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SIMPLE INFINITE DIMENSIONAL QUOTIENTS OF $C^*(G)$ FOR DISCRETE 5-DIMENSIONAL NILPOTENT GROUPS G

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Introduction

In each of 3 and 4 dimensions there is a unique (up to isomorphism) connected, simply connected, nilpotent Lie group, which we call G_3 and G_4, respectively (following Nielsen [11]). In [10] we showed that the simple C^*-algebras A_θ^4 arising from Anzai flows (with irrational θ), which had been studied in [7], [12], and [20], are isomorphic to simple infinite dimensional quotients of $C^*(H_4)$, where H_4 is the lattice subgroup of G_4, in the same way that the irrational rotation algebras A_θ^3 (as we call them to conform with our other notation) are isomorphic to such quotients of $C^*(H_3)$, where H_3 is the lattice subgroup of the Heisenberg group G_3. Also determined in [10; Theorem 2] were crossed product presentations of the A_θ^4's. The 5-dimensional case, which is studied in the present paper, is immediately complicated by the existence of 6 (non-isomorphic) connected and simply connected, nilpotent, Lie groups $G_{5,i}$, $1 \le i \le 6$ (see [11]). Following the Preliminaries, a section is devoted to each of these groups.

For each of the six Lie groups $G_{5,i}$ we identify a lattice subgroup $H_{5,i}$ (corresponding to $H_3 \subset G_3$) in a natural way. This subgroup is obtained from some operator equations (corresponding to $UV = \lambda VU$) that determine a faithful representation of $H_{5,i}$ which generates a simple C^*-algebra $\mathcal{A}$ with a unique tracial state (corresponding to the irrational rotation algebra A_θ^3). These algebras are infinite dimensional simple quotients of $C^*(H_{5,i})$. Each section concludes by identifying the other infinite dimensional simple quotients of $C^*(H_{5,i})$ – namely, those arising from a non-faithful representation of $H_{5,i}$, and we present them as matrix algebras over an irrational rotation algebra in most cases.

Analogues of the simple quotients arising from the non-faithful representations (as described above) exist for $C^*(H_4)$, but not for $C^*(H_3)$; the irrational rotation algebras A_θ^3 exhaust the infinite dimensional simple quotients of $C^*(H_3)$. (This situation for H_3 also holds for the group $H_{5,1}$; see Theorem 1.2.)

Here are some further comments about the structure of the paper. In the Preliminaries, notation is established for C^*-crossed products; also we give a brief summary of the results we need about the irrational rotation algebras A_θ^3 and, more especially,

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their 4-dimensional analogues A_θ^4. Furthermore, in each Section i, $1 \le i \le 6$, there appears Theorem $i.1$ establishing results about those C*-algebras $\mathcal{A}$ that arise from a faithful representation. (The proofs of these results are discussed in some detail in Section 1, the proofs in later sections being similar.) The infinite dimensional simple quotients of $C^*(\mathrm{H}_{5,i})$ are listed in Theorem $i.2$.

We take this opportunity to thank the referee for pointing out that the matrix presentations in the non-faithful situation would be possible, and for many other helpful suggestions.

0. Preliminaries

Terminology is to be consistent throughout. Thus, for example, G_3 is the connected, simply connected, nilpotent, Lie group of dimension 3, $\mathrm{G}_3 = \mathbb{R}^3$ with multiplication

$$(x, y, z)(x', y', z') = (x + x' + zy', y + y', z + z');$$

this notation is as in Nielsen [11]. Then $\mathrm{H}_3 = \mathbb{Z}^3$ is the lattice subgroup of G_3. Let $\lambda = e^{2\pi i\theta}$ for an irrational θ and let U and V be unitaries satisfying $UV = \lambda VU$; then the C^*-algebra generated by U and V is A_θ^3, as is the C^*-algebra generated by the representation $(k, m, n) \mapsto \lambda^k V^m U^n$ of H_3. From each of Nielsen's connected Lie groups G, we get analogously the lattice subgroup H, which is often a subgroup not of G, but of an isomorphic group with similar multiplication. (See the discussion of H_4 near the end of the Introduction in [10].) Then the simple C^*-algebras A studied in this paper come from representations of the (various) H's; see the remarks at the beginning of the section on $A_\theta^{5,1}$. Here, and throughout the paper, use is made of the 1-1 correspondence between (non-degenerate) representations of $C^*(\mathrm{H})$ and unitary representations of H [3; 13.9.3].

To present the results and proofs of the paper, it seems best to establish notation for C^*-crossed products; the discussion which follows is condensed from [10], where more detail is given. (Relevant references are [21, 22, 24].)

Zeller-Meier crossed product formulation. Let G be a discrete group with identity e and let A be a C^*-algebra. Assume that $s \mapsto \sigma_s$, $G \to \operatorname{Aut} A$, gives an action of G on A and that there is a cocycle α from $G \times G$ into the unitary group of the center of A, so that the following equations (analogous to those for Schreier group extensions) are satisfied:

$$\sigma_e(a) = a \qquad \text{and} \qquad \alpha(s, e) = \alpha(e, s) = 1,$$

$$\alpha(s, s')\alpha(ss', s'') = \sigma_s(\alpha(s', s''))\alpha(s, s's'')$$

for s, s', $s'' \in G$ and $a \in A$. Then, for f and g in the Banach space $\ell_1(G, A)$, the convolution product $f * g$ and involution f^* are defined by

$$f * g(s') = \sum_{s \in G} f(s)\, \sigma_s(g(s^{-1}s'))\, \alpha(s, s^{-1}s')$$

and $f^*(s) = \sigma_s(f(s^{-1})^*)\,\alpha(s, s^{-1})^*$; with these definitions, $\ell_1(G, A)$ becomes a Banach $*$-algebra. The C^*-crossed product $C^*(A, G, \alpha)$ is defined to be the enveloping C^*-algebra of $\ell_1(G, A)$; the notation is abbreviated to $C^*(A, G)$ when α is trivial, and to $C^*(G, \alpha)$ when $A = \mathbb{C}$. For $a \in A$ and $s \in G$, the functions a_s and δ_s in $\ell_1(G, A) \subset C^*(A, G)$ are defined by $a_s(s) = a$, $a_s(s') = 0$ otherwise, and $\delta_s(s) = 1$ (the identity of A), $\delta_s(s') = 0$ otherwise. (Thus $a_s = a\delta_s$.)

Notation. Depending on the context, the symbol v denotes the function $v \mapsto v$ in $\mathcal{C}(\mathbb{T})$ or the function $(w, v) \mapsto v$ in $\mathcal{C}(\mathbb{T}^2)$, and w denotes the function $(w, v) \mapsto w$ in $\mathcal{C}(\mathbb{T}^2)$.

We conclude this section with a discussion (from [10]) of the 4-dimensional case. The connected nilpotent group $\mathrm{G}_4 = \mathbb{R}^4$ and its lattice subgroup $\mathrm{H}_4 = \mathbb{Z}^4$ have the multiplication formula

$$(j, k, m, n)(j', k', m', n') = (j+j'+nk'+m'n(n-1)/2, k+k'+nm', m+m', n+n').$$

Let unitaries U, V and subsidiary operator W satisfy

$$UV = WVU, \quad UW = \lambda WU \text{ and } VW = WV, \tag{0.1}$$

and let A^4_θ denote the C^*-algebra generated by U and V. These operators give a representation π: $(j, k, m, n) \mapsto \lambda^j W^k V^m U^n$ of H_4 that also generates A^4_θ.

The reader should note that for these (and later) algebras, we have introduced the subsidiary operator(s) only to control the notation. Thus $W = [U, V] = UVU^{-1}V^{-1}$ here, and saying

$$U, V \text{ and } W \text{ satisfy (0.1)}$$

is equivalent to saying

$$U \text{ and } V \text{ satisfy } [U, [U, V]] = \lambda \text{ and } [V, [U, V]] = 1.$$

We will sometimes say (e.g., in the next theorem) merely that

$$U \text{ and } V \text{ satisfy (0.1);}$$

the reader must then recall that the first equation of (0.1) defines the subsidiary operator W in terms of U and V.

0.1 THEOREM. [12, 7, 20, 10] *The C^*-algebra A^4_θ is simple and is the unique (up to isomorphism) C^*-algebra generated by unitaries U and V satisfying* (0.1). *Furthermore, A^4_θ is a quotient of $C^*(\mathrm{H}_4)$.*

There are infinite dimensional simple quotients of $C^*(\mathrm{H}_4)$ apart from the A^4_θ's. They are given in the next result [10; Theorem 3].

0.2 THEOREM. *Let λ be a primitive qth root of unity ($q > 1$), let $\mathbb{Z}_q$ be the subgroup of $\mathbb{T}$ generated by λ, and let $\mu = e^{2\pi i\beta}$ for an irrational β. Define a flow on $\mathbb{Z}_q \times \mathbb{T}$ by $\psi(w, v) = (\lambda w, \mu w v)$, and denote the generated C^*-crossed product $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z})$ by ${}_q\mathrm{A}_\beta$. Then ${}_q\mathrm{A}_\beta$ is simple and is the unique (up to isomorphism) C^*-algebra generated by unitaries U and V satisfying*

$$UV = \mu WVU, \quad UW = \lambda WU, \quad VW = WV \text{ and } W^q = 1. \tag{0.2}$$

Furthermore, ${}_q\mathrm{A}_\beta$ is isomorphic to the matrix algebra $M_q(\mathrm{A}_\gamma^3)$ and is a simple quotient of $C^*(\mathrm{H}_4)$, where $e^{2\pi i\gamma} = (-1)^{q+1}\mu^q$.

1. The simple quotients $\mathrm{A}_\theta^{5,1}$ of $C^*(\mathrm{H}_{5,1})$

Let $\lambda = e^{2\pi i\theta}$ for an irrational θ, let unitaries U, V, W and X satisfy

$$\begin{cases} UV = \lambda VU, \quad WX = \lambda XW, \quad UW = WU, \\ UX = XU, \quad VW = WV \text{ and } VX = XV, \end{cases} \tag{1.1}$$

and let $\mathrm{A}_\theta^{5,1}$ denote the C^*-algebra generated by U, V, W and X.

A "discrete group construction" in [10] shows how to construct a group from some unitaries satisfying equations like (1.1); the essential property of the group is that it has a representation whose generated C^*-algebra is just the C^*-algebra generated by the unitaries. The result here is a group $\mathrm{H}_{5,1}$ ($= \mathbb{Z}^5$ as a set) with multiplication

$$(h, j, k, m, n)(h', j', k', m', n') = (h+h'+nm'+kj', j+j', k+k', m+m', n+n'), \tag{1.2}$$

and inverse $(h, j, k, m, n)^{-1} = (-h + nm + kj, -j, -k, -m, -n)$; it is the lattice subgroup of Nielsen's $\mathrm{G}_{5,1} = \mathbb{R}^5$ with multiplication (1.2) [11]. The representation of $\mathrm{H}_{5,1}$ is given by π: $(h, j, k, m, n) \mapsto \lambda^h X^j W^k V^m U^n$, and obviously generates $\mathrm{A}_\theta^{5,1}$.

The next theorem asserts that the C^*-algebra $\mathrm{A}_\theta^{5,1}$ is simple and has a unique tracial state, and Theorem $i.1$ has the same conclusion for the C^*-algebras in Section i, $2 \le i \le 6$. The existence of the unique tracial state is easy to verify directly in all these results (and can also be proved by citing results from the literature). The proof of simplicity can be achieved in a number of ways depending on which presentation as a C^*-crossed product one uses for the algebra.

Discussion of the proof of simplicity in Theorem $i.1$, $1 \le i \le 6$. The C^*-algebras $\mathcal{A}_i$ in Theorem $i.1$, $i = 1, 2, 3, 5$ (as well as the 'other' quotients at the end of Sections 2, 3 and 5) have minimal flow presentations analogous to $C^*(\mathcal{C}(\mathbb{T}), \mathbb{Z})$ for A_θ^3 and $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z})$ for A_θ^4. This situation is appealing because of its connection with geometry and topology; it yields the most attractive concrete representations of the algebras. (These representations are analogous to the representation of the irrational rotation algebra A_θ^3 on $L^2(\mathbb{T})$.) A classic result of Effros and Hahn [4; Corollary 5.16] asserts the simplicity of such C^*-algebras. (For $i = 2, 5$, where the flow is generated by a single homeomorphism, the special case of the Effros-Hahn result as proved by Power [18] can be used.)

The C^*algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ can also be proved to be simple with results of Slawny [23], Poguntke [17], or Baggett and Packer [2], while the simplicity of $\mathcal{A}_4$ and $\mathcal{A}_6$ follows from results of Pimsner-Voiculescu [16] and Kishimoto [8]. The simplicity of *all* of the C^*-algebras $\mathcal{A}_i$, $1 \le i \le 6$, can be established with Packer [13], or (as the referee suggests) Packer and Raeburn [15].

1.1 THEOREM. *Let $\lambda = e^{2\pi i\theta}$ for an irrational θ.*

(1) *There is a unique (up to isomorphism) simple C^*-algebra $A_\theta^{5,1}$ generated by unitaries U, V, W and X satisfying* (1.1). *Let $\mathbb{Z}^2$ act on $\mathcal{C}(\mathbb{T}^2)$ by (k, n): $f \mapsto f \circ \phi_2^k \circ \phi_1^n$, where the commuting homeomorphisms ϕ_1 and ϕ_2 are given by $\phi_1(w, v) = (w, \lambda v)$ and $\phi_2(w, v) = (\lambda w, v)$; then*

$$A_\theta^{5,1} \cong C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2).$$

(2) *Let π' be a representation of $H_{5,1}$ such that $\pi = \pi'$ (as scalars) on the center $(\mathbb{Z}, 0, 0, 0, 0)$ of $H_{5,1}$, and let A be the C^*-algebra generated by π'. Then $A \cong A_\theta^{5,1}$ via a unique isomorphism ω such that the following diagram commutes.*

$$\begin{tikzcd} H_{5,1} \arrow[rr, "\pi"] \arrow[dr, "\pi'"'] & & A_\theta^{5,1} \arrow[dl, "\omega"] \\ & A & \end{tikzcd}$$

(3) *The C^*-algebra $A_\theta^{5,1}$ has a unique tracial state.*

Proof. (1) Note that the flow $(\mathbb{Z}^2, \mathbb{T}^2)$ with action given by

$$(k, n)\colon\ (w, v) \mapsto \phi_2^k \circ \phi_1^n(w, v) = (\lambda^k w, \lambda^n v)$$

is minimal and effective; so $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ is simple, by Effros and Hahn [4; Corollary 5.16].

Once the simplicity of $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ is established, it is straightforward to show that any C^*-algebra A generated by 4 unitaries U, V, W and X satisfying the equations (1.1) is isomorphic to $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$. Since X and V commute, there is a $*$-homomorphism σ: $\mathcal{C}(\mathbb{T}^2) \to$ A such that $\sigma(w) = X$ and $\sigma(v) = V$; in fact, $\sigma(f) = f(X, V)$. Define a homomorphism ρ: $\mathbb{Z}^2 \to$ A by $\rho(k, n) = W^k U^n$, noting that $\sigma(f \circ \phi_2^k \circ \phi_1^n) = \rho(k, n)\sigma(f)\rho(k, n)^*$ holds for $f = w$ or v, and hence for all $f \in \mathcal{C}(\mathbb{T}^2)$. By the universal mapping property of $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ [24], the covariant pair (σ, ρ) yields a homomorphism of $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ onto A mapping $w_{(0,0)}$, $v_{(0,0)}$,

$\delta_{(1,0)}$ and $\delta_{(0,1)}$, respectively, to X, V, W and U; since $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ is simple, the homomorphism is an isomorphism.

(2) The hypotheses imply that (1.1) is satisfied by the unitaries X', W', V' and U' given by $\pi'(h, j, k, m, n) = \lambda^h X'^j W'^k V'^m U'^n$. Part 1 and its proof now yield the result. $\square$

Note. The normal subgroup $N = (\mathbb{Z}, 0, 0, 0, 0) \subset \mathrm{H}_{5,1}$ with $\mathrm{H}_{5,1}/N = \mathbb{Z}^4$ gives rise to a presentation of $\mathrm{A}_\theta^{5,1}$ that uses a cocycle α: $\mathbb{Z}^4 \times \mathbb{Z}^4 \to \mathbb{C}$ defined by

$$\alpha((j, k, m, n), (j', k', m', n')) = \lambda^{nm'+kj'};$$

namely $\mathrm{A}_\theta^{5,1} \cong C^*(\mathbb{Z}^4, \alpha)$.

This presentation makes it possible to view the algebra $\mathrm{A}_\theta^{5,1}$ as generated by a representation of canonical commutation relations (CCR) over $(\mathbb{Z}^4, b)$, where b is a bicharacter on $\mathbb{Z}^4$ (terminology as in Slawny [23]). Of course, b is just the cocycle α in the presentation above. The representation W of CCR over $(\mathbb{Z}^4, b)$, or b-representation of $\mathbb{Z}^4$, is given by

$$\mathrm{W}(j, k, m, n) = X^j W^k V^m U^n,$$

U, V, W and X satisfying (1.1), so that

$$\mathrm{W}(s)\mathrm{W}(s') = b(s, s')\mathrm{W}(s + s'), \qquad s, s' \in \mathbb{Z}^4.$$

Much as above, $\mathrm{A}_{\theta,\varphi}^{5,2}$ (in the next section) is generated by a b-representation of CCR over $\mathbb{Z}^3$.

Other infinite dimensional simple quotients of $C^*(\mathrm{H}_{5,1})$. When $\lambda = e^{2\pi i\theta}$ for irrational θ, $\mathrm{A}_\theta^{5,1}$ is an infinite dimensional simple quotient of $C^*(\mathrm{H}_{5,1})$; all such quotients are of this form and the homomorphism

$$(h, j, k, m, n) \mapsto \lambda^h X^j W^k V^m U^n, \quad \mathrm{H}_{5,1} \to \mathrm{A}_\theta^{5,1}$$

(as at the beginning of the section) is 1-1, in complete analogy with the situation for the A_θ^3's and H_3. To see this, note that any other simple quotient A of $C^*(\mathrm{H}_{5,1})$ has a faithful irreducible representation with $(1, 0, 0, 0, 0) \in \mathrm{H}_{5,1}$, the generator of the center of $\mathrm{H}_{5,1}$, mapping to a primitive qth (say) root λ of unity. When $q > 1$, one can modify the presentation $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ of $\mathrm{A}_\theta^{5,1}$ (in Theorem 1.1) and obtain from the minimal flow $(\mathbb{Z}_q^2, \mathbb{Z}_q^2)$ (the group $\mathbb{Z}_q^2$ acting on itself by left translation) a presentation of A as $C^*(\mathcal{C}(\mathbb{Z}_q^2), \mathbb{Z}_q^2)$, and so A is finite dimensional. (The symbol $\mathbb{Z}_q$ denotes the cyclic group with q elements; as used in $C^*(\mathcal{C}(\mathbb{Z}_q^2), \mathbb{Z}_q^2)$, think of it first as the subgroup $\{\lambda^r \mid 0 \le r < q\}$ of $\mathbb{T}$, and then as $\mathbb{Z}/q\mathbb{Z} = \{r \mid 0 \le r < q\}$, so that the action of $\mathbb{Z}_q^2$ on $\mathcal{C}(\mathbb{Z}_q^2)$ is given by $((k, n)f)(w, v) = f(\lambda^k w, \lambda^n v)$, just like the action of $\mathbb{Z}^2$ on $\mathcal{C}(\mathbb{T}^2)$ in Theorem 1.1.) When $q = 1$, A is generated by an irreducible representation of $\mathrm{H}_{5,1}$ that factors through $\mathrm{H}_{5,1}/(\mathbb{Z}, 0, 0, 0, 0) \cong \mathbb{Z}^4$, and so comes from a character of $\mathbb{Z}^4$.

1.2 THEOREM. *A C^*-algebra* A *is isomorphic to a simple infinite dimensional quotient of* $C^*(\mathrm{H}_{5,1})$ *if, and only if,* $\mathrm{A} \cong \mathrm{A}_\theta^{5,1}$ *for some irrational* θ.

2. The simple quotients $\mathrm{A}_{\theta,\varphi}^{5,2}$ of $C^*(\mathrm{H}_{5,2})$

Let unitaries U, V and W satisfy

$$UV = \lambda VU, \quad UW = \mu WU \text{ and } VW = WV, \tag{2.1}$$

where $\mu = e^{2\pi i\varphi}$ and $\lambda = e^{2\pi i\theta}$ are linearly independent, i.e., $\lambda^r \mu^{r'} \neq 1$ for $r, r' \in \mathbb{Z}$ unless $r = 0 = r'$; let $\mathrm{A}_{\theta,\varphi}^{5,2}$ denote the C^*-algebra generated by U, V and W.

By the same process as for the equations (1.1) in the previous section, the equations (2.1) yield a group with a representation whose generated C^*-algebra is $\mathrm{A}_{\theta,\varphi}^{5,2}$. The group is $\mathrm{H}_{5,2}$ ($= \mathbb{Z}^5$ as a set) with multiplication

$$(h, j, k, m, n)(h', j', k', m', n') = (h+h'+nk', j+j'+nm', k+k', m+m', n+n'), \tag{2.2}$$

and inverse $(h, j, k, m, n)^{-1} = (-h + nk, -j + nm, -k, -m, -n)$; it is the lattice subgroup of Nielsen's $\mathrm{G}_{5,2} = \mathbb{R}^5$ with multiplication (2.2). The representation of $\mathrm{H}_{5,2}$ is given by $\pi\colon (h, j, k, m, n) \mapsto \mu^h \lambda^j W^k V^m U^n$, and obviously generates $\mathrm{A}_{\theta,\varphi}^{5,2}$.

2.1 THEOREM. *Let* $\mu = e^{2\pi i\varphi}$ *and* $\lambda = e^{2\pi i\theta}$ *be linearly independent.*

(1) *There is a unique* (*up to isomorphism*) *simple C^*-algebra* $\mathrm{A}_{\theta,\varphi}^{5,2}$ *generated by unitaries U, V and W satisfying* (2.1). *Let* $\mathbb{Z}$ *act on* $C(\mathbb{T}^2)$ *by* $n\colon f \mapsto f \circ \phi^n$, *where* ϕ *is the homeomorphism of* $\mathbb{T}^2$ *given by* $\phi(w, v) = (\mu w, \lambda v)$; *then*

$$\mathrm{A}_{\theta,\varphi}^{5,2} \cong C^*(C(\mathbb{T}^2), \mathbb{Z}).$$

(2) *Let* π' *be a representation of* $\mathrm{H}_{5,2}$ *such that* $\pi = \pi'$ (*as scalars*) *on the center* $(\mathbb{Z}, \mathbb{Z}, 0, 0, 0)$ *of* $\mathrm{H}_{5,2}$, *and let* A *be the C^*-algebra generated by* π'. *Then* $\mathrm{A} \cong \mathrm{A}_{\theta,\varphi}^{5,2}$ *via a unique isomorphism* ω *such that the following diagram commutes*:

$$\begin{tikzcd} \mathrm{H}_{5,2} \arrow[r, "\pi"] \arrow[dr, "\pi'"'] & \mathrm{A}_{\theta,\varphi}^{5,2} \arrow[d, "\omega"] \\ & \mathrm{A} \end{tikzcd}$$

(3) *The C^*-algebra* $\mathrm{A}_{\theta,\varphi}^{5,2}$ *has a unique tracial state.*

Proof. As in Section 1, we note that the flow $(\mathbb{Z}, \mathbb{T}^2)$ with action given by

$$n\colon (w, v) \mapsto \phi^n(w, v) = (\mu^n w, \lambda^n v)$$

is minimal and effective; so $C^*(C(\mathbb{T}^2), \mathbb{Z})$ is simple [4], [18]. The rest of the proof is similar to that of Theorem 1.1. □

It seems that the $A^{5,2}_{\theta,\varphi}$'s are among the simple C^*-algebras on which the 3-torus $\mathbb{T}^3$ can act ergodically, as in Albeverio and Høegh-Krohn [1; p.16]; however, we have not checked the details.

Other infinite dimensional simple quotients of $C^*(\mathrm{H}_{5,2})$. When λ and μ are linearly independent ($\lambda^r\mu^{r'} \neq 1$ for any $r, r' \in \mathbb{Z}$ unless $r = 0 = r'$), $A^{5,2}_{\theta,\varphi}$ is an infinite dimensional simple quotient of $C^*(\mathrm{H}_{5,2})$ and the homomorphism

$$\pi\colon\ (h, j, k, m, n) \mapsto \mu^h\lambda^j W^k V^m U^n, \quad \mathrm{H}_{5,2} \to A^{5,2}_{\theta}$$

(as at the beginning of the section) is 1-1. But there are other infinite dimensional simple quotients of $C^*(\mathrm{H}_{5,2})$; for them the homomorphism is not 1-1. (Comments analogous to these hold in Sections 3–6.)

1. Suppose that just one of λ and μ is a root of unity; e.g., suppose that λ is a primitive qth root of unity, and suppose that A is a quotient of $C^*(\mathrm{H}_{5,2})$ that is irreducibly represented and generated by unitaries U, V and W satisfying

$$UV = \lambda VU, \quad UW = \mu WU \text{ and } VW = WV. \tag{2.1}$$

From (2.1) it follows that V^q commutes with U and W, and so by irreducibility equals $\eta' I$, a multiple of the identity. Since V is a generator of A, the substitution $V = \eta V_1$, where $\eta^q = \eta'$, gives $V_1^q = I$, while (2.1) still holds with V_1 replacing V. Now we can modify the presentation $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z})$ of $A^{5,2}_{\theta,\varphi}$ (in Theorem 2.1) and present A as $C^*(\mathcal{C}(\mathbb{T} \times \mathbb{Z}_q), \mathbb{Z})$ with the action of $\mathbb{Z}$ on $\mathcal{C}(\mathbb{T} \times \mathbb{Z}_q)$ generated by the minimal homeomorphism $\psi\colon (w, v) \mapsto (\mu w, \lambda v)$ of $\mathbb{T}\times\mathbb{Z}_q$; thus $C^*(\mathcal{C}(\mathbb{T}\times\mathbb{Z}_q), \mathbb{Z})$ is simple, and A is isomorphic to it.

2. If neither λ nor μ is a root of unity, but $\lambda^r\mu^{r'} = 1$ for some $r, r' \in \mathbb{Z}$ with $r \neq 0 \neq r'$, then $\mu^{pq} = \lambda^{p'q}$, where $(p, p') = 1$, i.e., $sp + s'p' = 1$ for some $s, s' \in \mathbb{Z}$. Thus we are starting with a C^*-algebra A generated by unitaries U, V and W satisfying (2.1), and set $\lambda_1 = \mu^p\lambda^{-p'}$, $\mu_1 = \mu^{s'}\lambda^s$, $V_1 = W^pV^{-p'}$ and $W_1 = W^{s'}V^s$. Then λ_1 is a primitive qth root of unity and μ_1 is not a root of unity, since

$$\mu = \mu^{sp+s'p'} = (\mu^p)^s\mu^{s'p'} = (\lambda_1\lambda^{p'})^s\mu^{s'p'} = \lambda_1^s(\lambda^s\mu^{s'})^{p'} = \lambda_1^s\mu_1^{p'};$$

also $UV_1 = \lambda_1V_1U$, $UW_1 = \mu_1W_1U$ and $V_1W_1 = W_1V_1$, so U, V_1 and W_1 generate a C^*-algebra isomorphic to a $C^*(\mathcal{C}(\mathbb{T} \times \mathbb{Z}_q), \mathbb{Z})$, as in comment 1 above. Since $W = W_1^{p'}V_1^s$ and $V = W_1^pV_1^{-s'}$, this C^*-algebra is A.

3. If both λ and μ are roots of unity, it follows from (2.1) that there is an $N \in \mathbb{N}$ such that U^N, V^N and W^N are scalars (since A is still assumed to be irreducibly represented); thus the C^*-algebra A consists of finite linear combinations of $\{W^kV^mU^n \mid 0 \leq k, m, n \leq N\}$ and so is finite dimensional.

The preceding comments are summarized in the next theorem.

2.2 THEOREM. *A C^*-algebra* A *is isomorphic to a simple infinite dimensional quotient of $C^*(\mathrm{H}_{5,2})$ if, and only if,* A *is isomorphic to $\mathrm{A}^{5,2}_{\theta,\varphi}$ for some linearly independent λ and μ, or to $C^*(\mathcal{C}(\mathbb{T}\times\mathbb{Z}_q),\mathbb{Z})$ (as in case* 1 *above) for some λ, a primitive qth root of unity, and μ not a root of unity.*

We thank the referee for pointing out that results of Rieffel and Green [19], [6] imply that the algebras $C^*(\mathcal{C}(\mathbb{T}\times\mathbb{Z}_q),\mathbb{Z})$ have matrix presentations. However, our approach here, and in later sections, has been to give explicit matrix presentations of the other quotients, as in [10; Theorem 3].

2.3 THEOREM. *Let λ be a primitive qth root of unity and suppose that $\mu = e^{2\pi i\varphi}$ is not a root of unity. Then the C^*-crossed product $C^*(\mathcal{C}(\mathbb{T}\times\mathbb{Z}_q),\mathbb{Z})$ (as above) is isomorphic to $M_q(\mathrm{A}^3_{q\varphi})$.*

Proof. Let unitaries U_0 and W_0 satisfy $U_0W_0 = \mu^q W_0U_0$, so that U_0 and W_0 generate $\mathrm{A}^3_{q\varphi}$. Then define the following 3 unitaries in $M_q(\mathrm{A}^3_{q\varphi})$ (all unspecified entries being 0).

U' has U_0 in the upper right hand corner and 1's on the subdiagonal.

W' has $W_0, \overline{\mu}W_0, \overline{\mu}^2W_0, \dots, \overline{\mu}^{q-1}W_0$ on the diagonal.

V' has $1, \overline{\lambda}, \overline{\lambda}^2, \dots, \overline{\lambda}^{q-1}$ on the diagonal.

Then U', V' and W' satisfy the equations (2.1) and generate $M_q(\mathrm{A}^3_{q\varphi})$. □

3. The simple quotients $\mathrm{A}^{5,3}_\theta$ of $C^*(\mathrm{H}_{5,3})$

Let $\lambda = e^{2\pi i\theta}$ for an irrational θ, let unitaries U, V, W and subsidiary operator X satisfy

$$\begin{cases} UV = XVU, & UW = WU, \quad UX = \lambda XU, \\ VW = \lambda WV, & VX = XV \text{ and } WX = XW, \end{cases} \tag{3.1}$$

and let $\mathrm{A}^{5,3}_\theta$ denote the C^*-algebra generated by U, V and W.

The equations (3.1) yield a group with a representation whose generated C^*-algebra is $\mathrm{A}^{5,3}_\theta$. The group is $\mathrm{H}_{5,3}$ ($=\mathbb{Z}^5$ as a set) with multiplication

$$\begin{aligned}(h,j,k,m,n)(h',j',k',m',n') = {} & (h+h'+mk'+nj'+m'n(n-1)/2, \\ & j+j'+nm', k+k', m+m', n+n'),\end{aligned} \tag{3.2}$$

and inverse

$$(h,j,k,m,n)^{-1} = (-h+mk+nj-mn(n+1)/2, -j+nm, -k, -m, -n);$$

we think of it as the lattice subgroup of Nielsen's $\mathrm{G}_{5,3} = \mathbb{R}^5$ with multiplication (3.2) [11] (although, in fact, Nielsen's group has a slightly different, but isomorphic, multiplication). The representation of $\mathrm{H}_{5,3}$ is given by π: $(h,j,k,m,n) \mapsto \lambda^h X^j W^k V^m U^n$, and obviously generates $\mathrm{A}^{5,3}_\theta$.

It seems worth pointing out that the equation $WX = XW$ follows from the other equations of (3.1). For $WX = W(UVU^{-1}V^{-1}) = (U \cdot \bar{\lambda}V \cdot U^{-1} \cdot \lambda V)W = XW$. An analogous remark holds for the 5th equation in each of (5.1) and (6.1) ahead. We also point out that $\mathbb{Z}^5$ is not a subgroup of Nielsen's $\mathrm{G}_{5,3}$, and that it is not obvious how to pick a lattice subgroup of $\mathrm{G}_{5,3}$ that is analogous to $\mathrm{H}_3 \subset \mathrm{G}_3$; the simplest isomorphism we have been able to devise of our $\mathrm{H}_{5,3}$ into Nielsen's $\mathrm{G}_{5,3}$ is

$$(h, j, k, m, n) \mapsto (h + j/2, j, k, m, n).$$

3.1 THEOREM. *Let $\lambda = e^{2\pi i\theta}$ for an irrational θ.*

(1) *There is a unique (up to isomorphism) simple C^*-algebra $\mathrm{A}_\theta^{5,3}$ generated by unitaries U, V and W satisfying* (3.1). *Let $\mathbb{Z}^2$ act on $\mathcal{C}(\mathbb{T}^2)$ by (k, n): $f \mapsto f \circ \phi_1^{-k} \circ \phi_2^n$, where the commuting homeomorphisms ϕ_1 and ϕ_2 of $\mathbb{T}^2$ are given by $\phi_1(w, v) = (w, \lambda v)$ and $\phi_2(w, v) = (\lambda w, wv)$. Then*

$$\mathrm{A}_\theta^{5,3} \cong C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2).$$

(2) *Let π' be a representation of $\mathrm{H}_{5,3}$ such that $\pi = \pi'$ (as scalars) on the center $(\mathbb{Z}, 0, 0, 0, 0)$ of $\mathrm{H}_{5,3}$, and let A be the C^*-algebra generated by π'. Then $\mathrm{A} \cong \mathrm{A}_\theta^{5,3}$ via a unique isomorphism ω such that the following diagram commutes*:

$$\begin{tikzcd} \mathrm{H}_{5,3} \arrow[rr, "\pi"] \arrow[dr, "\pi'"'] & & \mathrm{A}_\theta^{5,3} \arrow[dl, "\omega"] \\ & \mathrm{A} & \end{tikzcd}$$

(3) *The C^*-algebra $\mathrm{A}_\theta^{5,3}$ has a unique tracial state.*

Proof. Note that the flow $(\mathbb{Z}^2, \mathbb{T}^2)$ with action given by

$$(k, n)\colon (w, v) \mapsto \phi_1^{-k} \circ \phi_2^n(w, v) = (\lambda^n w, \lambda^{-k+n(n-1)/2} w^n v)$$

is minimal, since the Anzai flow $(\mathbb{Z}, \mathbb{T}^2)$ generated by ϕ_2 alone is [5; 3.3.12]; $(\mathbb{Z}^2, \mathbb{T}^2)$ is also effective, so $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ is simple. The rest of the proof can be modeled on that of Theorem 1.1; see also the Discussion in Section 1. □

In the next note, we shall need the presentation of $\mathrm{A}_\theta^{5,3}$ coming from the subgroup $N = (\mathbb{Z}, 0, 0, 0, 0) \subset \mathrm{H}_{5,3}$, for which $\mathrm{H}_{5,3}/N = \mathrm{H}_3 \times \mathbb{Z}$. For this presentation, define a cocycle α: $(\mathrm{H}_3 \times \mathbb{Z}) \times (\mathrm{H}_3 \times \mathbb{Z}) \to \mathbb{C}$ by

$$\alpha((j, k, m, n), (j', k', m', n')) = \lambda^{mk'+nj'+m'n(n-1)/2}.$$

Then $\mathrm{A}_\theta^{5,3} \cong C^*(\mathrm{H}_3 \times \mathbb{Z}, \alpha)$.

Note. As was indicated in the notes in Sections 1 and 2, the C^*-algebras $A_\theta^{5,1}$ and $A_{\theta,\varphi}^{5,2}$ can be thought of as generated by representations of canonical commutation relations (CCR) over $(\mathbb{Z}^r, b)$, where b is a bicharacter on $\mathbb{Z}^r$. The algebras $A_\theta^{5,i}$, $i = 3, 5, 6$, and $A_{\theta,\varphi}^{5,4}$ can be thought of as generated analogously by such representations only over *non-abelian* groups. From the presentation of $A_\theta^{5,3}$ just given, the group required is $H_3 \times \mathbb{Z}$ and b is no longer a bicharacter, but rather b is the cocycle α. Then the required representation W of CCR over $(H_3 \times \mathbb{Z}, b)$, or b-representation of $H_3 \times \mathbb{Z}$, is given by

$$W(j, k, m, n) = X^j W^k V^m U^n,$$

where U, V, W and X satisfy (3.1), so that

$$W(s)W(s') = b(s, s')W(ss'), \qquad s, s' \in H_3 \times \mathbb{Z}.$$

Now Packer's theorem [13] shows that the C^*-algebra $C^*(H_3 \times \mathbb{Z}, \alpha)$ generated by the b-representation is simple ([23] no longer being applicable).

Other infinite dimensional simple quotients of $C^*(H_{5,3})$. Suppose that λ is a primitive qth root of unity; suppose also that A is a simple quotient of $C^*(H_{5,3})$ that is irreducibly represented and generated by unitaries U, V, W and subsidiary operator X satisfying (3.1). Then W^q commutes with U and V and so by irreducibility equals $\gamma' I$, a multiple of the identity. Since W is a generator of A, we can substitute $W = \gamma W_1$, where $\gamma^q = \gamma'$, and have $W_1^q = I$, while (3.1) still holds with W_1 replacing W. Also, $X^q = \mu' I$, so if μ' is not a root of unity, substitute $X = \mu X_1$, where $\mu^q = \mu'$, in (3.1); then the following equations are satisfied:

$$\begin{cases} UV = \mu X_1 VU, \quad UW_1 = W_1 U, \quad UX_1 = \lambda X_1 U, \quad VW_1 = \lambda W_1 V, \\ VX_1 = X_1 V, \quad W_1 X_1 = X_1 W_1 \text{ and } W_1^q = X_1^q = I. \end{cases} \tag{3.3}$$

Now we can modify the presentation $C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ for $A_\theta^{5,3}$ (in Theorem 3.1) and present A as $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z}_q \times \mathbb{Z})$; the action of $\mathbb{Z}_q \times \mathbb{Z}$ on $\mathcal{C}(\mathbb{Z}_q \times \mathbb{T})$ is generated by the commuting homeomorphisms of $\mathbb{Z}_q \times \mathbb{T}$, $(x, v) \mapsto (x, \lambda v)$ and $(x, v) \mapsto (\lambda x, \mu x v)$. The unitaries

$$X_1 = x_{(0,0)}, \quad V = v_{(0,0)}, \; U = \delta_{(0,1)} \text{ and } W_1 = \delta_{(1,0)}$$

in $\ell_1(\mathbb{Z}_q \times \mathbb{Z}, \mathcal{C}(\mathbb{Z}_q \times \mathbb{T})) \subset C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z}_q \times \mathbb{Z})$ satisfy (3.3). (Here $x \in \mathcal{C}(\mathbb{Z}_q \times \mathbb{T})$ is the function $(x, v) \mapsto x$.) The flow $(\mathbb{Z}_q \times \mathbb{Z}, \mathbb{Z}_q \times \mathbb{T})$ is minimal [10; Theorem 3] and effective, so $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z}_q \times \mathbb{Z})$ is simple and isomorphic to A.

When μ is a root of unity (as well as λ), the C^*-algebra A is finite dimensional. The argument for this is analogous to that made in the previous section.

3.2 THEOREM. *A C^*-algebra* A *is isomorphic to a simple infinite dimensional quotient of $C^*(H_{5,3})$ if, and only if,* A *is isomorphic to $A_\theta^{5,3}$ for some irrational θ, or to $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z}_q \times \mathbb{Z})$ (as above) for some λ, a primitive qth root of unity, and μ not a root of unity.*

The referee pointed out that the method of proof of Proposition 1.6 of Lee and Packer [9], a result that is about 2-step groups, can be made to apply here and show that $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z}_q \times \mathbb{Z})$, and the analogous algebras in Sections 4–6, have matrix algebra presentations. We avoid the modification of this proof to our 3- and 4-step settings by giving explicit matrix presentations of the other quotients in Sections 3–6.

3.3 THEOREM. *When λ is a primitive qth root of unity and $\mu = e^{2\pi i\varphi}$ is not a root of unity, the C^*-crossed product $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z}_q \times \mathbb{Z})$ (of Theorem 3.2) is isomorphic to $M_{q^2}(\mathrm{A}^3_{q^2\varphi})$.*

Proof. Let unitaries U_0 and V_0 satisfy $U_0V_0 = \mu^{q^2}V_0U_0$, so that U_0 and V_0 generate $\mathrm{A}^3_{q^2\varphi}$. First define $X_2 \in M_q(\mathbb{C})$ to have $1, \overline{\lambda}, \overline{\lambda}^2, \dots, \overline{\lambda}^{q-1}$ on the diagonal; $U_2 \in M_q(\mathrm{A}^3_{q^2\varphi})$ to have U_0 in the upper right hand corner and 1's on the subdiagonal; $V_2 \in M_q(\mathrm{A}^3_{q^2\varphi})$ to have $V_0, \overline{\mu}^q V_0, \overline{\mu}^{2q} V_0, \dots, \overline{\mu}^{(q-1)q} V_0$ on the diagonal; so

$$U_2V_2 = \mu^q V_2U_2, \quad U_2X_2 = \lambda X_2U_2 \text{ and } V_2X_2 = X_2V_2,$$

and U_2, V_2 and X_2 generate $M_q(\mathrm{A}^3_{q^2\varphi})$. Now let I be the identity matrix in $M_q(\mathbb{C})$, and define unitaries U', V' and W' in $M_q(M_q(\mathrm{A}^3_{q^2\varphi})) = M_{q^2}(\mathrm{A}^3_{q^2\varphi})$, all unspecified entries of which are 0.

U' has $U_2, \mu X_2U_2, \mu^2X_2^2U_2, \mu^3X_2^3U_2, \dots, \mu^{q-1}X_2^{q-1}U_2$ on the diagonal,
V' has V_2 in the upper right hand corner and I's on the subdiagonal, and
W' has $I, \overline{\lambda}I, \overline{\lambda}^2I, \overline{\lambda}^3I, \dots, \overline{\lambda}^{q-1}I$ on the diagonal.

Then U', V', W' and subsidiary operator X' satisfy (3.3) and generate $M_{q^2}(\mathrm{A}^3_{q^2\varphi})$ (X' having X_2's on the diagonal). □

4. The simple quotients $\mathrm{A}^{5,4}_{\theta,\varphi}$ of $\mathrm{H}_{5,4}$

Let unitaries U, V and subsidiary operator W satisfy

$$UV = WVU, \quad UW = \lambda WU \text{ and } VW = \mu WV, \tag{4.1}$$

where $\mu = e^{2\pi i\varphi}$ and $\lambda = e^{2\pi i\theta}$ are linearly independent; let $\mathrm{A}^{5,4}_{\theta,\varphi}$ denote the C^*-algebra generated by U and V. The equations (4.1) yield a group with a representation whose generated C^*-algebra is $\mathrm{A}^{5,4}_{\theta,\varphi}$. The group is $\mathrm{H}_{5,4}$ ($= \mathbb{Z}^5$ as a set) with multiplication

$$\begin{aligned}(h, j, k, m, n)&(h', j', k', m', n') \quad (4.2)\\ &= (h + h' + mk' + mnm' + nm'(m' - 1)/2, j + j' + nk' + m'n(n - 1)/2,\\ &\quad k + k' + nm', m + m', n + n'),\end{aligned}$$

and inverse

$$(h, j, k, m, n)^{-1} = (-h + mk - nm(m+1)/2, -j + nk - mn(n+1)/2, \\ -k + nm, -m, -n);$$

we think of it as the lattice subgroup of Nielsen's $G_{5,4} = \mathbb{R}^5$ with multiplication (4.2) [11] (although, in fact, Nielsen's group has a slightly different, but isomorphic, multiplication). The representation of $H_{5,4}$ is given by π: $(h, j, k, m, n) \mapsto \mu^h \lambda^j W^k V^m U^n$, and obviously generates $A^{5,4}_{\theta,\varphi}$.

Most of the results in this section appear in Packer [14], where (among other things) the equations (4.1) are studied; the group $H_{5,4}$ is identified; the C^*-algebras $A^{5,4}_{\theta,\varphi}$ (called in [14] the algebras *of class* 3 associated with (4.1)) are classified, shown to be generated by representations of $H_{5,4}$, and shown to be simple with unique trace; and similarly for the algebras A_1 (called *of class* 2 in [14]) in Theorem 4.2 below.

For completeness, the results for $H_{5,4}$ are presented in the same format as for the other $H_{5,i}$'s. Aspects given here that are not dealt with in [14] include the connection of $H_{5,4}$ with the Lie group $G_{5,4}$, and also the algebras A_2 (in Theorem 4.2 below), which appear here as the *simple* (rather than universal) infinite dimensional C^*-algebras generated by unitaries satisfying (4.1) when λ and μ are both roots of unity. The algebras A_2 here are simple quotients of Packer's class 1 algebras. Also, the matrix presentation for the algebras A_2 appears here for the first time (Theorem 4.3 below).

4.1 THEOREM. *Let $\mu = e^{2\pi i\varphi}$ and $\lambda = e^{2\pi i\theta}$ be linearly independent.*

(1) *There is a unique (up to isomorphism) simple C^*-algebra $A^{5,4}_{\theta,\varphi}$ generated by unitaries U and V satisfying* (4.1). *Let A^3_φ be generated by unitaries U' and V' satisfying $U'V' = \mu V'U'$. Define an automorphism ν of A^3_φ by ν: $U' \mapsto V'U'$, $V' \mapsto \lambda V'$; ν determines an action of $\mathbb{Z}$ on A^3_φ. Then*

$$A^{5,4}_{\theta,\varphi} \cong C^*(A^3_\varphi, \mathbb{Z}).$$

(2) *Let π' be a representation of $H_{5,4}$ such that $\pi = \pi'$ (as scalars) on the center $(\mathbb{Z}, \mathbb{Z}, 0, 0, 0)$ of $H_{5,4}$, and let A be the C^*-algebra generated by π'. Then $A \cong A^{5,4}_{\theta,\varphi}$ via a unique isomorphism ω such that the following diagram commutes*:

$$\begin{tikzcd} H_{5,4} \arrow[r, "\pi"] \arrow[dr, "\pi'"'] & A^{5,4}_{\theta,\varphi} \arrow[d, "\omega"] \\ & A \end{tikzcd}$$

(3) *The C^*-algebra $A^{5,4}_{\theta,\varphi}$ has a unique tracial state.*

Proof. Note first that an argument as in [10; remark before Theorem 3] shows that each automorphism ν^n, $n \neq 0$, of the simple C^*-algebra A^3_φ is outer, and then

[8; Theorem 3.1] yields the conclusion that $C^*(\mathrm{A}^3_\varphi, \mathbb{Z})$ is simple. Specifically, if ν^n is inner and is implemented by some unitary $T \in \mathrm{A}^3_\varphi$, then in $K_1(\mathrm{A}^3_\varphi)$,

$$[\nu^n(U')] = [TU'T^*] = [U'] \text{ and } [\nu^n(U')] = [\lambda^{n(n-1)/2}V'^n U'] = n[V'] + [U'];$$

since $K_1(\mathrm{A}^3_\varphi) \cong \mathbb{Z}^2$ is generated by $[U']$ and $[V']$ (see [16]), we must have $n = 0$.

The rest of the proof can be modeled on the proof of Theorem 1.1. □

We need another presentation of $\mathrm{A}^{5,4}_{\theta,\varphi}$. Define a cocycle α: $\mathrm{H}_3 \times \mathrm{H}_3 \to \mathbb{C}$ by

$$\alpha((k,m,n),(k',m',n')) = \mu^{mk'+mnm'+nm'(m'-1)/2}\lambda^{nk'+m'n(n-1)/2}.$$

Then $\mathrm{A}^{5,4}_{\theta,\varphi} \cong C^*(\mathrm{H}_3, \alpha)$.

Note. The presentation $C^*(\mathrm{H}_3, \alpha)$ of $\mathrm{A}^{5,4}_{\theta,\varphi}$ above shows how the algebras $\mathrm{A}^{5,4}_{\theta,\varphi}$ can be thought of as generated by a representation of CCR over the non-abelian group H_3 (see the note in Section 3). Indeed, in [14; Example 1.9], Packer is generating a class of algebras including the $\mathrm{A}^{5,4}_{\theta,\varphi}$'s in this way.

Other infinite dimensional simple quotients of $C^*(\mathrm{H}_{5,4})$. 1. Suppose first that λ and μ are linearly dependent and that μ, at least, is not a root of unity; there is no other restriction on λ, which could be a root of unity. Let A be the C^*-algebra generated by unitaries U and V satisfying (4.1). Then the same formulas as in Theorem 4.1 give an action of $\mathbb{Z}$ on A^3_φ by outer automorphisms; so the generated crossed product $\mathrm{A}_1 = C^*(\mathrm{A}^3_\varphi, \mathbb{Z})$ is simple and A is isomorphic to it. In this case A_1 is an algebra of class 2 [14]. When $\lambda = 1$, the equations (4.1) are essentially equations (0.1) in the Preliminaries, and $\mathrm{A} \cong \mathrm{A}^4_\varphi$.

2. The situation is much the same when λ and μ are linearly dependent and λ, at least, is not a root of unity. Explicitly, Theorem 2.9 of [14] shows that then the algebra generated by unitaries satisfying (4.1) is isomorphic to an $\mathrm{A}_1 = C^*(\mathrm{A}^3_{\varphi_1}, \mathbb{Z})$ (as in case 1) for suitable irrational φ_1 and rational θ_1.

3. If λ and μ are primitive qth and q'th roots of unity, respectively, let $q'' = \operatorname{lcm}\{q, q'\}$. Then $W^{q''}$ commutes with both U and V, so assuming that A is irreducibly represented, we have $W^{q''} = \eta' I$, a multiple of the identity. If η' is not a root of unity, a modification of the second presentation given above for $\mathrm{A}^{5,4}_{\theta,\varphi}$ yields a simple C^*-algebra isomorphic to A. First, the substitution $W = \eta W_1$, where $\eta^{q''} = \eta'$, changes (4.1) to

$$UV = \eta W_1 VU, \quad UW_1 = \lambda W_1 U, \quad VW_1 = \mu W_1 V \text{ and } W_1^{q''} = 1. \tag{4.3}$$

Now use (4.3) to simplify the product $(W_1^k V^m U^n)(W_1^{k'} V^{m'} U^{n'})$ and get

$$\mu^{mk'+mnm'+nm'(m'-1)/2}\lambda^{nk'+m'n(n-1)/2}\eta^{nm'}W_1^{k+k'+nm'}V^{m+m'}U^{n+n'}.$$

Then the group we need is $\mathrm{H} = \mathbb{Z}_{q''} \times \mathbb{Z} \times \mathbb{Z}$ with multiplication

$$(k, m, n)(k', m', n') = ((k + k' + nm') \bmod p, m + m', n + n')$$

and the cocycle we need is $\alpha\colon \mathrm{H} \times \mathrm{H} \to \mathbb{C}$ defined by

$$\alpha((k, m, n), (k', m', n')) = \mu^{mk' + mnm' + nm'(m'-1)/2} \lambda^{nk' + m'n(n-1)/2} \eta^{nm'}.$$

Much as in the note in Section 3, the simplicity of $\mathrm{A}_2 = C^*(\mathrm{H}, \alpha)$ follows from [13], so A is isomorphic to A_2.

When $\mu = 1$ in this case, the algebra A is isomorphic to ${}_{q'}\mathrm{A}_\beta$, where $\eta = e^{2\pi i\beta}$; see Theorem 0.2 in the Preliminaries.

4. If η is also a root of unity (as well as λ and μ), then a C^*-algebra generated by unitaries U and V satisfying (4) is finite dimensional; see the argument for the analogous claim in Section 2.

4.2 THEOREM. *A C^*-algebra* A *is isomorphic to a simple infinite dimensional quotient of $C^*(\mathrm{H}_{5,4})$ if, and only if,* A *is isomorphic to* $\mathrm{A}^{5,4}_{\theta,\varphi}$ *for some linearly independent λ and μ, or to* $\mathrm{A}_1 = C^*(\mathrm{A}^3_\varphi, \mathbb{Z})$ *or* A_2 *(as in cases* 1 *and* 3 *above, respectively).*

Since none of the unitaries generating the algebra $\mathrm{A}_1 = C^*(\mathrm{A}^3_\varphi, \mathbb{Z})$ is unipotent, we conjecture that A_1 is not isomorphic to a matrix algebra; however it can be shown that A_1 is isomorphic to a subalgebra of $M_Q(\mathrm{A}^4_{\varphi'})$ for suitable Q and φ'.

4.3 THEOREM. *When λ, μ, η, q, q', q'' and* $\mathrm{A}_2 = C^*(\mathrm{H}, \alpha)$ *are as in case* 3 *above, the C^*-algebra* A_2 *is isomorphic to a matrix algebra* $M_{q''}(\mathrm{A}^3_\gamma)$, *where* $e^{2\pi i\gamma} = (-1)^{q''+1}\eta^{q''}$.

Proof. First choose a primitive q''th root of unity ζ, for which there are relatively prime integers c and d such that $\lambda = \zeta^d$ and $\mu = \zeta^{-c}$. Next choose $a, b \in \mathbb{Z}$ so that $ad - bc = 1$, and set

$$U' = U^a V^b \text{ and } V' = U^c V^d.$$

Then U' and V' generate the same C^*-algebra as U and V, and using (4.3) one verifies that they satisfy

$$\begin{aligned} U'V' &= \eta\chi W_1 V'U', \\ U'W_1 &= \lambda^a \mu^b W_1 U' = \zeta W_1 U' \end{aligned}$$

and

$$V'W_1 = \lambda^c \mu^d W_1 V' = W_1 V',$$

where χ is a power of λ times a power of μ, so χ is a q''th root of unity. Thus $W' = \chi W_1$ satisfies $W'^{q''} = 1$, and Theorem 0.2 in the Preliminaries implies that A_2 is isomorphic to $M_{q''}(\mathrm{A}^3_\gamma)$. $\square$

5. The simple quotients $A_\theta^{5,5}$ of $H_{5,5}$

Let $\lambda = e^{2\pi i\theta}$ for an irrational θ, let unitaries U, V and subsidiary operators W and X satisfy

$$\begin{cases} UV = WVU, & UW = XWU, \quad UX = \lambda XU, \\ VW = WV, & VX = XV \quad \text{and} \quad WX = XW, \end{cases} \tag{5.1}$$

and let $A_\theta^{5,5}$ denote the C^*-algebra generated by U and V.

The equations (5.1) yield a group with a representation whose generated C^*-algebra is $A_\theta^{5,5}$. The group is $H_{5,5}$ ($= \mathbb{Z}^5$ as a set) with multiplication

$$\begin{aligned} &(h, j, k, m, n)(h', j', k', m', n') \\ &\quad = (h + h' + nj' + k'n(n-1)/2 + m'n(n-1)(n-2)/6, \\ &\qquad j + j' + nk' + m'n(n-1)/2, k + k' + nm', m + m', n + n'), \end{aligned} \tag{5.2}$$

and inverse

$$\begin{aligned} (h, j, k, m, n)^{-1} &= (-h + nj - kn(n+1)/2 + mn(n+1)(n+2)/6, \\ &\quad - j + kn - mn(n+1)/2, -k + mn, -m, -n); \end{aligned}$$

we think of it as the lattice subgroup of Nielsen's $G_{5,5} = \mathbb{R}^5$ with multiplication (5.2) [11] (although, in fact, Nielsen's group has a slightly different, but isomorphic, multiplication). The representation of $H_{5,5}$ is given by π: $(h, j, k, m, n) \mapsto \lambda^h X^j W^k V^m U^n$, and obviously generates $A_\theta^{5,5}$.

5.1 THEOREM. *Let $\lambda = e^{2\pi i\theta}$ for an irrational θ.*

(1) *There is a unique (up to isomorphism) simple C^*-algebra $A_\theta^{5,5}$ generated by unitaries U and V satisfying* (5.1). *Define a homeomorphism ϕ of $\mathbb{T}^3$ by $\phi(x, w, v) = (\lambda x, xw, wv)$; iteration of ϕ gives an action of $\mathbb{Z}$ on $C(\mathbb{T}^3)$, n: $f \mapsto f \circ \phi^n$. Then*

$$A_\theta^{5,5} \cong C^*(C(\mathbb{T}^3), \mathbb{Z}).$$

(2) *Let π' be a representation of $H_{5,5}$ such that $\pi = \pi'$ (as scalars) on the center $(\mathbb{Z}, 0, 0, 0, 0)$ of $H_{5,5}$, and let A be the C^*-algebra generated by π'. Then $A \cong A_\theta^{5,5}$ via a unique isomorphism ω such that the following diagram commutes*:

$$\begin{tikzcd} H_{5,5} \arrow[r, "\pi"] \arrow[dr, "\pi'"'] & A_\theta^{5,5} \arrow[d, "\omega"] \\ & A \end{tikzcd}$$

(3) *The C^*-algebra $A_\theta^{5,5}$ has a unique tracial state.*

Proof. Ji [7] and Packer [12] and Rouhani [20] have noted that Anzai flows, like $(\mathbb{Z}, \mathbb{T}^3)$ generated by the homeomorphism ϕ, are minimal, so the crossed product $C^*(C(\mathbb{T}^3), \mathbb{Z})$ is simple and has a unique trace. The proof that $A_\theta^{5,5} \cong C^*(C(\mathbb{T}^3), \mathbb{Z})$ can be modeled on the proof of Theorem 1.1. □

Other infinite dimensional simple quotients of $C^*(\mathrm{H}_{5,5})$. Suppose that λ is a primitive qth root of unity and that A is a simple quotient of $C^*(\mathrm{H}_{5,5})$ that is irreducibly represented and generated by unitaries U and V and subsidiary operators W and X satisfying (5.1). Then X^q commutes with U and V and so by irreducibility equals $\mu' I$, a multiple of the identity. Put $X = \mu X_1$ for $\mu^q = \mu'$, so that $X_1^q = 1$, and substitute $X = \mu X_1$ in (5.1) to get

$$\begin{cases} UV = WVU, \quad UW = \mu X_1 WU, \quad UX_1 = \lambda X_1 U, \\ VW = WV, \quad VX_1 = X_1 V, \quad WX_1 = X_1 W \text{ and } X_1^q = 1. \end{cases} \tag{5.3}$$

1. If μ is not a root of unity, then much as for the other quotients of $C^*(\mathrm{H}_{5,3})$, or as in [10; Theorem 3], we can modify the presentation $C^*(\mathcal{C}(\mathbb{T}^3), \mathbb{Z})$ for $\mathrm{A}_\theta^{5,5}$ in Theorem 5.1 and present the operators U and V (and W and X_1) with the flow $\mathcal{F} = (\mathbb{Z}, \mathbb{Z}_q \times \mathbb{T}^2)$ generated by the homeomorphism ϕ_1 of $\mathbb{Z}_q \times \mathbb{T}^2$, $\phi_1(x, w, v) = (\lambda x, \mu x w, wv)$. To see that $\mathcal{F}$ is minimal, note that

$$\begin{aligned} \phi_1^{r+6qk}(x, w, v) &= (\lambda^{r+6qk}, \lambda^{(r+6qk)(r+6qk-1)/2}\mu^{r+6qk} w, \\ &\quad \lambda^{(r+6qk)(r+6qk-1)(r+6qk-2)/6}(\mu x)^{(r+6qk)(r+6qk-1)/2} w^{r+6qk} v) \\ &= (\lambda^r, \lambda^{r(r-1)/2}\mu^{r+6qk} w, \\ &\quad \lambda^{r(r-1)(r-2)/6} x^{r(r-1)/2} \mu^{(r+6qk)(r+6qk-1)/2} w^{r+6qk} v) \end{aligned}$$

and use the fact that, when $\xi \in \mathbb{T}$ is not a root of unity, $\{(\xi^k, \xi^{k^2}) \colon k \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$; this is the fact that yields the minimality of the Anzai flow on the 2-torus [5; 3.3.12, for example]. So the C^*-crossed product $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^2), \mathbb{Z})$ is simple and isomorphic to A with U and V corresponding to δ_1 and v_0 in $\ell_1(\mathbb{Z}, \mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^2))$, where v is the function $(x, w, v) \mapsto v$ in $\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^2)$.

2. Suppose that μ is also a root of unity, say a primitive q'th root of unity, and let $q'' = \operatorname{lcm}\{q, q'\}$, the least common multiple of q and q'. Then $W^{q''} = \eta' I$, a multiple of the identity. If η' is not a root of unity, substitute $W = \eta W_1$ (as well as $X = \mu X_1$) in (5.1), where $\eta^{q''} = \eta'$. Then $W_1^{q''} = 1$ and we can present A using the homeomorphism ϕ_2 on $\mathcal{X} = \mathbb{Z}_q \times \mathbb{Z}_{q''} \times \mathbb{T}$, $\phi_2(x, w, v) = (\lambda x, \mu x w, \eta w v)$. The flow $(\mathbb{Z}, \mathcal{X})$ that ϕ_2 generates is usually not minimal, so we restrict ϕ_2 to $\mathcal{Y} \times \mathbb{T} \subset \mathcal{X}$, where $\mathcal{Y} \subset \mathbb{Z}_q \times \mathbb{Z}_{q''}$ is the finite set

$$\begin{aligned} \mathcal{Y} &= \{(x, w) \mid (x, w, 1) \in \phi_2^r(1, 1, \mathbb{T}) \text{ for some } r \in \mathbb{N}\} \\ &= \{(\lambda^r, \lambda^{r(r-1)/2}\mu^r) \mid r \in \mathbb{N}\}. \end{aligned}$$

Then the flow $(\mathbb{Z}, \mathcal{Y} \times \mathbb{T})$ is minimal; the proof of this is similar to, but easier than, the minimality proof in case 1 above. So $C^*(\mathcal{C}(\mathcal{Y} \times \mathbb{T}), \mathbb{Z})$ is simple and isomorphic to A.

3. When η is a root of unity (as well as μ and λ), the C^*-algebra A is finite dimensional.

5.2 THEOREM. *A C*-algebra* A *is isomorphic to a simple infinite dimensional quotient of* $C^*(\mathrm{H}_{5,5})$ *if, and only if,* A *is isomorphic to* $\mathrm{A}_\theta^{5,5}$ *for some irrational* θ, *to* $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^2), \mathbb{Z})$, *or to* $C^*(\mathcal{C}(\mathcal{X}), \mathbb{Z})$ *(as in cases* 1 *and* 2 *above, respectively).*

The referee pointed out that the algebras $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^2), \mathbb{Z})$ and $C^*(\mathcal{C}(\mathcal{Y} \times \mathbb{T}), \mathbb{Z})$ above are isomorphic to matrix algebras over an irrational rotation algebra. For $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^2), \mathbb{Z})$ the simple C^*-algebra $\mathrm{A}'^4_\gamma(q)$ (for an irrational γ) needed for the matrix algebra is the analogue of A^4_β corresponding to a 'scaled' variant $\mathrm{H}'_4(q)$ of the 4-dimensional group H_4. That is, $\mathrm{A}'^4_\gamma(q)$ is a simple quotient of $C^*(\mathrm{H}'_4(q))$, and thus is the C^*-algebra generated by an irreducible representation of $\mathrm{H}'_4(q)$. This group and C^*-algebra were mentioned in [10; p.633]; the technical detail we need is that generators for $\mathrm{A}'^4_\gamma(q)$ are unitaries U_0, V_0 and W_0 satisfying

$$U_0V_0 = W_0^q V_0 U_0, \quad U_0 W_0 = \zeta W_0 U_0 \text{ and } V_0 W_0 = W_0 V_0, \tag{5.4}$$

where $\zeta = e^{2\pi i \gamma}$.

5.3 THEOREM. *When* λ *is a primitive* q*th root of unity and* μ *and* $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^2), \mathbb{Z})$ *are as in case* 1 *above, the* C^**-algebra* $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^2), \mathbb{Z})$ *is isomorphic to the matrix algebra* $M_q(\mathrm{A}'^4_\gamma(q))$, *where* $\zeta = e^{2\pi i \gamma} = (-1)^{q+1}\mu^q$.

Proof. Let unitaries U_0, V_0 and W_0 satisfy (5.4), thus generating $\mathrm{A}'^4_\gamma(q)$. Then define unitaries in $M_q(\mathrm{A}'^4_\gamma(q))$ as follows (all of whose unspecified entries are 0).

U' has 1's on the subdiagonal and U_0 in the upper right hand corner;
V' has $d_0 V_0, d_1 W_0^{-1} V_0, d_2 W_0^{-2} V_0, \ldots, d_{q-1} W_0^{-(q-1)} V_0$ on the diagonal;
W' has $b W_0, b\overline{\mu}\lambda W_0, b\overline{\mu}^2\lambda^3 W_0, \ldots, b\overline{\mu}^{q-1}\lambda^{q(q-1)/2} W_0$ on the diagonal, and
X' has $1, \overline{\lambda}, \overline{\lambda}^2, \ldots, \overline{\lambda}^{q-1}$ on the diagonal.

The constants must be chosen to make these matrices satisfy the equations (5.3); as defined, they already satisfy all but the first equation. We arrange $U'V' = W'V'U'$ by letting b be a qth root of

$$\overline{\lambda}^{(q^3-q)/6}\mu^{q(q-1)/2}\overline{\zeta}^{q-1} = \overline{\lambda}^{(q^3-q)/6}\overline{\mu}^{q(q-1)/2}(-1)^{q+1}$$

and setting $d_r = \overline{\lambda}^{(r^3-r)/6}\mu^{r(r-1)/2}\overline{b}^{(r-1)}$. Then the matrices satisfy (5.3) and generate $M_q(\mathrm{A}'^4_\gamma(q))$. □

To identify the algebra $C^*(\mathcal{C}(\mathcal{Y} \times \mathbb{T}), \mathbb{Z})$ with a matrix algebra over an irrational rotation algebra, we start by determining the cardinality $C = |\mathcal{Y}|$ of the set $\mathcal{Y}$ in $\mathcal{Y} \times \mathbb{T}$ (in 2 above).

5.4 LEMMA. (*a*) *If* q *is odd, then* $C = \mathrm{lcm}\{q, q'\}$, *the least common multiple of* q *and* q'.

(*b*) *If* q *is even and* q' *is odd, then* $C = 2\,\mathrm{lcm}\{q, q'\}$.

(c) If $q = 2^s t$ and $q' = 2^{s'} t'$ are both even and t and t' are odd, then

(i) $C = 2\,\mathrm{lcm}\{q, q'\}$, *if* $s \geq s'$,
(ii) $C = \frac{1}{2}\,\mathrm{lcm}\{q, q'\}$, *if* $s + 1 = s'$, *and*
(iii) $C = \mathrm{lcm}\{q, q'\}$, *if* $s + 1 < s'$.

Proof. Now $\mathcal{Y} = \{(\lambda^r, \lambda^{r(r-1)/2}\mu^r) \mid r \in \mathbb{N}\}$, and it follows from the definition of $(\mathbb{Z}, \mathcal{Y} \times \mathbb{T})$ as a minimal subflow of $(\mathbb{Z}, \mathcal{X})$ that C is the first $r \in \mathbb{N}$ for which

$$(\lambda^r, \lambda^{r(r-1)/2}\mu^r) = (1, 1),$$

so C is a multiple of q; also it is clear that $C \leq 2\,\mathrm{lcm}\{q, q'\}$.

Part (a) follows because $\lambda^{aq(aq-1)/2} = 1$ for all $a \in \mathbb{N}$, when q is odd.

For (b) and (c) where q is even, note that

$$\lambda^{aq(aq-1)/2} = \begin{cases} 1 & \text{if } a \text{ is even,} \\ -1 & \text{if } a \text{ is odd,} \end{cases}$$

so $C = 2\,\mathrm{lcm}\{q, q'\}$ for (b) and (i) of (c) because $\mathrm{lcm}\{q, q'\}$ is an odd multiple of q in these cases; also, for (b), $\mu^n \neq -1$ for any $n \in \mathbb{N}$. For case (ii) of (c), we have $C = \frac{1}{2}\mathrm{lcm}\{q, q'\}$, because this is an odd multiple both of q and of $q'/2$, so

$$\lambda^{C(C-1)/2} = \mu^C = -1.$$

Case (iii) of (c) follows similarly. $\square$

5.5 Lemma. *The flow $(\mathbb{Z}, \mathcal{Y} \times \mathbb{T})$ is isomorphic to a flow $(\mathbb{Z}, \mathbb{Z}_C \times \mathbb{T})$ generated by a homeomorphism $\psi\colon (w, v) \mapsto (\lambda_1 w, \eta_1 wv)$, where λ_1 is a primitive Cth root of unity, and $\eta_1 \in \mathbb{T}$ is chosen appropriately.*

Proof. We need to construct a homeomorphism

$$\tau\colon \mathcal{Y} \times \mathbb{T} = \{(\lambda^r, \lambda^{r(r-1)/2}\mu^r) \mid r \leq C\} \times \mathbb{T} \to \mathbb{Z}_C \times \mathbb{T}$$

that commutes with the actions of $\mathbb{Z}$, i.e., such that $\tau \circ \phi_2 = \psi \circ \tau$ on $\mathcal{Y} \times \mathbb{T}$. Define τ as follows for $v \in \mathbb{T}$:

$$\tau(1, 1, v) = (1, v),$$

$$\tau \circ \phi_2(1, 1, v) = \tau(\lambda, \mu, \eta v) = \psi \circ \tau(1, 1, v) = \psi(1, v) = (\lambda_1, \eta_1 v),$$

$$\tau \circ \phi_2^2(1, 1, v) = \tau(\lambda^2, \lambda\mu^2, \mu\eta^2 v) = \psi^2 \circ \tau(1, 1, v) = \psi^2(1, v) = (\lambda_1^2, \lambda_1\eta_1^2 v),$$

and so on down to

$$\begin{aligned}\tau \circ \phi_2^C(1,1,v) &= \tau(\lambda^C, \lambda^{C(C-1)/2}\mu^C, \lambda^{C(C-1)(C-2)/6}\mu^{C(C-1)/2}\eta^C v)\\ &= \tau(1,1,\lambda^{C(C-1)(C-2)/6}\mu^{C(C-1)/2}\eta^C v)\\ &= \psi^C \circ \tau(1,1,v) = \psi^C(1,v) = (\lambda_1^C, \lambda_1^{C(C-1)/2}\eta_1^C v)\\ &= (1, \lambda_1^{C(C-1)/2}\eta_1^C v).\end{aligned}$$

The definition of τ on $(1, 1, \mathbb{T})$ at this last step must coincide with the definition at the first step, so η_1 is chosen to satisfy the equation

$$\lambda^{C(C-1)(C-2)/6}\mu^{C(C-1)/2}\eta^C = \lambda_1^{C(C-1)/2}\eta_1^C,$$

e.g., $\eta_1 = \mu_1\eta$, where μ_1 is a Cth root of

$$\lambda_2 = \lambda^{C(C-1)(C-2)/6}\mu^{C(C-1)/2}/\lambda_1^{C(C-1)/2}.$$

To see that τ commutes with the actions of $\mathbb{Z}$, take a point $P \in \mathcal{Y} \times \mathbb{T}$. Then $P = \phi_2^r(1,1,v)$ for some $0 \le r < C$ and $v \in \mathbb{T}$, and

$$\tau \circ \phi_2(P) = \tau \circ \phi_2^{r+1}(1,1,v) = \psi^{r+1} \circ \tau(1,1,v)$$

$$= \psi \circ \tau \circ \phi_2^r(1,1,v) = \psi \circ \tau(P),$$

as required. □

5.6 THEOREM. *Let $\lambda, \mu, \eta, q, q'$, q'' and $C^*(\mathcal{Y} \times \mathbb{T}, \mathbb{Z})$ be as in case 2 above, and let $C = |\mathcal{Y}|$ be as in Lemma 5.4. Then the C^*-crossed product $C^*(\mathcal{Y} \times \mathbb{T}, \mathbb{Z})$ is isomorphic to the matrix algebra $M_C(\mathrm{A}_\gamma^3)$, where $e^{2\pi i\gamma} = (-1)^{C+1}\eta_1^C = (-1)^{C+1}\lambda_2\eta^C$ and η_1 and λ_2 are as in the proof of Lemma 5.5.*

Proof. The isomorphism of the flows $(\mathbb{Z}, \mathcal{Y} \times \mathbb{T})$ and $(\mathbb{Z}, \mathbb{Z}_C \times \mathbb{T})$ (Lemma 5.5) implies the isomorphism of the C^*-crossed products $C^*(\mathcal{C}(\mathcal{Y} \times \mathbb{T}), \mathbb{Z})$ and $C^*(\mathcal{C}(\mathbb{Z}_C \times \mathbb{T}), \mathbb{Z})$, the latter of which is isomorphic to the matrix algebra $M_C(\mathrm{A}_\gamma^3)$ (Theorem 0.2). □

6. The simple quotients $\mathrm{A}_\theta^{5,6}$ of $\mathrm{H}_{5,6}$

Let $\lambda = e^{2\pi i\theta}$ for an irrational θ, let unitaries U, V and subsidiary operators W and X satisfy

$$\begin{cases} UV = WVU, \quad UW = XWU, \quad UX = \lambda XU, \\ VW = \lambda WV, \quad VX = XV \quad \text{and} \quad WX = XW, \end{cases} \tag{6.1}$$

and let $\mathrm{A}_\theta^{5,6}$ denote the C^*-algebra generated by U and V.

The equations (6.1) yield a group with a representation whose generated C^*-algebra is $A_\theta^{5,6}$. The group is $H_{5,6}$ ($= \mathbb{Z}^5$ as a set) with multiplication

$$\begin{aligned}(h, j, k, m, n)&(h', j', k', m', n')\\ &= (h + h' + mk' + nj' + mnm' + nm'(m' - 1)/2\\ &\quad + k'n(n - 1)/2 + m'n(n - 1)(n - 2)/6,\\ &\quad j + j' + nk' + m'n(n - 1)/2, k + k' + nm', m + m', n + n'),\end{aligned} \tag{6.2}$$

and inverse

$$\begin{aligned}(h, j, k, m, n)^{-1} &= (-h + mk + nj - nm(m + 1)/2 - kn(n + 1)/2\\ &\quad + mn(n + 1)(n + 2)/6,\\ &\quad - j + kn - mn(n + 1)/2, -k + mn, -m, -n);\end{aligned}$$

we think of it as the lattice subgroup of Nielsen's $G_{5,6} = \mathbb{R}^5$ with multiplication (6.2) [11] (although, in fact, Nielsen's group has a slightly different, but isomorphic, multiplication). The representation of $H_{5,6}$ is given by π: $(h, j, k, m, n) \mapsto \lambda^h X^j W^k V^m U^n$, and obviously generates $A_\theta^{5,6}$. The simplest isomorphism we have been able to devise of our $H_{5,6}$ into Nielsen's $G_{5,6}$ is $(h, j, k, m, n) \mapsto (h + j + 2k/3, j + k/2, k, m, n)$.

6.1 THEOREM. *Let $\lambda = e^{2\pi i\theta}$ for an irrational θ.*

(1) *There is a unique (up to isomorphism) simple C^*-algebra $A_\theta^{5,6}$ generated by unitaries U and V satisfying* (6.1). *Let U' and V' be unitaries generating A_θ^4, i.e., U', V' and subsidiary operator W' satisfy*

(0.1) $\qquad U'V' = W'V'U', \quad U'W' = \lambda W'U'$ *and* $V'W' = W'V'$.

Define an automorphism ν of A_θ^4 by ν: $U' \mapsto V'^{-1}U'$ and $V' \mapsto \lambda V'$; ν determines an action of $\mathbb{Z}$ on A_θ^4. Then

$$A_\theta^{5,6} \cong C^*(A_\theta^4, \mathbb{Z}).$$

(2) *Let π' be a representation of $H_{5,6}$ such that $\pi = \pi'$ (as scalars) on the center $(\mathbb{Z}, 0, 0, 0, 0)$ of $H_{5,6}$, and let A be the C^*-algebra generated by π'. Then $A \cong A_\theta^{5,6}$ via a unique isomorphism ω such that the following diagram commutes:*

$$\begin{tikzcd} H_{5,6} \arrow[rr, "\pi"] \arrow[dr, "\pi'"'] & & A_\theta^{5,6} \arrow[dl, "\omega"] \\ & A & \end{tikzcd}$$

(3) *The C^*-algebra $A_\theta^{5,6}$ has a unique tracial state.*

Proof. The basic idea of the proof of simplicity is similar to that of Theorem 4.1; see also the Discussion in Section 1. The rest of the proof can be modeled on the proof of Theorem 1.1. □

Another presentation of $A_\theta^{5,6}$ will be useful below; it arises from the normal subgroup $N = (\mathbb{Z}, 0, 0, 0, 0) \subset H_{5,6}$ for which $H_{5,6}/N = H_4$. With cocycle α: $H_4 \times H_4 \to \mathbb{C}$ defined by

$$\alpha((j,k,m,n),(j',k',m',n')) = \lambda^{mk'+nj'+mnm'+nm'(m'-1)/2+k'n(n-1)/2+m'n(n-1)(n-2)/6},$$

we have $A_\theta^{5,6} \cong C^*(H_4, \alpha)$.

Other infinite dimensional simple quotients of $C^*(H_{5,6})$. Suppose that λ is a primitive qth root of unity and that A is a simple quotient of $C^*(H_{5,6})$ that is irreducibly represented and generated by unitaries U and V and subsidiary operators W and X satisfying (6.1). Guided by $C^*(A_\theta^4, \mathbb{Z})$ in Theorem 6.1, we note first that X^q commutes with U and V and so by irreducibility equals $\mu' I$, a multiple of the identity.

1. When μ' is not a root of unity, substitute $X = \mu X_1$, where $\mu^q = \mu'$, in (6.1) and get

$$\begin{cases} UV = WVU, & UW = \mu X_1 WU, \quad UX_1 = \lambda X_1 U, \\ VW = \lambda WV, & VX_1 = X_1 V, \quad WX_1 = X_1 W \text{ and } X_1^q = 1. \end{cases} \tag{6.3}$$

Then the second, third and last equations show that U, W and X_1, with the correspondence $(U, W, X_1) \sim (U, V, W)$, satisfy (0.2) of Theorem 0.2 in the Preliminaries, so these unitaries generate a simple C^*-algebra ${}_q A_\beta \subset$ A, where $\mu = e^{2\pi i\beta}$. The remaining unitary V provides an automorphism ν of ${}_qA_\beta$, ν: $U \mapsto W^{-1}U$ and $W \mapsto \lambda W$, and ν generates an action of $\mathbb{Z}$ on ${}_qA_\beta$. Since $K_1({}_qA_\beta) \cong \mathbb{Z}^2$ with generators $[U]$ and $[W]$, we can argue again as in the proof of Theorem 4.1 that the automorphisms ν^n, $n \neq 0$, are outer; so $C^*({}_qA_\beta, \mathbb{Z})$ is simple and A is isomorphic to it.

2. When μ is also a root of unity (as well as λ), say a primitive q'th root of unity, a modification of the presentation $C^*(H_4, \alpha)$ for $A_\theta^{5,6}$ (mentioned above) gives the simple C^*-algebra generated by U and V satisfying (6.1). If $q'' = \mathrm{lcm}\{q, q'\}$, then $W^{q''}$ commutes with both U and V and equals $\eta' I$, a multiple of the identity. Suppose that η' is not a root of unity, and substitute $W = \eta W_1$, where $\eta^{q''} = \eta'$ (as well as $X = \mu X_1$) in (6.1). The result is

$$\begin{cases} UV = \eta W_1 VU, & UW_1 = \mu X_1 W_1 U, \quad UX_1 = \lambda X_1 U, \\ VW_1 = \lambda W_1 V, & VX_1 = X_1 V, \quad W_1 X_1 = X_1 W_1, \quad X_1^q = 1 = W_1^{q''}. \end{cases} \tag{6.4}$$

Now use (6.4) to simplify the product

$$(X_1^j W_1^k V^m U^n)(X_1^{j'} W_1^{k'} V^{m'} U^{n'})$$

and get $cX_1^{j+j'+nk'+m'n(n-1)/2}W_1^{k+k'+nm'}V^{m+m'}U^{n+n'}$, where

$$c = \lambda^{mk'+nj'+mnm'+nm'(m'-1)/2+k'n(n-1)/2+m'n(n-1)(n-2)/6}\mu^{nk'}\eta^{nm'}.$$

Then the group we need is $\mathrm{H} = \mathbb{Z}_q \times \mathbb{Z}_{q''} \times \mathbb{Z} \times \mathbb{Z}$ with multiplication

$$\begin{aligned}&(j,k,m,n)(j',k',m',n')\\ &= ((j+j'+nk'+m'n(n-1)/2)\bmod q,\ (k+k'+nm')\bmod q'',\ m+m',\ n+n')\end{aligned}$$

and cocycle α: $\mathrm{H}\times\mathrm{H} \to \mathbb{C}$, $\alpha((j,k,m,n),(j',k',m',n')) = c$. To see that the crossed product $C^*(\mathrm{H},\alpha)$ is simple, one must check that Packer's condition is satisfied; but this is easy to do, and so A is isomorphic to $C^*(\mathrm{H},\alpha)$.

3. When η is a root of unity (as well as μ and λ), a simple C^*-algebra generated by U and V satisfying (6.1) is finite dimensional.

6.2 THEOREM. *A C^*-algebra A is isomorphic to a simple infinite dimensional quotient of $C^*(\mathrm{H}_{5,6})$ if, and only if, A is isomorphic to $\mathrm{A}_\theta^{5,6}$ for some irrational θ, to $C^*({}_q\mathrm{A}_\beta,\mathbb{Z})$ or to $C^*(\mathrm{H},\alpha)$ (as in cases 1 and 2 above, respectively).*

The referee suggested that the algebras $C^*({}_q\mathrm{A}_\beta,\mathbb{Z})$ and $C^*(\mathrm{H},\alpha)$ above are isomorphic to matrix algebras. We show next that $C^*({}_q\mathrm{A}_\beta,\mathbb{Z})$ is isomorphic to a matrix algebra over a simple C^*-algebra $\mathcal{A} = \mathcal{A}(q,\lambda,\zeta)$ that is a variant of $\mathrm{A}'^4_\gamma(q)$, as in the paragraph following Theorem 5.2, and is also a 'scaled' variant of the algebra A_1 in Theorem 4.2. To be specific, $\mathcal{A}$ is to be generated by unitaries U_0, V_0 and W_0 satisfying

$$U_0V_0 = W_0^qV_0U_0,\quad U_0W_0 = \zeta W_0U_0 \text{ and } V_0W_0 = \lambda W_0V_0. \tag{6.5}$$

We remark that, although the generating equations for $\mathcal{A}$ still involve the root of unity λ, they do not involve a unipotent operator.

6.3 THEOREM. *When λ is a qth root of unity and $C^*({}_q\mathrm{A}_\beta,\mathbb{Z})$ is as in case 1 above, and $\mathcal{A}$ is as in the preceding paragraph, the C^*-crossed product $C^*({}_q\mathrm{A}_\beta,\mathbb{Z})$ is isomorphic to the matrix algebra $M_q(\mathcal{A})$.*

Proof. Let unitaries U_0, V_0 and W_0 satisfy (6.5), thus generating $\mathcal{A}$, and define matrices U', V' and W' in $M_q(\mathcal{A})$ exactly as in Theorem 5.6. Then the only difference between the situation here and that in 5.6 is that the last equation of (6.5) gives the requirement $V'W' = \lambda W'V'$ (the 4th equation of (6.3)). □

As might be expected, the construction for the final theorem is quite complicated.

6.4 THEOREM. *Let λ, μ, η, q, q', q'' and $C^*(\mathrm{H},\alpha)$ be as in case 2 above. Then $C^*(\mathrm{H},\alpha)$ is isomorphic to $M_{qq''}(\mathrm{A}^3_{\varphi'})$ for $e^{2\pi i\varphi'} = \eta^{qq''}(-1)^{q+q''}$.*

Proof. The demonstration is in three steps; the first step is like the proof of 6.3, the simplification method in the second step has been used in the proof of 4.3, and the third step is much like the proof of Theorem 0.2.

Step I. Define $\alpha = \mu^q \overline{\lambda}^{q(q-1)/2} = \mu^q(-1)^{q+1}$ and

$$\zeta = \alpha^{q-1}\eta^q \overline{\mu}^{q(q-1)/2}\lambda^{(q^3-q)/6} = \eta^q \mu^{q(q-1)/2}(-1)^{q+1}\lambda^{(q^3-q)/6},$$

and let an algebra Q be generated by unitaries U_2, V_2 and W_2 satisfying

$$U_2V_2 = \zeta W_2^q V_2 U_2, \quad U_2W_2 = \alpha W_2U_2, \quad V_2W_2 = \lambda W_2V_2 \text{ and } W_2^{q''} = 1. \tag{6.6}$$

(A concrete representation of Q is given in step III.) Define unitaries U', V' and subsidiary operators W' and X' in $M_q(Q)$ as follows:

U' has 1's on the subdiagonal and U_2 in the upper right hand corner;
V' has $V_2, \overline{\eta}\mu\overline{\lambda}W_2^{-1}V_2, \overline{\eta}^2\mu^3\overline{\lambda}^4 W_2^{-2}V_2, \overline{\eta}^3\mu^6\overline{\lambda}^{10}W_2^{-3}V_2, \ldots,$

$$\overline{\eta}^{q-1}\mu^{q(q-1)/2}\overline{\lambda}^{(q^3-q)/6}W_2^{-(q-1)}V_2 \text{ on the diagonal;}$$

W' has $W_2, \overline{\mu}\lambda W_2, \overline{\mu}^2\lambda^3 W_2, \ldots, \overline{\mu}^{q-1}\lambda^{q(q-1)/2}W_2$ on the diagonal;
X' has $1, \overline{\lambda}, \overline{\lambda}^2, \ldots, \overline{\lambda}^{q-1}$ on the diagonal.

Then U', V' and subsidiary operators W' and X' satisfy (6.4) and generate $M_q(Q)$.

Step II. Choose relatively prime c and d such that $\alpha^c\lambda^d = 1$, and choose a and b such that $ad - bc = 1$. Then set $U_3 = U_2^a V_2^b$ and $V_3 = U_2^c V_2^d$. It follows from (6.6) that the unitaries U_3, V_3 and W_2 satisfy

$$U_3V_3 = \zeta' W_2^q V_3 U_3, \quad U_3W_2 = \alpha' W_2 U_3, \quad V_3W_2 = W_2V_3 \text{ and } W_2^{q''} = 1,$$

where $\zeta' = \zeta\chi$ for some q''th root of unity χ and $\alpha' = \alpha^a\lambda^b$ is a primitive q''th root of unity. Also U_3, V_3 and W_2 generate Q.

Step III. Let $\gamma = e^{2\pi i\varphi'} = \zeta'^{q''}\alpha'^{qq''(q''-1)/2} = \eta^{qq''}(-1)^{q+q''}$ and let U_0 and V_0 be unitaries generating $A^3_{\varphi'}$, i.e., $U_0V_0 = \gamma V_0U_0$. Then the algebra Q is isomorphic to $M_{q''}(A^3_{\varphi'})$. For the unitaries U_3, V_3 and W_2 in II can be represented in $M_{q''}(A^3_{\varphi'})$ by specifying that

U_3 has U_0 in the upper right hand corner and 1's on the subdiagonal,
V_3 has $V_0, \overline{\gamma}\alpha'^q V_0, \overline{\gamma}^2\alpha'^{3q}V_0, \ldots, \overline{\gamma}^{q''-1}\alpha'^{qq''(q''-1)/2}$ on the diagonal,
W_2 has $1, \overline{\alpha'}, \overline{\alpha'}^2, \ldots, \overline{\alpha'}^{q''-1}$ on the diagonal, and these matrix unitaries generate $M_{q''}(A^3_{\varphi'})$. □

Concluding remarks

Packer [14] has classified the quotients $A^{5,4}_{\theta,\varphi}$ (of class 3) and A_1 (of class 2) of $C^*(H_{5,4})$. We contemplate the analogous classification of the rest of the simple C^*-algebras considered here in a subsequent paper.

Another project to consider concerns the other lattice subgroups of the connected Lie groups $G_{5,i}$, $1 \leq i \leq 6$. In [10] we indicated that the 3-dimensional connected Lie group G_3 (the Heisenberg group) admits infinitely many non-isomorphic lattice subgroups and that the situation is more complicated for the 4-dimensional group. It is to be expected that the situation is even more complicated for the 5-dimensional groups.

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