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ON n-WIDTHS OF CLASSES OF HOLOMORPHIC FUNCTIONS WITH REPRODUCING KERNELS

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1. Introduction

Let A be a subset of a Banach space X. The Kolmogorov n-width of A in X is defined by

$$d_n(A, X) = \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|y - x\|$$

where X_n varies over all subspaces of X of dimension n.

The Gel'fand n-width is given by

$$d^n(A, X) = \inf_{Y_n} \sup_{x \in A \cap Y_n} \|x\|$$

where Y_n runs over all closed subspaces of X of codimension n.

The linear n-width is defined by

$$\delta_n(A, X) = \inf_{T_n} \sup_{x \in A} \|x - T_n x\|$$

where T_n runs over all linear operators of X into itself which have rank n or less. There are evident inequalities among these quantities; namely,

$$\delta_n(A, X) \geq d_n(A, X), \tag{1}$$

and

$$\delta_n(A, X) \geq d^n(A, X). \tag{2}$$

The concept of the n-width of a set was originally introduced by Kolmogorov [6] in 1936. Widths are important in approximation theory since knowledge of the exact or even the asymptotic value of the n-width can lead to best or near best methods of approximation and interpolation, as well as to the estimation of errors in these methods. Moreover, determination of an optimal subspace typically gives optimal methods of approximation, as well as fascinating

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interplay between approximation and the zeros of certain extremal functions. A bibliography of work on n-widths is in [9].

In recent years, the exact value of the n-width of sets of analytic functions has been the focus of attention. The first paper on this topic was Babenko [1]. The 1980 paper of Fisher and Micchelli [3] was the first to lay out general results in this area; other general results are in the paper of Osipenko and Stessin [8]. In [4] the present authors investigated the n-width of the unit ball of the Hardy space H_p in the L_q metric on a compact subset E in the unit disk of the complex plane. We were able to determine the precise value of this width in the case when $p \geq q$; when $p < q$, the situation is far more complicated and the answer is dependent, as well, on the "size" of the compact set.

In the present paper, we consider the n-width of the unit ball of a reproducing kernel Hilbert space of analytic functions in $C(E)$, the space of continuous complex-valued functions on a compact set E. We do this for three specific types of compact sets: a circle; a finite point set; and a subset of the open interval $(-1, 1)$. Our results in these settings are set forth in Sections
2, 3 and 4, respectively. The fifth and final section is devoted to investigations of the "skeleton" effect for the Hardy space H_2.

2. n-widths of the weighted Bergman space in the uniform metric on a circle

Let Δ be the open unit disk on the complex plane, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and let $\alpha > -1$ be a real number. The space $H_2^\alpha(\Delta)$ is the set of holomorphic functions on Δ which satisfy

$$\|f\|_\alpha^2 = \frac{1}{\pi} \int_\Delta |f(z)|^2 (1 - |z|^2)^\alpha \, dm(z) < \infty \tag{3}$$

where dm is Lebesgue area measure on Δ. When $\alpha = 0$, corresponding space is the Bergman space A_2 in the disk and when $\alpha \to -1$, H_2^α converges to the Hardy space H_2 in Δ. Let $0 < r < 1$ and denote by Γ_r the circle of radius r with center at the origin. In this section we determine the linear and Gel'fand n-widths $\delta_n(BH_2^\alpha, C(\Gamma_r))$ and $d^n(BH_2^\alpha, C(\Gamma_r))$ where BH_2^α is the unit ball of H_2^α in the norm (3). We shall need the following theorem from [4].

THEOREM 1. *Let X be a Hilbert space of holomorphic functions defined on the domain $\Omega \subset \mathbb{C}^m$ with reproducing kernel $K(z, w)$ and let E be a compact* *subset of* Ω. *Denote by* BX *the closed unit ball of* X. *Then*

$$\begin{aligned}\delta_n(BX, \mathbf{C}(E)) &= d^n(BX, \mathbf{C}(E)) \\ &= \inf_{\{\varphi_1, \ldots, \varphi_n\}} \sup_{z \in E} \left(K(z, z) - \sum_{i=1}^{n} |\varphi_i(z)|^2 \right)^{1/2}\end{aligned}$$

where $\{\varphi_1, \ldots, \varphi_n\}$ *runs over all orthonormal sets in* X *with* n *elements.*

This theorem was proved in [4] for the Hardy space H_2 in the disk Δ but the proof given there holds for an arbitrary domain in $\mathbb{C}^m$ and for an arbitrary Hilbert space with reproducing kernel.

It is well known that H_2^α has the reproducing kernel

$$K(z, w) = \frac{1}{(1 - z\overline{w})^{\alpha+2}}.$$

Put

$$a_k = \sqrt{\frac{\Gamma(\alpha + k + 2)}{k!\Gamma(\alpha + 1)}}\, r^k$$

and let $\{b_n\}_{n=0}^{\infty}$ be the decreasing rearrangement of the sequence $\{a_n\}_{n=0}^{\infty}$.

THEOREM 2.

$$\delta_n(BH_2^\alpha, \mathbf{C}(\Gamma_r)) = d^n(BH_2^\alpha, \mathbf{C}(\Gamma_r)) = \sqrt{\frac{1}{(1 - r^2)^{\alpha+2}} - \sum_{k=0}^{n-1} b_k^2}$$

To prove this theorem we need the following two lemmas.

LEMMA 1. *Let* $k_1, \ldots, k_m$ *be different natural numbers. If* $f \in BH_2^\alpha$ *and the derivatives of* f *of orders* $k_1, \ldots, k_m$ *at the origin are equal to zero, then*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \le \max_{k \ne k_1, \ldots, k_m} a_k^2$$

Proof. Let f^* be the extremal function in the problem

$$\sup\left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \colon \frac{1}{\pi} \int_\Delta |f(z)|^2 (1 - |z|^2)^\alpha \, dm(z) \le 1 \right\} \tag{4}$$

Write $f^* = \sum_{k=0}^{\infty} c_k z^k$ for $|z| < 1$. The problem (4) can be written in the equivalent form

$$\sup\left\{\sum_{k=0}^{\infty} |c_k|^2 r^{2k} \colon \sum_{k=0}^{\infty} a_k^{-2} |c_k|^2 \leq 1\right\} \tag{5}$$

The Lagrange principle for (5) implies that there is a number λ with

$$c_k r^{2k} = \lambda c_k \frac{k!\Gamma(\alpha+1)}{\Gamma(\alpha+k+2)}, \qquad k = 0, 1, \ldots. \tag{6}$$

It follows from (6) that if two coefficients of f^*, say c_m and c_k, are not zero, then

$$r = \left(\frac{k!\Gamma(\alpha+m+2)}{m!\Gamma(\alpha+k+2)}\right)^{1/2(k-m)}.$$

Hence, if we assume that r does not belong to the countable set

$$\left\{\left(\frac{k!\Gamma(m+\alpha+2)}{m!\Gamma(k+\alpha+2)}\right)^{1/2(k-m)}\right\}_{k,m=0,\, k>m}^{\infty} \tag{7}$$

then f^* must be a monomial. Note that if we add constraints $f^{(k_1)}(0) = 0, \ldots, f^{(k_n)}(0) = 0$, then for any extremal function $k \neq k_1, \ldots, k_n$. Since the supremum in (4) depends continuously on r, the result holds for all r. □

LEMMA 2. *If $\varphi_1, \ldots, \varphi_n$ is an orthonormal system in H_2^α, then*

$$\min_{|z|=r} \sum_{i=1}^{n} |\varphi_i(z)|^2 \leq \sum_{i=1}^{n} b_i^2$$

and equality holds if and only if span$(\varphi_1, \ldots, \varphi_n)$ *coincides with the span of the corresponding monomials.*

Proof. First we note that for any unitary transformation A of $\mathbb{C}^n$, the functions $\psi_1, \ldots, \psi_n$ defined by $\psi_i = (A\varphi)_i = \sum_{l=1}^{n} a_{il}\varphi_l$ form an orthonormal system in H_2^α and for every $z \in \Delta$, $\sum_{i=1}^{n} |\varphi_i(z)|^2 = \sum_{i=1}^{n} |\psi_i(z)|^2$. Note, too, that for every system $l_1, \ldots, l_n$ of linearly independent bounded complex-valued functionals on H_2^α, there exists a unitary transformation A of $\mathbb{C}^n$ such that corresponding system $\psi_1, \ldots, \psi_n$ satisfies

$$l_m(\psi_k) = 0, \qquad k > m.$$

Indeed, let $B = \|l_i(\varphi_j)\|_{i,\ j=1}^{n}$. There exists a unitary matrix A such that $C = A'BA$ is an upper-triangle matrix. Since C is the same transformation in the basis $\psi_i = (A\varphi)_i$ the system $\psi_1, \dots, \psi_n$ satisfies the required condition.

Now let $k_0, \dots, k_{n-1}$ be numbers such that $b_0 = a_{k_0}, \dots, b_{n-1} = a_{k_{n-1}}$. Put $l_i(f) = f^{(k_{i-1})}(0)$, $i = 1, \dots, n$. If A is the unitary transformation described above and $\psi_1, \dots, \psi_n$ is the corresponding orthonormal system, then from Lemma 1,

$$\frac{1}{2\pi}\int_0^{2\pi} |\psi_i(re^{i\theta})|^2\, d\theta \le \left(\max_{k \ne k_1, \dots, k_{i-1}} a_k^2\right) = a_{k_i}^2 = b_i^2$$

and therefore

$$\min_{z \in C_r} \sum_{i=1}^{n} |\varphi_i(z)|^2 = \min_{z \in C_r} \sum_{i=1}^{n} |\psi_i(z)|^2 \le \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=1}^{n} |\psi_i(re^{i\theta})|^2\, d\theta \le \sum_{i=0}^{n-1} b_i^2.$$

It is obvious that equality holds if and only if $\operatorname{span}(\varphi_1, \dots, \varphi_n) = \operatorname{span}(z^{k_0}, \dots, z^{k_{n-1}})$. □

Proof of Theorem 2. From Theorem 1 and Lemma 2 we have

$$\begin{aligned}
\delta_n(BH_2^\alpha, \mathrm{C}(\Gamma_r)) &= d^n(BH_2^\alpha, \mathrm{C}(\Gamma_r)) \\
&= \inf_{\varphi_1, \dots, \varphi_n} \sup_{z \in \Gamma_r} \sqrt{\frac{1}{(1 - |z|^2)^{2+\alpha}} - \sum_{i=1}^{n} |\varphi_i(z)|^2} \\
&= \inf_{\varphi_1, \dots, \varphi_n} \sqrt{\frac{1}{(1 - r^2)^{2+\alpha}} - \inf_{z \in \Gamma_r} \sum_{i=1}^{n} |\varphi_i(z)|^2} \\
&\ge \sqrt{\frac{1}{(1 - r^2)^{2+\alpha}} - \sum_{i=1}^{n} b_i^2}\,.
\end{aligned}$$

Further, equality is attained when

$$\varphi_i = \frac{\Gamma(\alpha + k_i + 2)}{k!\Gamma(\alpha + 1)} z^{k_i},\ i = 1, \dots, n. \qquad \square$$

Remark 1. Theorem 2 is an extension of the result [8], but in [8] the optimal n-dimensional space was the space of polynomials of degree at most $n - 1$. Here the situation is different; the best space is also the span of n monomials but degrees of these monomials depend on r.

Remark 2. It is easy to see that Theorem 2 can be extended to other Hilbert spaces of analytic functions. For instance, if X is a Hilbert space of functions holomorphic in Δ with the norm

$$\|f\|_X^2 = \frac{1}{\pi}\int_\Delta |f(z)|^2 \varphi(|z|)\, dm(z)$$

where φ is an integrable positive function on the interval $(0,1)$, then the same result holds. Namely, put

$$c_n = \left(\frac{1}{\pi}\int_\Delta |z|^{2n}\varphi(|z|)\, dm(z)\right)^{-1} = \left(|z^n|_X^2\right)^{-1}$$

and

$$K_X(z,w) = \sum_{n=0}^{\infty} c_n z^n \overline{w}^n.$$

The kernel $K_X(z,w)$ is evidently correctly defined. Let $a_k = \sqrt{c_k}\, r^k$ and $\{b_k\}_{k=0}^{\infty}$ be the decreasing rearrangement of the sequence $\{a_k\}$. Then

$$\delta_n(BX, \mathbf{C}(\Gamma_r)) = d^n(BX, \mathbf{C}(\Gamma_r)) = \sqrt{K_X(r,r) - \sum_{i=0}^{n-1} b_k^2}\,.$$

3. Finite point sets

Let Ω be a domain in $\mathbb{C}^n$, E a finite set of points $\{z_1, \dots, z_N\}$ in Ω and X a Hilbert space of functions holomorphic in Ω with reproducing kernel $K(z,w)$. Put $L(z_1, \dots, z_N) = \operatorname{span}\{K(z, z_1), \dots, K(z, z_N)\}$; we consider $K(z, z_i)$ as a function of its first argument.

Proposition 1. *For* $0 \le n \le N-1$,

$$\begin{aligned}\delta_n(BX, \mathbf{C}(E)) &= d^n(BX, \mathbf{C}(E)) \\ &= \inf_{\{\varphi_1, \dots, \varphi_{N-n}\} \in L(z_1, \dots, z_N)} \ \sup_{1 \le j \le N} \sqrt{\sum_{i=1}^{N-n} |\varphi_i(z_j)|^2}\end{aligned}$$

where $\{\varphi_1, \dots, \varphi_{N-n}\}$ *varies over all orthonormal systems in* $L(z_1, \dots, z_N)$ *of length* $N-n$.

Proof. If g is in X and g is orthogonal to $K(z, z_1), \dots, K(z, z_N)$, then $g(z_1) = \cdots = g(z_N) = 0$. Hence, if $\{\varphi_1, \dots, \varphi_N\}$ is an orthonormal basis for

$L(z_1, \ldots, z_N)$, then

$$K(z_j, z_j) = \sum_{i=1}^{N} |\varphi_i(z_j)|^2 \quad \text{for } j = 1, \ldots, N.$$

Thus, for every orthonormal system $\{\varphi_1, \ldots, \varphi_n\}$ in $L(z_1, \ldots, z_N)$, we have

$$K(z_j, z_j) - \sum_{i=1}^{n} |\varphi_i(z_j)|^2 = \sum_{i=n+1}^{N} |\varphi_i(z_j)|^2.$$

In view of Theorem 1 the last equality implies

$$\begin{aligned} \delta_n(BX, \mathbf{C}(E)) &= d^n(BX, \mathbf{C}(E)) \\ &\leq \inf_{\{\varphi_1, \ldots, \varphi_{N-n}\} \subset L(z_1, \ldots, z_N)} \sup_{1 \leq j \leq N} \sqrt{\sum_{i=1}^{N-n} |\varphi_i(z_j)|^2}. \end{aligned}$$

Now the let $\{\varphi_1, \ldots, \varphi_n\}$ be any orthonormal system in X. There exist $N - n$ functions $\varphi_{n+1}, \ldots, \varphi_N \in L(z_1, \ldots, z_N)$ such that $\{\varphi_1, \ldots, \varphi_N\}$ form an orthonormal system. Since at every point $z \in \Delta$, $K(z, z) \geq \sum_{i=1}^{N} |\varphi_i(z)|^2$ we have

$$\begin{aligned} &\sup_{1 \leq j \leq N} \sqrt{K(z_j, z_j) - \sum_{i=1}^{n} |\varphi_i(z_j)|^2} \\ &\quad \geq \sup_{1 \leq j \leq N} \sqrt{\sum_{i=1}^{N} |\varphi_i(z_j)|^2 - \sum_{i=1}^{n} |\varphi_i(z_j)|^2} \\ &\quad = \sup_{1 \leq j \leq N} \sqrt{\sum_{i=n+1}^{N} |\varphi_i(z_j)|^2} \\ &\quad \geq \inf_{\{\varphi_1, \ldots, \varphi_{N-n}\} \subset L(z_1, \ldots, z_N)} \sup_{1 \leq j \leq N} \sqrt{\sum_{i=1}^{N-n} |\varphi_i(z_j)|^2}. \end{aligned}$$

Now the proposition follows from Theorem 1. □

The equivalent statement of this proposition is the following.

PROPOSITION 1′. *If E is a subset of Δ and X is some Hilbert space of holomorphic functions with reproducing kernel $K(z, w)$, then the best*

n-dimensional subspace for linear approximation of BX belongs to the closure of

$$\operatorname{span}\{K(z,w): w \in E\}.$$

THEOREM 3. *Let $1 \le n \le N$. Suppose that $\varphi_1^*, \dots, \varphi_n^* \in L(z_1, \dots, z_N)$ satisfy*

$$\delta_{N-n}^2 = \max_{1 \le i \le N} \sum_{j=1}^{n} \left|\varphi_j^*(z_i)\right|^2 = \min_{(\varphi_1, \dots, \varphi_n)} \max_{1 \le i \le N} \sum_{j=1}^{n} \left|\varphi_j(z_i)\right|^2. \tag{8}$$

Then there are at least $N - n + 1$ points of E at which equality holds in (8).

Proof. Let E_0 be those points of $w \in E$ at which equality holds in (8). For each $w \in E_0$, the vector

$$v(w) = (\varphi_1^*(w), \dots, \varphi_n^*(w))^t$$

is non-zero. Hence, there is $n \times n$ unitary matrix U such that $Uv(w)$ does not lie in the hyperplane

$$\zeta_1 + \cdots + \zeta_n = 0$$

for all $w \in E_0$. The functions $\psi_j := \sum_{m=1}^{n} U_{jm}\varphi_m^*$ are still orthonormal, the set of points at which

$$\sum_{j=1}^{n} \left|\psi_j(w)\right|^2 = \delta_{N-n}^2$$

is still E_0, and, moreover, by the choice of U,

$$\sum_{j=1}^{n} \psi_j(w) \ne 0, \qquad w \in E_0. \tag{9}$$

Let $\varphi \in L(z_1, \dots, z_N)$ be orthogonal to $\psi_1, \dots, \psi_n$ and have norm one. The system

$$\frac{\psi_j + \varepsilon\varphi}{(1 + \varepsilon^2)^{1/2}}, \qquad j = 1, \dots, n \tag{10}$$

is still orthonormal and on E_0

$$\sum_{j=1}^{n} \left| \frac{\psi_j(w) + \varepsilon\varphi(w)}{(1+\varepsilon^2)^{1/2}} \right|^2$$

$$= \frac{1}{1+\varepsilon^2} \left\{ \delta_{N-n}^2 + 2\varepsilon \operatorname{Re}\left(\overline{\varphi(w)} \sum_{j=1}^{n} \psi_j(w) \right) + \varepsilon^2 n |\varphi(w)|^2 \right\}. \quad (11)$$

Assume now that E_0 has $p \leq N - n$ points. We shall show that the right-hand side of (11) may be made strictly less than δ_{N-n}^2 on E_0. Since

$$\sum_{j=1}^{n} |\psi_j(z)|^2 < \delta_{N-n}^2, \qquad z \in E \setminus E_0$$

it follows that the left-hand side of (11) is strictly less than δ_{N-n}^2 on E for sufficiently small ε. Thus, the system (10) is orthonormal and

$$\sup_E \sum_{j=1}^{n} \left| \frac{\psi_j(z) + \varepsilon\varphi(z)}{(1+\varepsilon^2)^{1/2}} \right|^2 < \delta_{N-n}^2,$$

a contradiction.

Let T be the linear operator from $L(z_1, \dots, z_N)$ to $\mathbb{C}^{n+p}$ defined by

$$Tf = \left(\{\langle f, \psi_i \rangle\}_{i=1}^{n}, \{f(w_j)\}_{j=1}^{p} \right)$$

where we have written $E_0 = \{w_1, \dots, w_p\}$, $p \leq N - n$. If T is not one-to-one, then there is an element φ of $L(z_1, \dots, z_N)$ of norm one for which $T\varphi = 0$. This φ clearly makes the right-hand side of (11) less than δ_{N-n}^2. On the other hand, if T is one-to-one (and so necessarily, $p = N - n$), then T is onto and therefore there is a $\varphi \in L(z_1, \dots, z_N)$ with norm one and

$$\langle \varphi, \psi_i \rangle = 0, \qquad i = 1, \dots, n$$

and

$$\overline{\varphi(w_j)} \sum_{l=1}^{n} \psi_l(w_j) = -\rho, \qquad j = 1, \dots, N - n$$

where ρ is some positive number. Once again, it follows that the right-hand side of (11) is less than δ_{N-n}^2. $\square$

DEFINITION. Let $\varphi_1, \dots, \varphi_n$ be an orthonormal system in X for which

$$\sup_E \sum_{j=1}^{n} |\varphi_j(z)|^2 = \delta_{N-n}^2.$$

The *deficiency of* $\{\varphi_1, \dots, \varphi_n\}$ is the number of points $z' \in E$ at which

$$\sum_{j=1}^{n} |\varphi_j(z')|^2 < \delta_{N-n}^2.$$

From Theorem 3 we know that the deficiency of any orthonormal system of n functions in $L(z_n, \dots, z_n)$ is never more than $n - 1$. This leads us to define the *n-th deficiency of* E as the largest deficiency of any orthonormal system of n elements.

PROPOSITION 2. *Let r be the n-th deficiency of E. Then there is a numbering $\{z_1, \dots, z_N\}$ of E and an orthonormal system $\psi_1, \dots, \psi_n$ in $L(z_1, \dots, z_N)$ such that*

$$\sup_{z \in E} \sum_{j=1}^{n} |\psi_j(z)|^2 = \delta_{N-n}^2$$

and

$$\psi_j(z_k) = 0, \qquad 1 \le j \le r, \qquad r + 1 \le k \le N.$$

Proof. By the definition of r there is a numbering $\{z_1, \dots, z_N\}$ of E and an orthonormal system $\varphi_1, \dots, \varphi_n$ in $L(z_1, \dots, z_N)$ such that

$$\sum_{j=1}^{n} |\varphi_j(z_k)|^2 < \delta_{N-n}^2, \qquad k = 1, \dots, r$$

and

$$\sum_{j=1}^{n} |\varphi_j(z_k)|^2 = \delta_{N-n}^2, \qquad k = r + 1, \dots, N.$$

Let $F_1, \dots, F_N$ be an orthonormal basis for $L(z_1, \dots, z_N)$ chosen so that

$$F_k(z_j) = 0, \qquad 1 \le k < j \le N. \tag{12}$$

Each φ_l has an expansion

$$\varphi_l = \sum_{j=1}^{N} a_{lj} F_j, \qquad l = 1, \dots, n.$$

There is $n \times n$ unitary matrix U such that

$$\psi_l := \sum_{j=1}^{n} U_{lj} \varphi_j, \qquad l = 1, \dots, n$$

has the representation

$$\psi_l = \sum_{i=1}^{n} b_{il} F_i, \qquad l = 1, \dots, n. \tag{13}$$

That is, the matrix B which represents $\psi_1, \dots, \psi_n$ in terms of $F_1, \dots, F_N$ is upper triangular:

$$B = \begin{pmatrix} b_{11} & \cdots & & \cdots & b_{1n} \\ 0 & & & & \\ \vdots & \ddots & & & \\ 0 & 0 & b_{nn} & \cdots & b_{nN} \end{pmatrix}.$$

We shall prove that

$$\psi_l(z_i) = 0, \qquad 1 \le l \le r, \qquad r + 1 \le i \le N, \tag{14}$$

which is, of course, the desired conclusion.

Fix l, $1 \le l \le r$, and let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l)$ be a unit vector in $\mathbb{C}^l$ such that

$$\sum_{j=1}^{l} \xi_j b_{pj} = 0, \qquad 1 \le p \le l - 1.$$

Define

$$\psi_l^{(\varepsilon)}(z) = \delta \sum_{j=1}^{l} \xi_j F_j(z) + (1 - \varepsilon) \psi_l(z)$$

where δ is the unique positive root of the quadratic

$$\delta^2 \left(1 - |\xi_l|^2\right) + \left(\delta + (1 - \varepsilon)\right)^2 |b_{ll}|^2 + (1 - \varepsilon)^2 \left(1 - |b_{ll}|^2\right) = 1.$$

This choice of δ gives $\psi_l^{(\varepsilon)}$ unit norm; note further that as $\varepsilon \to 0$, δ goes to zero and $\psi_l^{(\varepsilon)} \to \psi_l$ uniformly on Δ. By construction, the system

$$\{\psi_1, \dots, \psi_{l-1}, \psi_l^{(\varepsilon)}, \psi_{l+1}, \dots, \psi_n\}$$

is orthonormal. Moreover, the sum

$$S(z_k) := \sum_{j=1}^{l-1} |\psi_j(z_k)|^2 + |\psi_l^{(\varepsilon)}(z_k)|^2 + \sum_{j=l+1}^{n} |\psi_j(z_k)|^2$$

is certainly strictly less than δ_{N-n}^2 for ε sufficiently small when $k = 1, \dots, r$. Further, $S(z_k)$ is also strictly less than δ_{N-n}^2 for $k = r + 1, \dots, N$ unless

$$\psi_l^{(\varepsilon)}(z_k) = 0.$$

Since r is the deficiency of E, it is impossible to have $S(z_k) < \delta_{N-n}^2$ at any of the points z_k, $r + 1 \le k \le N$. It follows that $\psi_l^{(\varepsilon)}(z_k) = 0$ and so also $\psi_l(z_k) = 0$ for each k, $r + 1 \le k \le N$. □

Corollary. *The functions $\psi_1, \dots, \psi_r$ described in Proposition 2 are exactly $F_1, \dots, F_r$, respectively.*

Proof. We have

$$\psi_l = \sum_{j=l}^{N} b_{lj} F_j, \qquad l = 1, \dots, r.$$

However,

$$b_{lN} F_N(z_N) = \psi_l(z_N) = 0$$

and so $b_{lN} = 0$, $l = 1, \dots, r$. This implies

$$b_{l,N-1} F_{N-1}(z_{N-1}) = 0$$

and so $b_{l,N-1} = 0$, $l = 1, \dots, r$. Continuing we obtain

$$\psi_l = \sum_{j=l}^{r} b_{lj} F_j.$$

However, for $1 \leq l < m \leq r$, we also have

$$0 = \langle \psi_l, \psi_m \rangle = \sum_{j=m}^{r} b_{lj} \bar{b}_{mj}$$

and so $b_{lj} = 0$ when $l \neq j$. Finally,

$$1 = \|\psi_l\| = |b_{ll}| \, \|F_l\| = |b_{ll}|$$

and so we may assume that $b_{ll} = 1$, $l = 1, \ldots, r$. □

Remark. The functions $\psi_1, \ldots, \psi_r$ of Proposition 2 lie in $L(z_1, \ldots, z_N)$ and vanish at $z_{r+1}, \ldots, z_N$. Hence, they lie in

$$L(z_1, \ldots, z_N) \ominus L(z_{r+1}, \ldots, z_N)$$

From Theorem 2 we then conclude that there are n orthonormal functions $f_1, \ldots, f_n$ and points $z_1, \ldots, z_m$, $m \geq n + 1$, in E such that

$$\text{(a)} \qquad \delta_n^2 = \sup\nolimits_E \left\{ K(z, z) - \sum_{j=1}^{n} |f_j(z)|^2 \right\}$$

$$\text{(b)} \qquad \delta_n^2 = K(z_k, z_k) - \sum_{j=1}^{n} |f_j(z_k)|^2, \qquad k = 1, \ldots, m \tag{15}$$

$$\text{(c)} \qquad f_1, \ldots, f_n \in L(z_1, \ldots, z_m).$$

We now use (15) to establish a result similar to Theorem 3 for any compact set E.

In Theorem 4 we shall assume that for each $w \in \Omega$ the kernel function $K(z, w)$ is analytic on an open set (depending on w) which contains the closure of Ω and that $|K(z, w)|$ is bounded above by a constant depending only on the distance from w to the boundary of Ω. This conditions are satisfied, for instance, if the boundary of Ω is analytic and X is the Bergman space.

THEOREM 4. *Let the above hypotheses on $K(z, w)$ hold. Let E be a compact set in Ω. Then there are orthonormal functions $f_1, \ldots, f_n$ in X such that*

$$\sup_E \left\{ K(z, z) - \sum_{j=1}^{n} |f_j(z)|^2 \right\} = \delta_n^2 \tag{16}$$

and equality holds at $n + 1$ or more points of E.

Proof. We may assume that E has infinitely many points. Let $\{\xi_k\}_{k=1}^{\infty}$ be a countable dense set in E and let $E_N = \{\xi_1, \ldots, \xi_N\}$, $N = 1, 2, \ldots,$. We note first that the numbers

$$\delta_{nN} = n - \text{width of the unit ball of } X \text{ in } \mathbf{C}(E_N)$$

increase with N and have as their limit the n-width of the unit ball of X is $\mathbf{C}(E)$.

We apply (15) to E_N: for each N, there are n orthonormal functions $f_{1N}, \ldots, f_{nN}$ and $m = m(N) \geq n + 1$ points $\{z_{1N}, \ldots, z_{mN}\} \subset E_N$ such that

$$K(z, z) - \sum_{j=1}^{n} |f_{jN}(z)|^2 \leq \delta_{nN}^2, \qquad z \in E_N$$

$$K(z_{kN}, z_{kN}) - \sum_{j=1}^{n} |f_{jN}(z_{kN})|^2 = \delta_{nN}^2, \qquad k = 1, \ldots, m.$$

Furthermore, $f_{1N}, \ldots, f_{nN}$ lie in $L(z_{1N}, \ldots, z_{mN})$. Each kernel function $K(z, w)$ is holomorphic in a neighborhood of the closure of Ω and bounded there by a bound depending on the distance of w to $\partial\Omega$. Hence, there is a fixed open set U containing the closure of Ω and a constant M such that

$$|f_{jN}(z)| \leq M, \quad z \in U; \quad j = 1, \ldots, n, \quad N = 1, 2, \ldots .$$

This implies that $\{f_{jN}\}_{N=1}^{\infty}$ forms a normal family on a neighborhood of $\overline{\Omega}$ and so a subsequence coverges uniformly as $N \to \infty$ on $\overline{\Omega}$ and in the norm of X to a function f_j. This gives n functions $f_1, \ldots, f_n$ which are orthonormal and for which

$$K(z, z) - \sum_{j=1}^{n} |f_j(z)|^2 \leq \delta_n^2, \qquad z \in E. \tag{17}$$

Moreover, equality holds in (17) for any point z' which is the limit point of a sequence drawn from $\{z_{kN}\}$. If this collection of points has n or fewer elements, say $\{w_1, \ldots, w_p\}$, $p \leq n$, then

$$f_1, \ldots, f_n \in L(w_1, \ldots, w_p).$$

Hence, $p = n$; but also any element of X which is orthogonal to $L(w_1, \ldots, w_n)$ must vanish at all of $w_1, \ldots, w_n$. This implies that

$$K(w_k, w_k) = \sum_{j=1}^{n} |f_j(w_k)|^2, \qquad k = 1, \ldots, n.$$

But then $\delta_n = 0$, a contradiction. □

4. Compact subsets of the real axis

In this section we consider the case when compact set E lies in the interval $(-1,1)$. We shall assume that the kernel $K(z,w)$ satisfies

$$K(x, y) \text{ is strictly totally positive for } -1 < x,\, y < 1. \tag{18}$$

PROPOSITION 3. *Let E be a finite set in $(-1,1)$ and suppose that* (18) *holds. Then there is a solution $\{\varphi_1^*, \dots, \varphi_n^*\}$ of* (8) *such that each φ_j^* is real on the interval $(-1,1)$.*

Proof. Let χ_j, $1 \le j \le N$, be the function in $L(x_1, \dots, x_N)$ with

$$\chi_j(x_k) = (-1)^j \delta_{jk}, \qquad 1 \le j, k \le N.$$

Let

$$A_{ij} = \langle \chi_i, \chi_j \rangle, \qquad 1 \le i, j \le N.$$

Then

$$A_{ij} = D_{ij}/D$$

where

$$D = \det\| K(x_r, x_s) \|_{r, s=1,\dots,N} \quad \text{and} \quad D_{ij} = \det\| K(x_r, x_s) \|_{r \ne i,\, s \ne j}.$$

The functions χ_j are real on $(-1,1)$. The matrix A is strictly totally positive; see [5], formulas (0.10) and (9.1).

In terms of these numbers and the basis $\{\chi_j\}$, the width problem given in (8) may be rephrased in this way:

$$\min_C \left\{ \max_{1 \le k \le N} \left(\sum_{j=1}^{n} |c_{jk}|^2 \right)^{1/2} : CAC^* = I \right\}$$

where C is an $n \times N$ matrix, C^* is its adjoint, and I is the $n \times n$ identity matrix.

We shall find it more convenient to solve the related extremal problem for $p < \infty$:

$$\nu_p := \min_C \left\{ \left(\frac{1}{N} \sum_{k=1}^{N} \left(\sum_{j=1}^{n} |c_{jk}|^2 \right)^{p/2} \right)^{1/p} : CAC^* = I \right\} \tag{19}$$

and then pass to the limit as $p \nearrow \infty$. The details related to the limit argument are not elaborate and we postpone them to later in the proof.

Suppose that $C = (c_{jk})$ is a solution of (19). The numbers

$$\sum_{j=1}^{n} |c_{jk}|^2, \qquad k = 1, \ldots, N$$

play an important role in what follows. If necessary, we may renumber the points $x_1, \ldots, x_N$ so that

$$\sum_{j=1}^{n} |c_{jk}|^2 > 0 \quad \text{for } 1 \leq k \leq R$$

and

$$\sum_{j=1}^{n} |c_{jk}|^2 = 0 \quad \text{for } R + 1 \leq k \leq N.$$

Let A_R be the $R \times R$ block in the upper left corner of A. Then among all $n \times R$ matrices B satisfying

$$BA_R B^* = I \tag{20}$$

the $n \times R$ matrix $C = (c_{jk})_{j=1,\ldots,n}^{k=1,\ldots,R}$ minimizes the functional

$$H(B) = \left[\sum_{k=1}^{R} \left(\sum_{j=1}^{n} |b_{jk}|^2 \right)^{p/2} \right]^{1/p}. \tag{21}$$

The constraints in (20) may be rephrased as

$$G_{lm}(B) := \sum_{i,j} A_{ij} c_{li} \bar{c}_{mj} - \delta_{lm} = 0, \qquad 1 \leq l \leq m \leq n. \tag{22}$$

This allows us to use Lagrange multipliers to conclude that there are scalars μ_{lm}, $1 \leq l \leq m \leq n$, such that

$$(\nabla H)(C) = \sum_{l \leq m} \mu_{lm} (\nabla G_{lm})(C). \tag{23}$$

From (23) we obtain two sets of equations:

$$\nu_p^{1-p} \left(\sum_{j=1}^{n} |c_{js}|^2 \right)^{p/2-1} \bar{c}_{rs} = \sum_{m \geq r} \mu_{rm} \sum_{j=1}^{R} A_{sj} \bar{c}_{mj} \tag{24}$$

and

$$\nu_p^{1-p}\left(\sum_{j=1}^{n}|c_{js}|^2\right)^{p/2-1} c_{rs} = \sum_{l\le r}\mu_{lr}\sum_{i=1}^{R}A_{is}c_{li}. \tag{25}$$

Set

$$\lambda_s = 2\nu_p^{1-p}\left(\sum_{j=1}^{n}|c_{js}|^2\right)^{p/2-1}, \qquad s = 1,\dots,R,$$

$$\Lambda = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_R\end{pmatrix}$$

and for l, $m = 1,\dots,n$,

$$M_{lm} = \begin{cases}\mu_{lm} & \text{if } l < m\\ \mu_{ml} & \text{if } l > m\\ \mu_{ll} + \bar{\mu}_{ll} & \text{if } l = m.\end{cases}$$

Then (24) and (25) may be combined to yield

$$C\Lambda = MCA.$$

Since M is self-adjoint, there is unitary matrix U and a real diagonal matrix D with $M = U^{-1}DU$. Since $\lambda_s > 0$ for $s = 1,\dots,R$, the matrix Λ is invertible. Thus we obtain

$$UC = DUCA\Lambda^{-1}.$$

Put $C_1 = (UC)^t$, $A_1 = (A\Lambda^{-1})^t$. Then

$$C_1 = A_1C_1D.$$

Equivalently, if $w_1,\dots,w_n$ are the columns of C_1 and $\gamma_1,\dots,\gamma_n$ are the diagonal entries of D, we have

$$w_k = \gamma_k A_1 w_k, \qquad k = 1,\dots,n. \tag{26}$$

Since γ_k and A_1 are real, the equation (26) implies that both $\mathrm{Re}(w_k)$ and $\mathrm{Im}(w_k)$ are eigenvectors for A_1 with the same eigenvalue. However, A_1 is strictly totally positive and so its eigen-values have multiplicity one. Equivalently, w_k is scalar multiple of a real vector. That is, up to a unitary transformation, the rows of C are real.

We now show how to pass to the limit as $p \to \infty$. For $p < \infty$,

$$\left[\frac{1}{N}\sum_{k=1}^{N}\left(\sum_{j=1}^{n}|c_{jk}|^2\right)^{p/2}\right]^{1/p} \leq \max_{1\leq k\leq N}\left(\sum_{j=1}^{n}|c_{jk}|^2\right)^{1/2}$$

and hence

$$\nu_p \leq \inf_C \max_{1\leq k\leq n}\left(\sum_{j=1}^{n}|c_{jk}|^2\right)^{1/2} = \delta_{N-n}.$$

Now let $\{c_{jk}^{(p)}\}$, $1 \leq j \leq n$, $1 \leq k \leq N$, be a solution of (19) with the property that

$$\varphi_j^{(p)} := \sum_{k=1}^{N} c_{jk}^{(p)}\chi_k, \qquad 1 \leq j \leq n,$$

is real-valued on the interval $(-1, 1)$. A compactness argument shows that there is a sequence $p_i \to \infty$ and numbers $\{c_{jk}\}$ such that

$$c_{jk}^{(p)} \to c_{jk} \quad \text{as } p = p_i \to \infty.$$

Since

$$C^{(p)}A(C^{(p)})^* = I$$

we have

$$CAC^* = I.$$

Moreover,

$$\delta_{N-n} \geq \nu_p = \left(\frac{1}{N}\sum_{k=1}^{N}\left(\sum_{j=1}^{n}\left|c_{jk}^{(p)}\right|^2\right)^{p/2}\right)^{1/p} \to \max_{1\leq k\leq N}\left(\sum_{j=1}^{n}|c_{jk}|^2\right)^{1/2} \geq \delta_{N-n}.$$

It follows that the functions

$$\varphi_j := \sum_{k=1}^{N} c_{jk}\chi_k$$

are real-valued on $(-1, 1)$, orthonormal in X, and solve (8). We are done. $\square$

For a Hilbert space X of analytic functions with strictly totally positive kernel $K(z,w)$ we let $X_{\mathbb{R}} = \{f \in X\colon \operatorname{Im} f(x) = 0 \text{ for } x \text{ real}\}$. When we defined n-widths we did not specify over what field, $\mathbb{R}$ or $\mathbb{C}$, we considered the dimensions of subspaces. Normally when considering classes of holomorphic functions we mean the complex-dimension but for the following theorem we separate these two cases and use the notation $\delta_n^{\mathbb{R}}, d_{\mathbb{R}}^n$ or $\delta_n^{\mathbb{C}}, d_{\mathbb{C}}^n$, respectively.

THEOREM 5. *Let X be a Hilbert space of analytic functions on Δ with a strictly totally positive reproducing kernel and let E be a compact subset of the interval* $(-1,1)$. *Then*

$$\delta_n^{\mathbb{R}}(BX_{\mathbb{R}}, \mathbf{C}(E)) = d_{\mathbb{R}}^n(BX_{\mathbb{R}}, \mathbf{C}(E)) = \delta_n^{\mathbb{C}}(BX, \mathbf{C}(E)) = d_{\mathbb{C}}^n(BX, \mathbf{C}(E)).$$

Proof. Let $\varphi_1, \dots, \varphi_n$ be an orthonormal system in $X_{\mathbb{R}}$. It is well-known in optimal recovery theory (see [7]) that the linear operator

$$A\colon BX \to \operatorname{span}_{\mathbb{C}}\{\varphi_1, \dots, \varphi_n\}$$

that minimizes $\sup_{f \in BX} \|f - Af\|_{\mathbf{C}(E)}$ is defined by

$$Af = \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i$$

and the worst function is

$$f^* = \frac{K(z,x) - \sum_{i=1}^n \varphi_i(x)\varphi_i(z)}{\sqrt{K(x,x) - \sum_{i=1}^n |\varphi_i(x)|^2}}$$

for some $x \in E$. Since the same operator and the same function are the best operator and the worst function for the problem of approximation $BX_{\mathbb{R}}$ by $\operatorname{span}_{\mathbb{R}}\{\varphi_1, \dots, \varphi_n\}$, we have $\delta_n^{\mathbb{C}}(BX, \mathbf{C}(E)) \le \delta_n^{\mathbb{R}}(BX_{\mathbb{R}}, \mathbf{C}(E))$. On the other hand we know from Lemma 3 that for any finite set $\{x_1, \dots, x_N\} \subset E$, the Gel'fand n-width $d_{\mathbb{C}}^n(BX, \mathbf{C}(\{x_1, \dots, x_N\}))$ is attained on the subspace spanned by the orthonormal system from $X_{\mathbb{R}}$. Since

$$d_{\mathbb{C}}^n(BX, \mathbf{C}(E)) = \sup_{\{x_1, \dots, x_N\}} d_{\mathbb{C}}^n(BX, \mathbf{C}(\{x_1, \dots, x_N\})),$$

we have

$$d_{\mathbb{C}}^n(BX, \mathbf{C}(E)) \ge d_{\mathbb{R}}^n(BX_{\mathbb{R}}, \mathbf{C}(E)).$$

The theorem now follows from (1) and Theorem 1. □

THEOREM 6. *Let X be a Hilbert space of holomorphic functions on the unit disk Δ with a strictly totally positive kernel $K(x,w)$ and E be a compact set in the interval $(-1,1)$. Then for every n there exists a finite subset $\{x_1^*,\dots,x_r^*\} \subset E$ and n functions $\varphi_1^*,\dots,\varphi_n^*$ which form an orthonormal system in* $\operatorname{span}\{K(z,x_1^*),\dots,K(z,x_r^*)\}$ *that* $\operatorname{span}\{\varphi_1^*,\dots,\varphi_n^*\}$ *is the best linear subspace; that is*

$$\begin{aligned} d^n(BX, \mathrm{C}(E)) &= \delta_n(BX, \mathrm{C}(E)) \\ &= \max_{x\in E} \sup_{f\in BX} \left| f(x) - \sum_{i=1}^{n} \langle f, \varphi_i^* \rangle \varphi_i^*(x) \right| \\ &= \max_{x\in\{x_1^*,\dots,x_r^*\}} \sqrt{K(x,x) - \sum_{i=1}^{n} |\varphi_i^*(x)|^2} \end{aligned}$$

Proof. Consider the extremal problem

$$\sup_{x_1,\dots,x_N \subset E} \delta_n(BX, \mathrm{C}(\{x_1,\dots,x_N\})).$$

Let $\{\hat{x}_1,\dots,\hat{x}_N\}$ be an extremal set of points in this problem and let

$$\hat{\varphi}_1^N,\dots,\hat{\varphi}_n^N \in L(x_1,\dots,x_N)$$

be the corresponding orthonormal system, that is

$$\delta_n^N = \delta_n(BX, \mathrm{C}(\{\hat{x}_1,\dots,\hat{x}_N\}) = \max_{1\le i\le N} \sqrt{K(x_i,x_i) - \sum_{l=1}^{n} |\hat{\varphi}_l^N(x_i)|^2}.$$

In accordance with Lemma 3, we may assume that all the functions $\hat{\varphi}_1^N,\dots,\hat{\varphi}_n^N$ are real on the real axis. Let $\hat{x}_{i_1},\dots,\hat{x}_{i_{m_N}}$ be $\{\hat{\varphi}_1^N,\dots,\hat{\varphi}_n^N\}$ points of (8). It is easy to check that $\hat{\varphi}_1^N,\dots,\hat{\varphi}_n^N \in L(\hat{x}_{i_1},\dots,\hat{x}_{i_{m_N}})$. Since $\hat{\varphi}_l^N$ are real on the real axis, we have

$$K\left(\hat{x}_{i_m}^N, \hat{x}_{i_m}^N\right) - \sum_{l=1}^{n} \left(\hat{\varphi}_l^N\left(\hat{x}_{i_m}^N\right)\right)^2 = \left(\delta_n^N\right)^2. \tag{27}$$

The functions $\{\hat{\varphi}_1^N,\dots,\hat{\varphi}_n^N\}$ converge uniformly on compact subsets to a system $\{\varphi_1^*,\dots,\varphi_n^*\}$ and all these functions are real on the real axis. For any point $x \in E$ that is the limit point for some sequence $\{\hat{x}_{i_m}^N\}_{N=n+1}^\infty$, we have

$$K(x,x) - \sum_{l=1}^{n} (\varphi_l^*(x))^2 = (\delta_n)^2$$

where $\delta_n = \lim_{N\to\infty} \delta_n^N = \delta_n(BX, \mathbf{C}(E))$. Now it follows immediately from the uniqueness theorem that the number of such limit points is finite because otherwise we would have that

$$K(z, \bar{z}) = \sum_{l=1}^{n} (\varphi_l^*(z))^2 + (\delta_n)^2 \tag{28}$$

for all $z \in \Delta$. But $\varphi_l^*(z)$ is bounded in Δ and $K(x, x)$ is unbounded when $x \to 1$, $x \in (-1, 1)$. (We assume that $\dim X = \infty$.) So (28) is impossible.

Denote this finite number of limit points by $x_1^*, \ldots, x_r^*$. Then $\varphi_1^*, \ldots, \varphi_n^*$ is evidently an orthonormal system in $L(x_1^*, \ldots, x_r^*)$. $\square$

Remark. If $r = n + 1$ this theorem is a classical "skeleton" theorem. In the general case we have the "extended skeleton" property. Theorem 2 from Section 2 shows that when the compact set does not lie in the real axis, this "extended skeleton" property may not hold, even in the case of the simplest compact set like a circle.

5. Skeleton effect in H_2

Let $x_1, \ldots, x_m$ be points from the interval $(-1, 1)$ and $\varphi_{x_1,\ldots,x_m}^*(z)$ be the solution of the extremal problem

$$\inf_{f\in L(x_1,\ldots,x_m),\, \|f\|_{H_2}=1} \ \max_{1\le i\le m} |f(x_i)|.$$

It follows from Theorem 3 that $|\varphi_{x_1,\ldots,x_m}^*(x_i)| = |\varphi_{x_1,\ldots,x_m}^*(x_j)|$, for all $1 \le i$, $j \le m$ and from Lemma 3 we know that $\varphi_{x_1,\ldots,x_m}^*$ is real on the real axis.

THEOREM 7. *Let $x_1 \le x_2 \le \cdots \le x_m$; then*

$$\varphi_{x_1,\ldots,x_m}^*(x_i) = -\varphi_{x_1,\ldots,x_m}^*(x_{i+1}), \quad \textit{for } i = 1, \ldots, m-1. \tag{29}$$

Proof. Fix some $x_{m+1} > x_m$ and consider the extremal problem

$$\sup\{|f(x_{m+1})| : f \in L(x_1, \ldots, x_m),\ \ |f(x_i)| = 1, i = 1, \ldots, m\}. \tag{30}$$

We shall show that the extremal function, $\tilde{\varphi}$, of this problem satisfies $\tilde{\varphi}(x_i) = -\tilde{\varphi}(x_{i+1})$. Suppose that for some i, $\tilde{\varphi}(x_i) = \tilde{\varphi}(x_{i+1})$. Without loss of generality, we may assume that $\tilde{\varphi}(x_i) = \tilde{\varphi}(x_{i+1}) = 1$. Put

$$\chi_i(x) = \frac{1}{1 - x_i x} \prod_{j=1,\, j\ne i}^{m} \frac{x - x_i}{1 - x_i x}. \tag{31}$$

Note that $\chi_i(x_i)\chi_{i+1}(x_{i+1}) \le 0$ and that $\chi_i(x) \ge 0$ for $x \ge x_m$. Therefore, one of the numbers $\chi_i(x_i), \chi_{i+1}(x_{i+1})$ has the opposite sign of $\tilde{\varphi}(x_{m+1})$. Let it be $\chi_i(x_i)$. Put

$$\lambda = -\frac{2}{(\operatorname{sign}(\tilde{\varphi}(x_{m+1}))\chi_i(x_i)}$$

and consider the function

$$\hat{\varphi}(x) = \tilde{\varphi}(x) + \lambda(\operatorname{sign}(\tilde{\varphi}(x_{m+1}))\chi_i(x).$$

Then $\hat{\varphi} \in L(x_1, \dots, x_m)$, $\hat{\varphi}(x_j) = \tilde{\varphi}(x_j)$ for $j = 1, \dots, m$, $j \ne i$, $\hat{\varphi}(x_i) = -1$ and

$$\begin{aligned} |\hat{\varphi}(x_{m+1})| &= \left|\tilde{\varphi}(x_{m+1}) - \frac{2\chi_i(x_{m+1})}{\chi_i(x_i)}\right| \\ &= \left|\tilde{\varphi}(x_{m+1}) + \frac{2\chi_i(x_{m+1})}{|\chi_i(x_i)|} \operatorname{sign}(\tilde{\varphi}(x_{m+1})\right| \\ &> |\tilde{\varphi}(x_{m+1})|. \end{aligned}$$

This contradiction proves our assertion.

To prove the theorem, first, we note that

$$\frac{\varphi^*_{x_1, \dots, x_m}}{\varphi_{x_1, \dots, x_m}(x_1)}$$

is the extremal function for the problem

$$\sup\{\|f\|_{H_2} \colon f \in L(x_1, \dots, x_m), |f(x_i)| = 1, i = 1, \dots, m\} \tag{32}$$

Put

$$\psi_i = \frac{\sqrt{1 - x_i^2}}{1 - x_i x} \prod_{j \le i} \frac{x - x_i}{1 - x_i x}, \quad i = 1, \dots, m.$$

These functions form an orthonormal basis of $L(x_1, \dots, x_m)$ and any $f \in L(x_1, \dots, x_m)$ can be represented as $f = \sum_{i=1}^m \beta_i \psi_i$. Thus the equivalent form of (32) is

$$\sup\left\{\sum_{i=1}^m |\beta_i|^2 : |f(x_i)| = 1, i = 1, \dots, m, f = \sum_{i=1}^m \beta_i \psi_i\right\}. \tag{33}$$

Now we prove the theorem by induction. The case $m = 1$ is obvious. Suppose we have already proved (29) for m and let us prove it for $m + 1$. Consider the extremal function $\tilde{\varphi}$ for the problem (30). In accordance with the induction hypothesis, $\tilde{\varphi}$ is also the extremal function for the problem (33) and, therefore, for every $f \in L(x_1, \dots, x_m)$, $f = \sum_{i=1}^{m} \beta_i \psi_i$ such that $|f(x_i)| = 1$, $i = 1, \dots, m$, we have

$$\sum_{i=1}^{m} |\beta_i|^2 \leq \sum_{i=1}^{m} |\tilde{\beta}_i|^2, \quad \text{where } \tilde{\varphi} = \sum_{i=1}^{m} \tilde{\beta}_i \psi_i.$$

Since $\tilde{\varphi}$ has $m - 1$ zeros in the interval (x_1, x_m) it does not change sign in the interval (x_m, x_{m+1}). Let

$$\tilde{\beta}_{m+1} = \frac{\tilde{\varphi}(x_{m+1}) + \operatorname{sign}(\tilde{\varphi}(x_{m+1}))}{\psi_{m+1}(x_{m+1})}.$$

Then the function $\tilde{\tilde{\varphi}} = \tilde{\varphi} - \tilde{\beta}_{m+1}\psi_{m+1}$ is obviously the extremal function for (32) and $\tilde{\tilde{\varphi}}(x_{m+1}) = -\tilde{\varphi}(x_m) = -\tilde{\tilde{\varphi}}(x_{m+1})$. □

Example 1.

$$d^1(BH_2, C(-r, r)) = \frac{r}{\sqrt{1 - r^4}}.$$

Consider the two-point set $\{-r, r\}$. The function

$$\varphi(-z) = \frac{z\sqrt{1 - r^4}}{1 - r^2 z^2}$$

satisfies

$$\|\varphi\|_{H_2} = 1, \quad \varphi \in L(-r, r), \quad \varphi(-r) = -\varphi(r).$$

We conclude from Theorem 7 that

$$d^1(BH_2, \mathbf{C}(\{-r, r\})) = |\varphi(r)| = \frac{r}{\sqrt{1 - r^4}}$$

and therefore

$$d^1(BH_2, \mathbf{C}(-r, r)) \geq \frac{r}{\sqrt{1 - r^4}}.$$

The upper estimate follows from Theorem 1 and the following. Note that

$$\varphi^{\perp}(z) = \frac{\sqrt{1-r^4}}{1-r^2z^2}$$

satisfies $\|\varphi^{\perp}\|_{H_2} = 1$ and for $x \in (-1, 1)$,

$$\begin{aligned} K(x,x) - |\varphi^{\perp}(x)|^2 &= \frac{1}{1-x^2} - \frac{1-r^4}{(1-r^2x^2)^2} \\ &= \sum_{n=0}^{\infty} (1-(1-r^4)r^{2n}(n+1))x^{2n}. \end{aligned} \tag{34}$$

Since

$$(n+1)(1-r^4)r^{2n} \le (n+1)\left(1 - \frac{n}{n+2}\right)\left(\frac{n}{n+2}\right)^{n/2} < 1,$$

all coefficients of the series (34) are positive and, hence

$$\max_{-r\le x\le r} K(x,x) - |\varphi^{\perp}(x)|^2 \le K(r,r) - |\varphi^{\perp}(r)|^2 = |\varphi(r)| = \frac{r}{\sqrt{1-r^4}}.$$

DEFINITION. Let E be a compact set and $x_1, \dots, x_{n+1} \in E$. We say that $\{x_1, \dots, x_{n+1}\}$ forms an "n-skeleton" of E if

$$d^n(BH_2, \mathbf{C}(E)) = d^n(BH_2, \mathbf{C}(\{x_1, \dots, x_{n+1}\})).$$

Example 1 shows that the couple $\{-r, r\}$ forms a "1-skeleton" of the interval $(-r, r)$. Note that this problem is not conformally invariant and the end-points do not necessarily form a "1-skeleton" for an interval $[a, b]$ in $(-1, 1)$.

LEMMA 3. *The $(n+1)$-tuple $\{x_1, \dots, x_{n+1}\}$ forms an "n-skeleton" of the compact set $E \subset (-1,1)$ if and only if*

$$\begin{aligned} &\max_{x\in E}\left((\varphi^*_{x_1,\dots,x_{n+1}}(x))^2 + \frac{(B_{x_1,\dots,x_{n+1}}(x))^2}{1-x^2} \right) \\ &\quad = (\varphi^*_{x_1,\dots,x_{n+1}}(x_1))^2 = (d^n(BH_2, \mathbf{C}(\{x_1,\dots,x_{n+1}\})))^2 \end{aligned} \tag{35}$$

where $B_{x_1,\dots,x_{n+1}}$ is Blaschke product with zeros at $x_1, \dots, x_{n+1}$.

Proof. Suppose that $x_1, \dots, x_{n+1}$ form an "n-skeleton" of E. It implies that

$$d^n(BH_2, \mathbf{C}(\{x_1, \dots, x_{n+1}\})) = d^n(BH_2, \mathbf{C}(\{x_1, \dots, x_{n+1}, x\}))$$

where x is an arbitrary point of E.

Let $\varphi_1^*, \varphi_2^* \in L((x_1, \dots, x_{n+1}, x)$ be the extremal pair of orthonormal functions in the sense of $\min_{(\varphi_1, \varphi_2)} \max_{z \in \{x_1, \dots, x_{n+1}\}}(|\varphi_1(z)|^2 + |\varphi_2(z)|^2)$. We may assume that $\varphi_1^* \in L(x_1, \dots, x_{n+1})$ and, therefore,

$$\begin{aligned} \max_{z \in \{x_1, \dots, x_{n+1}\}} |\varphi_1^*(z)|^2 &\geq \min_{\varphi \in L(x_1, \dots, x_{n+1})} \max_{z \in \{x_1, \dots, x_{n+1}\}} |\varphi(z)| \\ &= d^n(BH_2, \mathbf{C}(E)). \end{aligned} \tag{36}$$

It follows from Theorem 3 that if equality holds in (36), then

$$|\varphi_1^*(x_i)| = d^n(BH_2, \mathbf{C}(E)), \qquad i = 1, \dots, n+1$$

and, hence,

$$\varphi_2^*(x_i) = 0, \qquad i = 1, \dots, n+1,$$

which implies that

$$\varphi_2^*(z) = B_{x_1, \dots, x_{n+1}}(z) \frac{\sqrt{1 - x^2}}{1 - xz}.$$

Thus

$$(\varphi_1^*(x))^2 + (\varphi_2^*(x))^2 = (\varphi_{x_1, \dots, x_{n+1}}^*(x))^2 + \frac{B_{x_1, \dots, x_{n+1}}(x))^2}{1 - x^2}.$$

Conversely, suppose that (35) holds. It implies that for every $x \in E$,

$$d^n(BH_2, \mathbf{C}(\{x_1, \dots, x_{n+1}\})) = d^n(BH_2, \mathbf{C}(\{x_1, \dots, x_{n+1}, x\})). \tag{37}$$

Let us show that this condition is sufficient for the $(n+1)$-tuple $\{x_1, \dots, x_{n+1}\}$ to form an "n-skeleton." Suppose that

$$d^n(BH_2, \mathbf{C}(E)) > d^n(BH_2, \mathbf{C}\{x_1, \dots, x_{n+1}\})$$

and let $\varphi_1, \dots, \varphi_n$ be the orthonormal basis of the orthogonal complement of $\varphi_{x_1, \dots, x_{n+1}}^*$ in $L(\{x_1, \dots, x_{n+1})$. Then

$$\begin{aligned} &\max_{x \in E} \left(K(x, x) - \sum_{i=1}^n |\varphi_i(x)|^2 \right) \\ &\quad > \max_{x \in \{x_1, \dots, x_{n+1}\}} \left(K(x, x) - \sum_{i=1}^n |\varphi_i(x)|^2 \right) = (\varphi_{x_1, \dots, x_{n+1}}^*(x))^2. \end{aligned}$$

The last inequality contradicts (37). □

Remark. It immediately follows from the lemma that the $(n+1)$-tuple $\{x_1, \dots, x_{n+1}\}$, which satisfies (35), is the extremal $(n+1)$-tuple, i.e.,

$$d^n\big(BH_2, \mathbf{C}(\{x_1, \dots, x_{n+1}\})\big) = \max_{\{z_1, \dots, z_{n+1}\} \subset E} = d^n\big(BH_2, \mathbf{C}(\{z_1, \dots, z_{n+1}\})\big).$$

Example 2. Let r_0 be the root of the equation $3r^6 - 4r^4 - 4r^2 + 4 = 0$ which lies between 0 and 1. If $r \le r_0$, then

$$d^2(BH_2, \mathbf{C}(-r, r)) = \frac{r^2}{\sqrt{4 - 3r^4}}$$

and the points $\{-r, 0, r\}$ form a "2-skeleton" of the interval $(-r, r)$.

Proof. The function $\varphi^* \in L(-r, 0, r)$ which has norm 1 and equioscillates at the points $\{-r, 0, r\}$ is

$$\varphi^* = \frac{(2 - r^4)z^2 - r^2}{\sqrt{4 - 3r^3}\,(1 - r^2 z^2)}.$$

Let us check (35). We have

$$B(z) = z\frac{r^2 - z^2}{1 - z^2 r^2},$$

$$\begin{aligned}\psi(x) = (\varphi^*(x))^2 + \frac{(B(x))^2}{1 - x^2} &= \frac{[(2 - r^4)x^2 - r^2]^2}{(4 - 3r^4)(1 - r^2x^2)^2} \\ &\quad + \frac{x^2(r^2 - x^2)^2}{(1 - r^2x^2)^2(1 - x^2)} \\ &= \frac{x^6(r^4 - r^8) + x^4(r^8 + 4r^6 - 4r^4 r - 4r^2 + 4) + x^2(-3r^8 + 2r^6 + 3r^4 - 4r^2) + r^4}{(4 - 3r^4)(1 - r^2x^2)^2(1 - x^2)},\end{aligned}$$

$$\begin{aligned}\psi(x) - (\varphi^*(0))^2 = \psi(x) - \frac{r^4}{4 - 3r^4} \\ = \frac{x^2}{(4 - 3r^4)(1 - r^2x^2)(1 - x^2)} \cdot P(x),\end{aligned}$$

where $P(x) = r^4x^4 + x^2(2r^6 - 4r^4 - 4r^2 + 4) - (3r^8 - 4r^6 - 4r^4 + 4r^2)$.

$P(x)$ is a quadratic polynomial in x^2 which has only two real roots, at $x = r$ and $x = -r$ (since $3r^8 - 4r^6 - 4r^4 + 4r^2 = r^2(3r^6 - 4r^4 - 4r^2 + 4) > 0$ if $x < r_0$) and, therefore $P(x) < 0$ for $x \in (-r, r)$.

Problem. Is it true that there is an "n-skeleton" for the interval $(-r, r)$ for every n?

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