\})$. Let $a_i, b_i, i = 1, \ldots, g$, $x_i, i = 1, \ldots, s$, $e_i, i = 1, \ldots, v$ be a canonical system of generators of Γ_1. As in subcase 1, $\theta(x_i)$ and $\theta(e_i)$ are determined by the conjugation class of ϕ^2 and the action of ϕ^2 on $\mathrm{Fix}\, \phi^p$. Using the automorphisms ω, ξ, ν_j, μ and σ we can choose the generators a_i, b_i in order to obtain $\theta(a_j) = \theta(b_i) = \bar{1}$ (compare with [H]).

Case 1. $s = 0$

Subcase 1. $v > 0$ and there exists a generator e_i of Γ_1 such that $\theta(e_i) \neq \bar{0}$. Using the automorphisms λ_i and δ_i we can assume $\theta(e_t) \neq \bar{0}$. Then the proof of the two subcases of case 1 can be modified for this subcase replacing the automorphism σ by π if the sign is $+$ in the signature of Γ_1 and the automorphism γ by ε if the sign is $-$.

Subcase 2. $v = 0$ or $\theta(e_i) = \bar{0}$ for every generator e_i of Γ_1. Since ϕ is orientation reversing the sign in the signature of Γ_1 must be $-$ in order for θ to be an epimorphism. Then the signature of Γ_1 is $(g, -, [\ ], \{(\ )^v\})$. Let $d_1, \ldots, d_g, e_1, \ldots, e_v$ be a system of generators. In this case we have $\theta(e_1) = \cdots = \theta(e_v) = \bar{0}$. If $g = 2$, $\theta(d_1)$ and $\theta(d_2)$ are completely determined by the long relation and the fact that $\theta(d_i) = e$ with e odd, up automorphism in Z_{2p}. Assume now that $g \geq 3$. Since θ is an epimorphism there is a d_i such that $\theta(d_i)$ is a generator of Z_{2p} and by automorphism of Z_{2p} we can assume that $\theta(d_i) = \bar{1}$. After use of the automorphisms α_i we can assume $\theta(d_m) = \cdots = \theta(d_g) = \bar{1}$ and $\theta(d_j) \neq \bar{1}$ for every j from 1 to $m - 1$. There exists $e \in \{1, \ldots, (p-1)\}$ such that $\theta(d_{m-1}) + \overline{2e} = \bar{1}$; then the automorphism $(\beta_{m-1} \cdot \beta_{m-2} \cdot \alpha_{m-2} \cdot \alpha_{m-1})^e$ gives us a new system of generators $d_1, \ldots, d_g, e_1, \ldots, e_v$ such that $\theta(d_{m-1}) = \cdots = \theta(d_g) = \bar{1}$. Repeating the process we obtain a system of generators such that $\theta(e_1) = \cdots = \theta(e_v) = \bar{0}$, $\theta(d_2) = \cdots = \theta(d_g) = \bar{1}$ and $\theta(d_1)$ is determined by the long relation and the fact that $\theta(d_1) = \bar{e}$ where e is odd.

In Zarrow's paper [Z2] the condition 3 of Theorem 3 is replaced by ϕ_1^p and ϕ_2^p are conjugate. The next example shows the problems of his condition:

Example. Let Γ be an NEC group of signature $(0, +, [\ ], \{(\ )^3\})$ and let $e_1, e_2, e_3, c_{10}, c_{20}, c_{30}$ be a canonical system of generators for Γ. Consider the epimorphism θ_1: $\Gamma \to Z_{10}$ defined by $\theta_1(e_1) = \bar{0}$, $\theta_1(e_2) = \bar{2}$, $\theta_1(e_3) = \bar{8}$, $\theta_1(c_{i0}) = \bar{5}$, $i = 1, 2, 3$, and θ_2: $\Gamma \to Z_{10}$ defined by $\theta_2(e_1) = \bar{0}$, $\theta_2(e_2) = \bar{4}$, $\theta_2(e_3) = \bar{6}$, $\theta_2(c_{i0}) = \bar{5}$, $i = 1, 2, 3$. Then $\ker \theta_1 = \ker \theta_2$ and let X be $H^2/\ker \theta_1 = H^2/\ker \theta_2$. The epimorphisms θ_1 and θ_2 define two orientation reversing automorphisms ϕ_1 and ϕ_2 of order 10 on X. The automorphisms

ϕ_1 and ϕ_2 satisfy $X/\langle\phi_1\rangle \approx X/\langle\phi_2\rangle \approx H^2/\Gamma_1$ (sphere with three holes), ϕ_1^2 and ϕ_2^2 are conjugate (they are two fixed point free automorphisms of order five on X) and ϕ_1^5 and ϕ_2^5 are conjugate because they are two orientation reversing involutions on X with seven fixed curves; i.e. ϕ_1 and ϕ_2 satisfy the condition of Zarrow. The action of ϕ_1 on Fix ϕ_1^5 permutes cyclically five fixed curves of ϕ_1^5, there is a fixed curve of ϕ_1^5 rotating $2\pi/5$ and the other one rotating $-2\pi/5$. The action of ϕ_2 on Fix ϕ_2^5 permutes cyclically five curves, there is a curve of Fix ϕ_2^5 rotating $4\pi/5$ and the other one rotating $-4\pi/5$. Then the action of ϕ_1 on Fix ϕ_1^5 is not conjugate to the action of ϕ_2 on Fix ϕ_2^5 and ϕ_1 is not conjugate to ϕ_2.

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